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EUCLIDEAN YANG – MILLS THEORY IN CONSTANT BACKGROUND FIELDS AND UNSTABLE MODES

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EUCLIDEAN YANG-MILLS THEORY IN CONSTANT BACKGROUND FIELDS AND UNSTABLE MODES

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Abstract

Within Euclidean SU(2) Yang-Mills theory considered in certain constant non-abelian background fields the explicit construction of gluon and ghost Green functions is presented paying special attention to unstable modes. The propagators obtained turn out to be purely real. The propagator expressions given for two different background fields are related to each other in the coincidence limit of these background configurations. On the basis of the derived results the imaginary part of the Euclidean Yang-Mills theory effective action is discussed showing that no higher loop corrections beyond the well known 1-loop result appear. Also arguments are given that the 1-loop imaginary part should not be trusted in.

1. Introduction

Non-Abelian gauge theory in background fields has been an issue by now for over a decade. Started by the investigation of Batalin, Matinyan and Savvidi [1] numerous papers dealing with the subject appeared. Continued interest in these background field problems is tightly connected with the search for the ground state in QCD — a problem which remained unsolved so far — as well as with certain classes of problems in electro-weak theory. One of the peculiar features of non-abelian gauge theories in (constant) background fields is the appearance of unstable modes as has been pointed out by Nielsen and Olesen [2]. These unstable modes lead to a 1-loop effective action $\Gamma^{[1]}$ which calculated on the basis of the master formula $\Gamma^{[1]} \sim \ln \det K$ exhibits an imaginary part. This is said to signal an instability of the trial ground state modelled by the given background field configuration. Over the time there has been much trouble about these unstable modes and their impact. Different lines of thought have been tried to tackle the problem (The few selected references [2]-[11] given should represent here the subject only.). So, we felt motivated to consider the whole problem on a broader footing [12].

Considering Yang-Mills theory in constant background fields let us remind you that the imaginary part in the 1-loop effective action already shows up in the Euclidean version of the theory in opposition to the imaginary part of the QED effective action in the case of a constant electric field [13]. Therefore, in the present article we focus on the Euclidean Yang-Mills theory effective action. We are going to report on the explicit construction of gluon and ghost Green functions in certain constant non-abelian background fields paying special attention to the unstable modes. On the basis of these results we discuss the imaginary part of the Euclidean effective action.

We are considering Euclidean SU(2) Yang-Mills theory in the covariant background gauge (SU(2) has been chosen for simplicity). Before starting the explicit construction of Green functions let us give a few general properties of SU(N) gluon and ghost Green functions in non-abelian background fields obeying the classical field equation $\left({}_x\partial_\mu = \frac{\partial}{\partial x^\mu} \right)$

$$D^{ab}_\mu \, F^b_{\mu\nu} = 0 \quad , \tag{1}$$

$$
\begin{aligned}
{}_xD^{ab}_\mu &= {}_xD^{ab}_\mu(B) = \delta^{ab} \, {}_x\partial_\mu + g f^{abc} B^c_\mu(x) \quad , \\
{}_xF^a_{\mu\nu} &= {}_xF^a_{\mu\nu}(B) = {}_x\partial_\mu B^a_\nu(x) - {}_x\partial_\nu B^a_\mu(x) + g f^{abc} B^b_\mu(x) B^c_\nu(x) \quad .
\end{aligned}
$$

The gluon and ghost Green functions are defined as the inverse of the respective quadratic kernels of the action in the given background.

$$K^{ab}_{\mu\nu}(\alpha; x) \, G_0{}^{ba'}_{\nu\mu'}(\alpha; x, x') = \delta^{aa'} \, \delta_{\mu\mu'} \, \delta^{(4)}(x - x') \tag{2}$$

$$K^{ab}(x) \, G_0{}^{ba'}(x, x') = \delta^{aa'} \, \delta^{(4)}(x - x') \tag{3}$$

$$K^{ab}_{\mu\nu}(\alpha; x) = \delta_{\mu\nu} \, K^{ab}(x) + (1 - \alpha^{-1}) \, D^{ac}_\mu \, D^{cb}_\nu - 2g \, f^{abc} \, F^c_{\mu\nu} \tag{2a}$$

$$K^{ab}(x) = - \, D^{ac}_\sigma \, D^{cb}_\sigma \tag{3a}$$

The operator relation (4) which holds due to eq. (1).

$${}_xD^{ac}_\mu \, K^{cb}_{\mu\nu}(\alpha; x) = \alpha^{-1} \, K^{ac}(x) \, {}_xD^{cb}_\nu \tag{4}$$

yields the following identities.

$${}_xD^{ac}_\mu \, G_0{}^{cb}_{\mu\nu}(\alpha; x, x') = \alpha \, {}_{x'}D^{bc}_\nu \, G_0{}^{ac}(x, x') \tag{5}$$

$${}_xD^{ac}_\mu \, {}_{x'}D^{bd}_\nu \, G_0{}^{cd}_{\mu\nu}(\alpha; x, x') = \alpha \, \delta^{ab} \, \delta^{(4)}(x - x') \tag{6}$$

Using the above equations one finds

$$G_0{}^{ab}_{\mu\nu}(\alpha; x, x') = G_0{}^{ab}_{\mu\nu}(1; x, x') + (1 - \alpha) \, {}_xD^{ac}_\mu \, {}_{x'}D^{be}_\nu \int d^4z \, G_0{}^{cd}(x, z) \, G_0{}^{de}(z, x') \quad . \tag{7}$$

Eq. (7) justifies the restriction to the case $\alpha = 1$ in the further consideration inasmuch as expressions for general α can be derived from the knowledge of this special case.

