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A Grassmann integral equation

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The present study introduces and investigates a new type of equation which is called *Grassmann integral equation* in analogy to integral equations studied in real analysis. A Grassmann integral equation is an equation which involves Grassmann (Berezin) integrations and which is to be obeyed by an unknown function over a (finite-dimensional) Grassmann algebra $\mathcal{G}_m$ (i.e., a sought after element of the Grassmann algebra $\mathcal{G}_m$). A particular type of Grassmann integral equations is explicitly studied for certain low-dimensional Grassmann algebras. The choice of the equation under investigation is motivated by the effective action formalism of (lattice) quantum field theory. In a very general setting, for the Grassmann algebras $\mathcal{G}_{2n}$, $n=2,3,4$, the finite-dimensional analogues of the generating functionals of the Green functions are worked out explicitly by solving a coupled system of nonlinear matrix equations. Finally, by imposing the condition $G[\{\bar{\Psi}\},\{\Psi\}] = G_0[\{\lambda\bar{\Psi}\},\{\lambda\Psi\}] + \text{const}$, $0<\lambda\in\mathbf{R}$ ($\bar{\Psi}_k$, Ψ_k, $k=1,\ldots,n$, are the generators of the Grassmann algebra $\mathcal{G}_{2n}$), between the finite-dimensional analogues G_0 and G of the (“classical”) action and effective action functionals, respectively, a special Grassmann integral equation is being established and solved which also is equivalent to a coupled system of nonlinear matrix equations. If $\lambda\neq 1$, solutions to this Grassmann integral equation exist for $n=2$ (and consequently, also for any even value of n, specifically, for $n=4$) but not for $n=3$. If $\lambda=1$, the considered Grassmann integral equation (of course) has always a solution which corresponds to a Gaussian integral, but remarkably in the case $n=4$ a further solution is found which corresponds to a non-Gaussian integral. The investigation sheds light on the structures to be met for Grassmann algebras $\mathcal{G}_{2n}$ with arbitrarily chosen n. [DOI: 10.1063/1.1612896]

I. INTRODUCTION

The problem to be studied in the present paper is a purely mathematical one and one might arrive at it within various research programmes in mathematics and its applications. Our starting point will be (lattice) quantum field theory[1–4] and for convenience we will mainly use its terminology throughout the study (incidentally, for a finite-dimensional problem). However, one could equally well rely on the terminology of statistical mechanics or probability theory throughout. We will be interested in certain aspects of differential calculus in Grassmann (Graßmann) algebras[5] and in particular in Grassmann analogues to integral equations studied in real analysis which we will call *Grassmann integral equations*. A Grassmann integral equation is an equation which involves Grassmann (Berezin) integrations and which is to be obeyed by an unknown function over a (finite-dimensional) Grassmann algebra $\mathcal{G}_m$ (i.e., a sought after element of the Grassmann algebra $\mathcal{G}_m$). To the best of our knowledge this problem is considered for the first time in this paper. Of course, the following comment is due. Bearing in mind that in a Grassmann algebra taking a (Grassmann) derivative and an integral are equivalent operations we could equally well

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denote any Grassmann integral equation as a Grassmann differential equation. There is an extensive literature on supersymmetric extensions of differential equations. Corresponding research has been performed in areas such as supersymmetric field theory (see, e.g., Ref. 6, Vol. 3), superconformal field theory, the study of supersymmetric integrable models (see, e.g., Refs. 7, 8), and superanalysis (for a review of the latter see the recent book by Khrennikov,[9] in particular Chap. 2, and references therein). Only few mathematical references exist which treat pure Grassmann differential equations (understood in the narrow sense, i.e., in a nonsupersymmetric setting).[10–12] In the physics literature, specifically in the quantum field theoretic literature, such equations (in general, for infinite-dimensional Grassmann algebras) can be found in studies of purely fermionic models by means of the Schwinger–Dyson equations[13–20] or the Schrödinger representation (Refs. 21, 22 and follow-up references citing these). Within the framework of supersymmetric generalizations of conventional analysis, it is customary to consider all structures in strict analogy to real (complex) analysis. Consequently, as we will be lead to the problem of Grassmann integral equations from the corresponding problem in real analysis the choice of this term should not lead to any objection. Incidentally, it might be interesting to note that Khrennikov[9] mentions [at the end of Chap. 2, p. 102 (p. 106 of the English translation)] integral equations (item 9) among the subjects which have not yet been studied in superanalysis.