2. Propagator construction for two special background fields

Now, we will sketch the explicit construction of gluon and ghost Green functions for two special background configurations. As mentioned above we restrict our consideration to the gauge group SU(2) (i.e. $f^{abc} = \epsilon^{abc}$) and take the gauge parameter $\alpha = 1$. Configuration I is purely colormagnetic while configuration II has a colorelectric component too (These labels are inspired by the Minkowskian version of the theory.),

configuration I

$$B^a_\mu(x) = -\frac{1}{2}\, F^a_{\mu\nu} x_\nu = -\frac{1}{2}\, \delta^{a3}\, B\, \epsilon^{\perp}_{\mu\nu} x_\nu, \qquad \epsilon^{\perp}_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

configuration II

$$B^a_\mu(x) = -\frac{1}{2}\, F^a_{\mu\nu} x_\nu = -\frac{1}{2}\, \delta^{a3}\, [B\, \epsilon^{\perp}_{\mu\nu} x_\nu + B'\, \epsilon^{\parallel}_{\mu\nu} x_\nu], \quad \epsilon^{\parallel}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{8'}$$

$B > B' > 0$.

Note, that the gauge potentials (8), (8') are chosen within the Fock-Schwinger gauge

$$x_\mu\, B^a_\mu(x) = 0 \quad . \tag{9}$$

The diagonalization of the gluon and ghost kernels (2a), (3a) for configurations I and II is a straightforward task. Let us give a few details for case I and II (Wherever formulas for case II differ from those of case I the corresponding equation number carries a prime.). Color space and coordinate space diagonalization is performed with the help of the matrices U and R respectively.

$$K = U \begin{pmatrix} h^+ & & \\ & h^- & \\ & & \Delta \end{pmatrix} U^{-1} \tag{10}$$

$$K_{\mu\nu} = U \begin{pmatrix} \delta_{\mu\nu}\, h^+ + 2igF^3_{\mu\nu} & & \\ & \delta_{\mu\nu}\, h^- - 2igF^3_{\mu\nu} & \\ & & \delta_{\mu\nu}\, \Delta \end{pmatrix} U^{-1} \quad , \tag{11}$$

$$\Delta = -\partial_\sigma \partial_\sigma$$

Here, the following notation is used.

$$\delta_{\mu\nu}\, h^{\pm} \pm 2igF^3_{\mu\nu} = R \begin{pmatrix} h^{\pm\mp} & & & \\ & h^{\pm\pm} & & \\ & & g^{\pm\mp} & \\ & & & g^{\pm\pm} \end{pmatrix} R^{-1} \tag{12}$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \tag{13}$$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix} \tag{14),(14'}$$

$$h^{+\pm} = h^+ \pm 2gB, \quad h^{-\pm} = h^- \pm 2gB, \quad g^{+\pm} = h^+, \quad g^{-\pm} = h^- \tag{15}$$

$$h^{+\pm} = h^+ \pm 2gB, \quad h^{-\pm} = h^- \pm 2gB, \quad g^{+\pm} = h^+ \pm 2gB', \quad g^{-\pm} = h^- \pm 2gB' \tag{15'}$$

Now, using the creation and annihilation operators

$(z_{1,2} = \sqrt{gB/2}\, x_{1,2},\ z_{3,4} = \sqrt{gB'/2}\, x_{3,4})$

$$\begin{aligned} d^\dagger &= \frac{1}{2}\left[-\partial_1 + z_1 + i(-\partial_2 + z_2)\right], \quad d = \frac{1}{2}\left[\partial_1 + z_1 - i(\partial_2 + z_2)\right] \\ \tilde{d}^\dagger &= \frac{1}{2}\left[-\partial_1 + z_1 - i(-\partial_2 + z_2)\right], \quad \tilde{d} = \frac{1}{2}\left[\partial_1 + z_1 + i(\partial_2 + z_2)\right] \end{aligned} \tag{16}$$

(and in the case II in addition

$$\begin{aligned} c^\dagger &= \frac{1}{2}\left[-\partial_3 + z_3 + i(-\partial_4 + z_4)\right], \quad c = \frac{1}{2}\left[\partial_3 + z_3 - i(\partial_4 + z_4)\right] \\ \tilde{c}^\dagger &= \frac{1}{2}\left[-\partial_3 + z_3 - i(-\partial_4 + z_4)\right], \quad \tilde{c} = \frac{1}{2}\left[\partial_3 + z_3 + i(\partial_4 + z_4)\right] \quad) \end{aligned} \tag{16'}$$

obeying the commutation relations

$$[d, d^\dagger] = [\tilde{d}, \tilde{d}^\dagger] = 1$$

$$[d, \tilde{d}] = [d^\dagger, \tilde{d}^\dagger] = [d, \tilde{d}^\dagger] = [\tilde{d}, d^\dagger] = 0 \tag{17}$$

(and the same relations hold for the c-operators) we rewrite the operator $h^{\pm}$

$(x_\perp = (x_1, x_2), x_\parallel = (x_3, x_4))$

$$h^{\pm} = \Delta - \frac{1}{4} g^2 B^2 x_\perp^2 \pm igB(x_1\partial_2 - x_2\partial_1) \tag{18}$$

$$h^{\pm} = \Delta - \frac{1}{4}g^2(B^2 x_\perp^2 + B'^2 x_\parallel^2) \pm ig[B(x_1\partial_2 - x_2\partial_1) + B'(x_3\partial_4 - x_4\partial_3)] \tag{18'}$$

in the following manner

$$\begin{aligned} h^+(x) &= gB(2d^\dagger d + 1) + \Delta_\parallel \\ h^-(x) &= gB(2\tilde{d}^\dagger \tilde{d} + 1) + \Delta_\parallel \quad , \end{aligned} \tag{19}$$

$$\Delta_\parallel = -(\partial_3^2 + \partial_4^2) \quad ,$$

$$\begin{aligned} h^+(x) &= gB(2d^\dagger d + 1) + gB'(2c^\dagger c + 1) \\ h^-(x) &= gB(2\tilde{d}^\dagger \tilde{d} + 1) + gB'(2\tilde{c}^\dagger \tilde{c} + 1) \quad . \end{aligned} \tag{19'}$$

In the case I we end up with a 2-dimensional harmonic oscillator problem for the 1,2-direction and a free field situation in the 3,4-space. On the other hand, in case II we are dealing with two independent 2-dimensional harmonic oscillator problems. So, we find immediately eigenvalues and eigenfunctions for the operators (18) and (18') ($m, n, p, q \in I\!N$).

$$\lambda^+_{mnk_\parallel} = gB(2m+1) + k_\parallel^2, \quad \lambda^-_{mnk_\parallel} = gB(2n+1) + k_\parallel^2 \quad , \tag{20}$$

$$\lambda^+_{mnpq} = gB(2m+1) + gB'(2p+1), \quad \lambda^-_{mnpq} = gB(2n+1) + gB'(2q+1) \quad , \tag{20'}$$

$$\begin{aligned} u_{mnk_\parallel}(x) &= e^{ik_\parallel x_\parallel} \sqrt{\frac{gB}{2\pi m!\, n!}}\ (d^\dagger)^m\ (\tilde{d}^\dagger)^n\ e^{-\frac{1}{2}z_\perp^2} \\ &= e^{ik_\parallel x_\parallel} \sqrt{\frac{gB}{2\pi m!\, n!}}\ \frac{e^{-\frac{1}{2}z_\perp^2}}{2^{m+n}} \cdot \\ &\quad \cdot \sum_{\mu,\nu} \binom{m}{\mu} \binom{n}{\nu}\ i^{m-n+\mu-\nu} H_{\mu+\nu}(z_1)\ H_{m+n-\mu-\nu}(z_2) \quad , \end{aligned} \tag{21}$$

$$\begin{aligned} u_{mnpq}(x) &= \frac{g}{2\pi} \sqrt{\frac{BB'}{m!\, n!\, p!\, q!}}\ (d^\dagger)^m\ (\tilde{d}^\dagger)^n\ (c^\dagger)^p\ (\tilde{c}^\dagger)^q\ e^{-\frac{1}{2}z^2} \\ &= \frac{g}{2\pi} \sqrt{\frac{BB'}{m!\, n!\, p!\, q!}}\ \frac{e^{-\frac{1}{2}z^2}}{2^{m+n+p+q}} \cdot \\ &\quad \cdot \left[\sum_{\mu,\nu} \binom{m}{\mu} \binom{n}{\nu} i^{m-n+\mu-\nu}\ H_{\mu+\nu}(z_1)\ H_{m+n-\mu-\nu}(z_2) \right] \cdot \\ &\quad \cdot \left[\sum_{\alpha,\beta} \binom{p}{\alpha} \binom{q}{\beta} i^{p-q+\alpha-\beta} H_{\alpha+\beta}(z_3)\ H_{p+q-\alpha-\beta}(z_4) \right] \quad . \end{aligned} \tag{21'}$$

Using the integral representation of the Hermite polynomials

$$H_n(x) = \frac{(-1)^n}{2\sqrt{\pi}}\, e^{x^2} \int_{-\infty}^{\infty} dk\; e^{-\frac{1}{4}k^2 + ikx} \tag{22}$$

we rewrite eqs. (21), (21') in the following manner.

$$u_{mnk_\parallel}(x) = e^{ik_\parallel x_\parallel} \sqrt{\frac{gB}{2}}\, \frac{e^{\frac{1}{2}x_\perp^2}}{4\pi^{3/2}\sqrt{m!\,n!}} \cdot \int d^2k_\perp e^{-\frac{1}{4}k_\perp^2 + ik_\perp x_\perp} \left(\frac{-ik_1+k_2}{2}\right)^m \left(\frac{-ik_1-k_2}{2}\right)^n \tag{23}$$

$$u_{mnpq}(x) = g\sqrt{BB'}\frac{e^{\frac{1}{2}x^2}}{32\pi^3\sqrt{m!\,n!\,p!\,q!}} \cdot \int d^4k\; e^{-\frac{1}{4}k^2 + ikx} \left(\frac{-ik_1+k_2}{2}\right)^m \left(\frac{-ik_1-k_2}{2}\right)^n \cdot \left(\frac{-ik_3+k_4}{2}\right)^p \left(\frac{-ik_3-k_4}{2}\right)^q \tag{23'}$$

The eigenfunctions (23), (23') have been normalised properly in order to fulfil conditions (24), (24').

$$\int \frac{d^2k_\parallel}{(2\pi)^2} \sum_{m,n} u_{mnk_\parallel}(x)\; u^*_{mnk_\parallel}(x') = \delta^{(4)}(x-x') \tag{24}$$

$$\sum_{m,n,p,q} u_{mnpq}(x)\; u^*_{mnpq}(x') = \delta^{(4)}(x-x') \tag{24'}$$

Let us comment here on the unstable modes present in situation I and II. Inspection of eqs. (15), (15'), (20), (20') shows that $h^{\pm-}$ in both cases exhibits negative eigenvalues, namely ($[l] = l' \in \mathbb{N}$, where $1 \geq l' - l > 0$)

$$\text{configuration } I \qquad \lambda^+_{0nk_\parallel} = \lambda^-_{m0k_\parallel} = k_\parallel^2 - gB, \quad k_\parallel^2 < gB\,, \tag{25}$$

$$\text{configuration } II \qquad \lambda^+_{0npq} = gB'(2p+1) - gB, \quad p < \left[\frac{1}{2}\left(\frac{B}{B'} - 1\right)\right]\,,$$

$$\lambda^-_{m0pq} = gB'(2q+1) - gB, \quad q < \left[\frac{1}{2}\left(\frac{B}{B'} - 1\right)\right]\,. \tag{25'}$$