Having characterized in general the subject of the present study we will now explain in somewhat greater detail the problem we are interested in and where it arises from. Our motivation for the present investigation derives from quantum field theory. Quantum field theory is a rich subject with many facets and is being studied on the basis of a number of approaches and methods. For the present purpose, we rely on the functional integral approach to Lagrangian quantum field theory (see, e.g., Ref. 14, Ref. 15, Chap. 9, p. 425, Ref. 16, Ref. 6, Vol. I, Chap. 9, p. 376). To begin with, consider the theory of a scalar field ϕ in k-dimensional Minkowski space–time. By the following equations one defines generating functionals for various types of Green functions of this field (see, e.g., Ref. 14, Ref. 15, *loc. cit.*, Ref. 16, Ref. 17, Chap. 6, Ref. 6, Vol. II, Chap. 16, p. 63),

$$Z[J]=C\int D\phi e^{i\Gamma_0[\phi]+i\int \mathrm{d}^kx\, J(x)\phi(x)}, \tag{1}$$

$$W[J]=-i\ln Z[J], \tag{2}$$

$$\Gamma[\overline{\phi}]=W[J]-\int \mathrm{d}^kx\, J(x)\overline{\phi}(x), \tag{3}$$

$$\overline{\phi}(x)=\frac{\delta W[J]}{\delta J(x)}. \tag{4}$$

From Eq. (3) one finds the relation

$$\frac{\delta\Gamma[\overline{\phi}]}{\delta\overline{\phi}(x)}=-J(x). \tag{5}$$

In Eq. (1), $\int D\phi$ denotes the (infinite-dimensional) functional integration over the scalar field ϕ. $Z[J]$ is the generating functional of the Green functions,[23,24] $W[J]$ is the generating functional of the connected Green functions while the (first) Legendre transform $\Gamma[\overline{\phi}]$ of $W[J]$ is the generating functional of the one-particle-irreducible (1PI) Green functions. $\Gamma_0[\phi]$ is the so-called classical action of the theory and C some fixed normalization constant. $\Gamma[\overline{\phi}]$ is also called the effective action of the theory and, in principle, any information one might ever be interested in can be derived from it.

Equation (1) defines a map, $g_1:\Gamma_0[\phi]\rightarrow Z[J]$, from the class of functionals called classical actions to the class of functionals Z. Furthermore, we have mappings, $g_2:Z[J]\rightarrow W[J]$, [Eq. (2)], and $g_3 : W[J] \to \Gamma[\overline{\phi}]$ [Eq. (3)]. These three maps together define a map $g_3 \circ g_2 \circ g_1 = f : \Gamma_0[\phi] \to \Gamma[\overline{\phi}]$ from the set of so-called classical actions to the set of effective actions (we will call f the "action map"). In general, the action map is mathematically not well-defined in quantum field theory due to the occurrence of ultraviolet divergencies and one has to apply a regularization procedure for making proper mathematical sense of the above equations. A widely applied approach which is very natural from a mathematical point of view consists in studying quantum field theory not on a space–time continuum but on a space–time lattice (see, e.g., Refs. 1–4). The map f can be represented by the following single equation which can be derived from the Eqs. (1)–(3):

$$e^{i\Gamma[\overline{\phi}]} = C \int D\phi e^{i\Gamma_0[\phi + \overline{\phi}] + i\int \mathrm{d}^k x\, J(x)\phi(x)}. \tag{6}$$

$J(x)$ is given here by Eq. (5), consequently, Eq. (6) is only an implicit representation of the map f. For any quantum field theory, the properties of the action map f are of considerable interest but are hard if not impossible to study in general. In the simplest case, Γ_0 is a quadratic functional of the field ϕ (reasonably chosen to ensure that the functional integral is well defined). Then, the functional integral is Gaussian and one immediately finds (free field theory; const is some constant depending on the choice of C)

$$\Gamma[\phi] = \Gamma_0[\phi] + \text{const.} \tag{7}$$

There are very few other cases in which the formalism can explicitly be studied beyond perturbation theory. A number of exact results exist in quantum mechanics (which can be understood as quantum field theory in $0+1$-dimensional space-time; see, e.g., Refs. 25, 26). For some quantum field theoretic results see, e.g., Ref. 27.