In constructing the inverse of the gluon kernel (2a) negative modes do not impose any problem but zero modes do. The latter may be prevented in the case II without serious loss of generality by choosing $B/B' \neq (2p+1), p \in \mathbb{N}$. Furthermore, to deal with an isolated zero mode as it may occur in case II is a manageable problem [14]. On the other hand, configuration I shows a zero mode in any case for $gB > 0$. So, the Green function construction has to be supplemented by a prescription to deal with the pole connected with the zero mode. A few lines below we will discuss this point in greater detail.

Now, we are constructing the inverse of the operator $(h^{\pm} + b)$, $b \in \mathbb{C}$ in the case I.

$$
\begin{aligned}
&[h^{\pm} + b]^{-1}(x, x') = \\
&= \int \frac{d^2 k_{\|}}{(2\pi)^2} \sum_{m,n} \frac{u_{mnk_{\|}}(x)\, u^*_{mnk_{\|}}(x')}{\lambda_{mnk_{\|}} + b} \\
&= \int \frac{d^2 k_{\|}}{(2\pi)^2} \frac{e^{ik_{\|}(x-x')_{\|}}}{4\pi}\, e^{\pm \frac{i}{2} gB\,(x_1 x_2' - x_2 x_1')} \,. \\
&\qquad \cdot \int_0^1 \frac{d\alpha}{1-\alpha}\, \alpha^{\left(\frac{1}{2gB}(k_{\|}^2 + b) - \frac{1}{2}\right)}\, e^{-\frac{gB}{4}\frac{1+\alpha}{1-\alpha}(x-x')_{\perp}^2} .
\end{aligned}
\tag{26}
$$

Introducing a Fourier transformation in the 1,2-space and substituting $\alpha = e^{-s}$ we find (for $\rho(x, x')$ see eq. (30))

$$
\begin{aligned}
&[h^{\pm} + b]^{-1}(x, x') = \\
&= e^{\pm i\rho(x,x')} \int \frac{d^4 k}{(2\pi)^4}\, e^{ik(x-x')} \int_0^\infty \frac{ds}{\cosh gBs}\, e^{-(k_{\|}^2 + b)s - \frac{k_{\perp}^2}{gB}\tanh gBs} &(27a)\\
&= \frac{gB}{16\pi^2}\, e^{\pm i\rho(x,x')} \int_0^\infty \frac{ds\, e^{-bs}}{s \sinh gBs}\, e^{-(x-x')_{\|}^2 \frac{s}{4} - \frac{gB}{4}(x-x')_{\perp}^2 \coth gBs} \,. &(27b)
\end{aligned}
$$

This expression is valid for $Re b > -gB$. In order to construct the inverse of the gluon kernel (2a) we need a valid expression of the left hand side of eq. (26) for $Re b \geq -2gB$ inasmuch as we have to deal with the negative modes (25) already discussed. To achieve this goal we continue the k-space expression in eq. (27a) analytically and supplement the k-integration performed by a prescription to deal with the created pole in $k_{\|}^2$. We find

$$[h^{\pm} + b]^{-1}(x, x') =$$

$$= e^{\pm i\rho(x,x')} \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')} e^{\frac{-k_\perp^2}{gB}} \left\{ \int_0^\infty \frac{ds}{\cosh gBs}\, e^{-(k_\parallel^2 + b)s} \cdot \right.$$

$$\left. \cdot \left[e^{\frac{k_\perp^2}{gB}(1-\tanh gBs)} - 1 - e^{-2gBs} \right] + \frac{2}{k_\parallel^2 + gB + b} \right\} \tag{28a}$$

$$= \frac{gB}{16\pi^2}\, e^{\pm i\rho(x,x')}\, e^{-\frac{gB}{4}(x-x')_\perp^2} \left\{ \int_0^\infty \frac{ds}{s}\, e^{-bs}\, e^{-(x-x')_\parallel^2 \frac{s}{4}} \cdot \right.$$

$$\cdot \left[\frac{e^{\frac{gB}{4}(x-x')_\perp^2\,(1-\coth gBs)}}{\sinh gBs} - 2\, e^{-gBs} \right] +$$

$$\left. + 4 \left[\begin{matrix} K_0\left(\sqrt{(x-x')_\parallel^2 (gB+b)}\right) \\ -\frac{\pi}{2}\left(N_0\left(\sqrt{(x-x')_\parallel^2 (gB-b)}\right) + \kappa\, J_0\left(\sqrt{(x-x')_\parallel^2 (-gB-b)}\right)\right) \end{matrix} \right] \right\},$$

$$\begin{matrix} Re\, b > -gB \\ -gB > Re\, b > -3gB' \end{matrix} \quad \kappa \in C \tag{28b}$$

This representation specified to $b = -2gB$ effectively corresponds to the separation of a term containing the contribution of the negative modes of the gluon kernel (2a).

$$\int \frac{d^2 k_\parallel}{(2\pi)^2} \sum_{n=0}^{\infty} \frac{u_{0nk_\parallel}(x) u^*_{0nk_\parallel}(x')}{k_\parallel^2 - gB} =$$

$$= \int \frac{d^2 k_\parallel}{(2\pi)^2} \sum_{n=0}^{\infty} u_{0nk_\parallel}(x) u^*_{0nk_\parallel}(x') \left\{ \text{p.v.}\left[\frac{1}{k_\parallel^2 - gB}\right] - \pi\, \kappa\, \delta(k_\parallel^2 - gB) \right\}$$

$$= 2\, e^{i\rho(x,x')} \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')}\, e^{-\frac{k_\perp^2}{gB}} \left\{ \text{p.v.}\left[\frac{1}{k_\parallel^2 - gB}\right] - \pi\, \kappa\, \delta(k_\parallel^2 - gB) \right\}$$

$$= -\frac{gB}{8\pi} e^{i\rho(x,x')}\, e^{-\frac{gB}{4}(x-x')_\perp^2} \left[N_0\left(\sqrt{gB(x-x')_\parallel^2}\right) + \kappa\, J_0\left(\sqrt{gB(x-x')_\parallel^2}\right) \right] \tag{29}$$

The coefficient κ is arbitrary here because in eq. (29) the term proportional to κ is a solution $f(x,x')$ of the homogeneous equation $h^{\pm-} f(x,x') = 0$ with an acceptable asymptotic behaviour. The inverse of $h^{\pm-}$ is not uniquely determined. Practically, the prescription mentioned above to deal with the pole $k_\parallel^2 = gB$ has to be specified by choosing κ.

In the case II the same procedure leads to (we do not display eq. (26'))

$$
\begin{aligned}
&[h^{\pm}+b]^{-1}(x,x') \\
&= e^{\pm i\rho(x,x')}\int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')}\int_0^\infty \frac{ds\ e^{-bs}}{\cosh gB's\cosh gBs}\,\cdot \\
&\qquad \cdot\, e^{-\frac{k_\parallel^2}{gB'}\tanh gB's-\frac{k_\perp^2}{gB}\tanh gBs}
\end{aligned}
\tag{27'a}
$$

$$
\begin{aligned}
&= \frac{g^2BB'}{16\pi^2}\, e^{\pm i\rho(x,x')}\int_0^\infty \frac{ds\ e^{-bs}}{\sinh gB's\sinh gBs}\,\cdot \\
&\qquad \cdot\, e^{-\frac{gB'}{4}(x-x')_\parallel^2\coth gB's-\frac{gB}{4}(x-x')_\perp^2\coth gBs}\,.
\end{aligned}
\tag{27'b}
$$

This representation is valid for $Re\ b > -g(B+B')$ only. Again, for the same reason as in case I we have to perform an analytic continuation for the right hand side of eq. (27'). Employing the property that

$$
\frac{1}{1+z}\, e^{\frac{\chi z}{1+z}} = \sum_{p=0}^{\infty}(-z)^p L_p(\chi), \quad z=-e^{-2gB's},\ \chi=\frac{gB'}{2}(x-x')_\parallel^2 \tag{28'a}
$$

is a generating functional of the Laguerre polynomials we end up with

$$
\begin{aligned}
&[h^{\pm}+b]^{-1}(x,x') = \\
&= \frac{g^2BB'}{16\pi^2}e^{\pm i\rho(x,x')}\, e^{-\frac{gB}{4}(x-x')_\perp^2}\left\{\int_0^\infty \frac{ds\ e^{-bs}}{\sinh gB's}e^{-\frac{gB'}{4}(x-x')_\parallel^2\coth gB's}\,\cdot\right. \\
&\qquad \cdot\left[\frac{e^{\frac{gB}{4}(x-x')_\perp^2\,(1-\coth gBs)}}{\sinh gBs}-2e^{-gBs}\right]+ \\
&\qquad \left. +\ 4e^{-\frac{gB'}{4}(x-x')_\parallel^2}\sum_{p=0}^{\infty}\frac{L_p\left(\frac{gB'}{2}(x-x')_\parallel^2\right)}{gB+gB'(2p+1)+b}\right\}, \\
&\qquad -g(B+B') > Re\,b > -g(3B+B')
\end{aligned}
\tag{28'b}
$$

Again, this representation corresponds to the separation of a term containing the contribution of the negative modes.

Expressions (27), (27') are translational invariant up to a phase

$$\rho(x,x') = \frac{gB}{2}(x_1x_2' - x_2x_1') \tag{30}$$

$$\rho(x,x') = \frac{gB}{2}(x_1x_2' - x_2x_1') + \frac{gB'}{2}(x_3x_4' - x_4x_3') \tag{30'}$$

which may be written as an integral over a straight line connecting x and x'.

$$\rho(x,x') = \int_x^{x'} dy_\mu \; B_\mu^3(y) \tag{31}$$

Collecting all contributions we obtain the desired final result for the ghost and gluon Green functions in the background I and II. The ghost propagator reads

$$G_0^{ab}(x,x') = \Phi^{ab}(x,x')D^0(x-x') + \frac{\delta^{a3}\,\delta^{b3}}{4\pi(x-x')^2} \tag{32}$$

$$G_0^{ab}(x,x') = \Phi^{ab}(x,x')D^{00}(x-x') + \frac{\delta^{a3}\,\delta^{b3}}{4\pi(x-x')^2} \tag{32'}$$

and the gluon propagator

$$\begin{aligned} G_0{}^{ab}_{\mu\nu}(1;x,x') = \Phi^{ab}(x,x')&\left\{\frac{1}{2}\delta^{\perp}_{\mu\nu}\left[D^+(x-x') + D^-(x-x')\right] + \delta^{\parallel}_{\mu\nu}D^0(x-x')\right\} + \\ &+ \Phi^{ac}(x,x')\epsilon^{cb3}\,\frac{1}{2}\epsilon^{\perp}_{\mu\nu}\left[D^+(x-x') - D^-(x-x')\right] + \frac{\delta^{a3}\,\delta^{b3}\,\delta_{\mu\nu}}{4\pi(x-x')^2} \end{aligned} \tag{33}$$