It is common and successful practice in mathematics and physics to approach difficult infinite-dimensional problems from their finite-dimensional analogues. For example, in numerical studies within the framework of lattice quantum field theory the infinite-dimensional functional integral as present in Eq. (6) is replaced by a multidimensional multiple integral. The simplest finite-dimensional analogue of Eq. (6) is being obtained by replacing the infinite-dimensional functional integral by an one-dimensional integral over the real line. [More precisely, we obtain it from the Euclidean field theory version of Eq. (6) where the imaginary unit i in the exponent is replaced by $(-)1$. g' denotes here the first derivative of the function g.]

$$e^{\,g(y)} = C \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{\,g_0(x+y) - g'(y)x}. \tag{8}$$

Still, even the study of Eq. (8) represents a formidable task. The consideration of the (one-dimensional) analogues of the Eqs. (1)–(6) is often pursued under the name of zero-dimensional field theory [Refs. 28–42, Ref. 15, Subsec. 9-4-1, p. 463, Refs. 43–46, Refs. 18, 47–58, Ref. 59, Chap. 9, p. 211, Refs. 60–64; we have included into the list of reference also articles on the static ultralocal single-component scalar model but left aside papers on the corresponding $O(N)$ symmetric model].

For simplicity, the above discussion has been based on the consideration of a bosonic quantum field. However, fermionic (Grassmann valued) quantum fields are also of considerable physical interest (for a general discussion of Grassmann variables see Ref. 5). The analogue of Eq. (6) for a purely fermionic field theory of the Grassmann field Ψ, $\overline{\Psi}$ reads as follows:

$$e^{i\Gamma[\overline{\Psi},\Psi]} = C \int D(\chi,\overline{\chi})\;\; e^{\,i\Gamma_0[\overline{\chi}+\overline{\Psi},\chi+\Psi] + i\int \mathrm{d}^k x\;\; (\overline{\eta}(x)\chi(x) + \overline{\chi}(x)\eta(x))}, \tag{9}$$

$$\overline{\eta}(x) = \frac{\delta\Gamma[\overline{\Psi},\Psi]}{\delta\Psi(x)}, \qquad \eta(x) = -\frac{\delta\Gamma[\overline{\Psi},\Psi]}{\delta\overline{\Psi}(x)}. \tag{10}$$

Here, $D(\chi,\overline{\chi})$ denotes the infinite-dimensional Grassmann integration and the functional derivatives used in (10) are left Grassmann derivatives. The finite-dimensional (fermionic) analogues of the Eqs. (1)–(5) and (9), (10) consequently read[65–69]

$$Z[\{\overline{\eta}\},\{\eta\}]=C\int\prod_{l=1}^{n}(\mathrm{d}\chi_l\,\mathrm{d}\overline{\chi_l})e^{G_0[\{\overline{\chi}\},\{\chi\}]+\Sigma_{l=1}^{n}(\overline{\eta_l}\chi_l+\overline{\chi_l}\eta_l)},\tag{11}$$

$$W[\{\overline{\eta}\},\{\eta\}]=\ln Z[\{\overline{\eta}\},\{\eta\}],\tag{12}$$

$$G[\{\overline{\Psi}\},\{\Psi\}]=W[\{\overline{\eta}\},\{\eta\}]-\sum_{l=1}^{n}(\overline{\eta_l}\Psi_l+\overline{\Psi}_l\eta_l),\tag{13}$$

$$\overline{\Psi}_l=-\frac{\partial W[\{\overline{\eta}\},\{\eta\}]}{\partial\eta_l},\quad \Psi_l=\frac{\partial W[\{\overline{\eta}\},\{\eta\}]}{\partial\overline{\eta_l}},\tag{14}$$

and

$$e^{G[\{\overline{\Psi}\},\{\Psi\}]}=C\int\prod_{l=1}^{n}(\mathrm{d}\chi_l\,\mathrm{d}\overline{\chi_l})e^{G_0[\{\overline{\chi}+\overline{\Psi}\},\{\chi+\Psi\}]+\Sigma_{l=1}^{n}(\overline{\eta_l}\chi_l+\overline{\chi_l}\eta_l)},\tag{15}$$

$$\overline{\eta_l}=\frac{\partial G[\{\overline{\Psi}\},\{\Psi\}]}{\partial\Psi_l},\quad \eta_l=-\frac{\partial G[\{\overline{\Psi}\},\{\Psi\}]}{\partial\overline{\Psi}_l},\tag{16}$$