$$\begin{aligned} G_0{}^{ab}_{\mu\nu}(1;x,x') = \Phi^{ab}(x,x')&\left\{\frac{1}{2}\delta^{\perp}_{\mu\nu}\left[D^{+0}(x-x') + D^{-0}(x-x')\right] + \right. \\ &\left. + \frac{1}{2}\delta^{\parallel}_{\mu\nu}\left[D^{0+}(x-x') + D^{0-}(x-x')\right]\right\} + \\ &+ \Phi^{ac}(x,x')\epsilon^{cb3}\left\{\frac{1}{2}\epsilon^{\perp}_{\mu\nu}\left[D^{+0}(x-x') - D^{-0}(x-x')\right] + \right. \\ &\left. + \frac{1}{2}\epsilon^{\parallel}_{\mu\nu}\left[D^{0+}(x-x') - D^{0-}(x-x')\right]\right\} + \\ &+ \frac{\delta^{a3}\,\delta^{b3}\,\delta_{\mu\nu}}{4\pi(x-x')^2}\,. \end{aligned} \tag{33'}$$

The phase factor $\Phi^{ab}(x,x')$ is given by the following expression.

$$\Phi^{ab}(x,x') = \left(e^{\,i\tau\rho(x,x')}\right)^{ab} - \delta^{a3}\,\delta^{b3} = \delta^{ab}_{(12)}\cos\rho(x,x') + \epsilon^{ab3}\sin\rho(x,x'), \qquad (34)$$

$$\delta^{ab}_{(12)} = \delta^{ab} - \delta^{a3}\,\delta^{b3}\ ,\quad \tau^{ab} = -i\epsilon^{ab3}$$

Note, that the phase factor (34) is a purely real object. The used D- functions in the case I read explicitly

$$D^0(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx} \int_0^\infty \frac{ds}{\cosh gBs} e^{-k_\parallel^2 s - \frac{k_\perp^2}{gB}\tanh gBs} \qquad (35a)$$

$$= \frac{gB}{16\pi^2}\int_0^\infty \frac{ds}{s\sinh gBs}\, e^{-x_\parallel^2\frac{s}{4} - \frac{gB}{4}x_\perp^2\coth gBs} \qquad (35b)$$

$$D^+(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx} e^{-\frac{k_\perp^2}{gB}} \left\{ \int_0^\infty \frac{ds}{\cosh gBs}\, e^{2gB - k_\parallel^2 s} \cdot \right.$$

$$\cdot \left[e^{\frac{k_\perp^2}{gB}(1-\tanh gBs)} - 1 - e^{-2gBs} \right] +$$

$$\left. + \frac{2}{k_\parallel^2 - gB} \right\} \qquad (36a)$$

$$= \frac{gB}{16\pi^2}\, e^{-\frac{gB}{4}x_\perp^2} \left\{ \int_0^\infty \frac{ds\ e^{2gBs}}{s}\, e^{-x_\parallel^2\frac{s}{4}} \cdot \right.$$

$$\cdot \left[\frac{e^{\frac{gB}{4}x_\perp^2(1-\coth gBs)}}{\sinh gBs} - 2\,e^{-gBs} \right] -$$

$$\left. - 2\pi\left[N_0\left(\sqrt{gBx_\parallel^2}\right) + \kappa\, J_0\left(\sqrt{gBx_\parallel^2}\right) \right] \right\} \qquad (36b)$$

$$D^-(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx} \int_0^\infty \frac{ds}{\cosh gBs}\, e^{-2gBs}\, e^{-k_\parallel^2 s - \frac{k_\perp^2}{gB}\tanh gBs} \qquad (37a)$$

$$= \frac{gB}{16\pi^2}\int_0^\infty \frac{ds\ e^{-2gBs}}{s\sinh gBs}\, e^{-x_\parallel^2\frac{s}{4} - \frac{gB}{4}x_\perp^2\coth gBs} \qquad (37b)$$

For configuration II we are using the following D-functions whereby for $D^{+0}(x)$ the analytic continuation of $I(b;x)$ (lower line) has to be used.

$$D^{00}(x) = I(0;x), \qquad (35')$$

$$D^{+0}(x) = I(-2gB;x), \qquad D^{0+}(x) = I(-2gB';x) \tag{36'}$$

$$D^{-0}(x) = I(2gB;x), \qquad D^{0-}(x) = I(2gB';x) \tag{37'}$$

$$I(b,x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \int_0^\infty \frac{ds\ e^{-bs}}{\cosh gB's \cosh gBs}\ e^{-\frac{k_\parallel^2}{gB'} \tanh gB's - \frac{k_\perp^2}{gB} \tanh gBs} \tag{38}$$

$$= \frac{g^2 BB'}{16\pi^2}\ e^{-\frac{gB}{4}x_\perp^2} \left\{ \int_0^\infty \frac{ds\ e^{-bs}}{\sinh gB's}\ e^{-\frac{gB'}{4}x_\parallel^2 \coth gB's} \cdot \right.$$

$$\cdot \left[\frac{e^{\frac{gB}{4}x_\perp^2 (1-\coth gBs)}}{\sinh gBs} - 2e^{-gBs} \right] +$$

$$\left. + \ 4e^{-\frac{gB'}{4}x_\parallel^2} \sum_{p=0}^\infty \frac{L_p\left(\frac{gB'}{2}x_\parallel^2\right)}{gB + gB'(2p+1) + b} \right\} \tag{39}$$

3. Ambiguity for the gauge field propagator at $B' \to 0$

Having obtained the gluon and ghost Green functions for configuration I and II let us discuss these results next. As is obvious configuration I (eq. (8)) can be obtained from configuration II (eq. (8')) by performing the limit $B' \to 0$. Therefore, it is legitimate to ask for the limit $B' \to 0$ of the Green functions in case II. Naively one would expect getting the Green functions in case I from the case II Green functions. But, as we have seen by discussing the negative and zero modes of the gluon kernel (2a) obviously the Yang-Mills theory is better defined in the background II than in the background I. The only function one has to worry about is $D^{+0}(x)$ contained in the gluon Green function. Does $\lim_{B' \to 0} D^{+0}(x) = D^+(x)$ hold? We have seen that $D^+(x)$ contains an arbitrary parameter κ but $D^{+0}(x)$ does not and is completely well defined. So, we are going to define now the Green functions in case I by taking the $B' \to 0$ limit of the case II Green functions.

Consider the essential part of the last term in expression (39) for $Re\ b > -gB$. We find in terms of the confluent hypergeometric function Ψ $(\gamma = 1/2 + (gB+b)/2gB',\ \chi =$

$gB'x_\parallel^2/2)$

$$\sum_{p=0}^{\infty}\frac{L_p(\chi)}{p+\gamma}=\sum_{p=0}^{\infty}\int_0^1 dz\, z^{p+\gamma-1}L_p(\chi)=\int_0^1 dz\frac{z^{\gamma-1}}{1-z}\, e^{\frac{\chi z}{z-1}}=\Gamma(\gamma)\,\Psi(\gamma,1,\chi) \quad (40)$$

$$=-\Phi(\gamma,1,\chi)\ln\chi-\sum_{m=0}^{\infty}\frac{(\gamma)_m\chi^m}{(m!)^2}\left[\psi(\gamma+m)-2\psi(1+m)\right]. \quad (41)$$

The left hand side of eq. (40) for $Re\, b < -gB$ is defined here by analytic continuation of the right hand side of eq. (40). Let us consider $b=-2gB$ and construct the asymptotic expansion of the right hand side of eq. (40) for $B'\to 0$ using eq. (41). Define $\gamma = -N-\delta$, $N\in\mathbb{N}$, $0<Re\,\delta<1$ and $\bar{\chi}=gBx_\parallel^2/4=(N+\delta+1/2)\chi$. Then, performing the limit $N\to 0$ is equivalent to $B'\to 0$. We expand the confluent hypergeometric function Φ in powers of $1/N$.

$$\Phi(\gamma,1,\chi)=\left(1-\bar{\chi}+\frac{\bar{\chi}^2}{(2!)^2}-\frac{\bar{\chi}^3}{(3!)^2}+\cdots\right)+\frac{1}{2N}\left(\bar{\chi}-\frac{2^2\bar{\chi}^2}{(2!)^2}+\frac{3^2\bar{\chi}^3}{(3!)^2}-\cdots\right)+O\left(\frac{1}{N^2}\right)$$
$$=\left(1+\frac{\bar{\chi}}{2N}\right)J_0\left(2\sqrt{\bar{\chi}}\right)+O\left(\frac{1}{N^2}\right) \quad (42)$$

Next, we expand the digamma function for large argument.

$$\psi(\gamma+m)=\psi(-N-\delta+m)=\psi(N+\delta+1-m)+\pi\cot\pi\delta$$
$$=\ln N+\pi\cot\pi\delta+\frac{\delta+\frac{1}{2}-m}{N}+O\left(\frac{1}{N^2}\right) \quad (43)$$

Furthermore,

$$\sum_{m=0}^{\infty}\frac{(\gamma)_m\chi^m}{(m!)^2}\psi(\gamma+m)=\left\{\ln N+\pi\cot\pi\delta+\frac{1}{N}\left[\frac{\bar{\chi}}{2}(\ln N+\pi\cot\pi\delta)+\delta+\frac{1}{2}\right]\right\}J_0(2\sqrt{\bar{\chi}})+$$
$$+\frac{\sqrt{\bar{\chi}}}{N}J_1(2\sqrt{\bar{\chi}})+O\left(\frac{1}{N^2}\right), \quad (44)$$

and

$$\sum_{m=0}^{\infty}\frac{(\gamma)_m\chi^m}{(m!)^2}\psi(1+m)=$$
$$=\sum_{m=0}^{\infty}\frac{(-\bar{\chi})^m}{(m!)^2}\left[\psi(1+m)+\frac{\bar{\chi}}{2N}\psi(2+m)\right]+O\left(\frac{1}{N^2}\right)$$
$$=\left(1+\frac{\bar{\chi}}{2N}\right)\left[J_0(2\sqrt{\bar{\chi}})\ln\sqrt{\bar{\chi}}-\frac{\pi}{2}N_0(2\sqrt{\bar{\chi}})\right]+\frac{\sqrt{\bar{\chi}}}{2N}J_1(2\sqrt{\bar{\chi}})+O\left(\frac{1}{N^2}\right). \quad (45)$$

Finally,

$$\sum_{p=0}^{\infty} \frac{L_p\left(\frac{\bar{X}}{p-N-1/2}\right)}{p-N-\delta} = -\pi\left(1+\frac{\bar{X}}{2N}\right)\left[N_0(2\sqrt{\bar{X}}) + \cot\pi\delta J_0(2\sqrt{\bar{X}})\right] + O\left(\frac{1}{N^2}\right) \ . \tag{46}$$

After all we find

$$\lim_{B'\to 0} D^{+0}(x) = D^{+}(x) \ , \tag{46}$$

where $\delta = \pi^{-1} \arctan \kappa$. So, this procedure yields purely real Green functions in case I too, but the parameter κ depends on the kind of limit we are performing. As a problem presently open we mention that it is not clear whether physical quantities like the effective action are really independent of κ. The singularity for $\delta = 0, 1$ corresponds to the appearance of zero modes in case II.

Summarizing the presented study we would like to point out that the gluon and ghost propagators in cases I and II are real functions determined by convergent real proper time representations (in spite of the negative modes). So far the consideration of the analytical results — let us discuss now their impact on the imaginary part of the Euclidean effective action.

4. Imaginary part of the effective action

First, let us remind you the 1-loop result. It is given by the following expression [2],[14].

$$Im\, \Gamma^{[1]}[B, B']/V^4 = \pm \frac{g^2\, BB'}{4\pi}\left[\frac{1}{2}\left(\frac{B}{B'} - 1\right)\right] \tag{47}$$

Now, we are asking for higher loop corrections to the imaginary part (47). As we have mentioned already the gluon and ghost Green functions (and, of course, the vertices) in the considered constant background are purely real. Therefore, we find that no higher order corrections to the 1-loop imaginary part (47) exist. Certainly, this inevitable conclusion generates serious doubts in a quite often applied interpretation of eq. (47) as a physical decay probability . It would indeed be rather astonishing if a quantity like a decay probability would not be influenced by higher loop corrections.

The rather surprising result stated above encourages us to reconsider the 1-loop imaginary part itself. Let us underscore that the well known 1-loop result (47) has been obtained by exploiting the Gaussian approximation in the functional integral by neglecting the rules of its applicability — the quadratic kernel (2a) of the gluon action in a constant background has negative, i.e. unstable, modes as mentioned above. Therefore, the application of the standard formula for the Gaussian approximation $\Gamma^{[1]} \sim \ln \det K$ has to be considered here more as a kind of recipe or working rule rather than as a theoretically completely well based procedure. In difference to the imaginary part of the QED effective action in the case of a constant electric field [13] which has clear physical implications the imaginary part of the effective action in Yang-Mills theory found in accordance to usual wisdom in 1-loop approximation is already present in the Euclidean version of the theory. For the above mentioned reasons the 1-loop imaginary part in the Yang-Mills effective action looks suspicious and also deserves a fresh look from an alternative point of view.

In the following argument we are using a recently obtained special representation of the effective action in terms of the polarization tensor in a background field [15]. Specified to the 1-loop case this representation reads

$$\Gamma^{[1]}_{ren}[B] = \Gamma_{cl}[B] - \int_0^1 d\tau (1-\tau) \int d^4z \, d^4z' \, B^{a\mu}(z) \Pi^{[1]}_{ren}{}^{ab}_{\mu\nu}(\tau B; z, z') B^{b\nu}(z') \qquad (48)$$

Here $\Pi^{[1]}_{ren}{}^{ab}_{\mu\nu}(\tau B; z, z')$ is the renormalized 1-loop gluon polarization tensor in the given background which as usual can be depicted by the diagrams shown in figure 1.

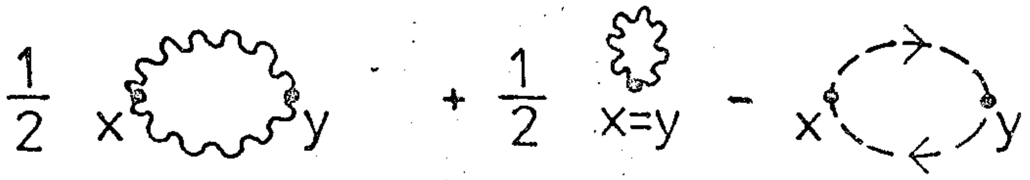

Figure 1. Diagrams contributing to the 1-loop gluon polarization tensor.

Here, the vertices and the propagators are those considered in the given background (in addition, for the representation (48) to be valid one has to fix the gauge parameter to $\alpha = -3$ in the 1-loop case). The representation (48) holds at least for background fields respecting the classical field equation (1) what is just the case in our consideration.

Now, the argument goes as follows. On the basis of the gluon and ghost propagators constructed explicitly in the present paper it is possible to build up the 1-loop gluon polarization tensor in the background (8), (8'). Although this is a quite tedious task and has not been done explicitly up to now it can be concluded that the polarization tensor in the given background is a purely real object inasmuch as all involved propagators (and vertices) have been shown to be purely real. $\Pi^{[1]}_{\tau c h}{}^{ab}_{\mu\nu}(\tau B, \tau B'; x, x')$ is a well defined object for all $\tau \in [0,1]$. Therefore, the application of (48) leads to the conclusion that the 1-loop effective action of Euclidean Yang-Mills theory is purely real in contradiction to usual wisdom.

Where does the difference come from? As we already mentioned above the usual formula

$$\Gamma^{[1]} \sim \ln \det K \tag{49}$$

is applicable for bosonic kernels K only if they are positive definite. In the presence of negative modes formula (49) has to be considered merely as a kind of recipe only. On the other hand, formula (48) is not necessarily connected with the functional integral approach and may equally be derived in some canonical quantization scheme. Test calculations in simpler theories have been shown that eq. (48) may be relied on [15]. Furthermore, eq. (48) is not invalidated in the presence of negative modes.

So, summarizing the results obtained we may conclude that the effective action of Euclidean Yang-Mills theory considered for a constant colormagnetic and colorelectric background does not exhibit any imaginary part (Although we have considered the gauge group SU(2) only it should be pointed out that the argument given equally applies to the general SU(N) case). In order to be cautious let the final lesson derived from our investigation be expressed by the message: don't trust in the imaginary part of the effective action of Euclidean Yang-Mills theory.

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