respectively. $\{\overline{\Psi}\}$, $\{\Psi\}$ denote the sets of Grassmann variables $\overline{\Psi}_l$, $l=1,\ldots,n$ and Ψ_l, $l=1,\ldots,n$, respectively, which are the generators of the Grassmann algebra $\mathcal{G}_{2n}$ [more precisely, we are considering a Grassmann algebra $\mathcal{G}_{4n}$ as the χ_l, $\overline{\chi_l}$ in Eq. (15) are also Grassmann variables, but we will ignore this mathematical subtlety in the following]. These generators obey the standard relations

$$\Psi_l\Psi_m+\Psi_m\Psi_l=\overline{\Psi}_l\Psi_m+\Psi_m\overline{\Psi}_l=\overline{\Psi}_l\overline{\Psi}_m+\overline{\Psi}_m\overline{\Psi}_l=0.\tag{17}$$

In this paper, we will concentrate on the explicit study of the Eqs. (15), (16) for small values of n ($n=2,3,4$) (some of the calculations have been performed by means of a purpose designed Mathematica program[70]). The Eqs. (15), (16) define (implicitly) a map f between the elements G_0 and G of the Grassmann algebra $\mathcal{G}_{2n}$ (in analogy to the infinite-dimensional case, we call the map f the action map). As we will see, the Eqs. (15), (16) are equivalent to a coupled system of nonlinear matrix equations which however can successively be solved completely (for a general exposition of matrix equations see, e.g., Refs. 71 and 72). This way, we will explicitly work out the action map f for the following fairly general ansatz for G_0:

$$\begin{aligned}G_0[\{\overline{\Psi}\},\{\Psi\}]=A^{(0)}&+\sum_{l,m=1}^{n}A^{(2)}_{l,m}\overline{\Psi}_l\Psi_m+\left(\frac{1}{2!}\right)^2\sum_{l_1,l_2,m_1,m_2=1}^{n}A^{(4)}_{l_1l_2,m_1m_2}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\Psi_{m_1}\Psi_{m_2}\\&+\left(\frac{1}{3!}\right)^2\sum_{l_1,l_2,l_3,m_1,m_2,m_3=1}^{n}A^{(6)}_{l_1l_2l_3,m_1m_2m_3}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\overline{\Psi}_{l_3}\Psi_{m_1}\Psi_{m_2}\Psi_{m_3}+\cdots\\&+\left(\frac{1}{n!}\right)^2\sum_{l_1,\ldots,l_n,m_1,\ldots,m_n=1}^{n}A^{(2n)}_{l_1\cdots l_n,m_1\cdots m_n}\overline{\Psi}_{l_1}\cdots\overline{\Psi}_{l_n}\Psi_{m_1}\cdots\Psi_{m_n}.\end{aligned}\tag{18}$$

Here, $A^{(0)}$ is some constant and the coefficients $A^{(2k)}_{\ldots}$, $k>1$, are chosen to be completely antisymmetric in the first and in the second half of their indices, respectively.

Although the explicit determination of the action map f between G_0 and G for low-dimensional Grassmann algebras represents previously unknown information, it may seem that the study of the map f for low-dimensional Grassmann algebras is a mathematical exercise of purely academic nature as quantum field theory and statistical mechanics are concerned with infinitely many degrees of freedom. To some extent this view may be justified for the time being but one should also take note of the fact that results for the Grassmann algebras $\mathcal{G}_{2n}$ and $\mathcal{G}_{2(n-1)}$ are closely related. To see this observe the following. Put in Eq. (18) considered in the case of the Grassmann algebra $\mathcal{G}_{2n}$ the coefficient $A^{(2)}_{n,n}$ equal to one but all other coefficients $A^{(2k)}_{\ldots}$, $k>1$, equal to zero whose index set {...} contains at least one index with value n.

$$A^{(2)}_{n,n}=1, \tag{19}$$

$$A^{(2k)}_{\ldots n\ldots}=0, \quad k>1. \tag{20}$$

Then, perform in Eq. (15) the Grassmann integrations with respect to χ_n, $\overline{\chi_n}$. Up to the factor $(\exp \overline{\Psi}_n\Psi_n)$ present on both sides (no summation with respect to n here) Eq. (18) then coincides with Eq. (18) considered in the case of the Grassmann algebra $\mathcal{G}_{2(n-1)}$. Consequently, results obtained for low-dimensional Grassmann algebras tightly constrain structures to be found for Grassmann algebras $\mathcal{G}_{2n}$ with arbitrarily chosen n. In fact, we will use this observation in two ways. On the one hand, we will rely on it in order to check the explicit results obtained for $n=4$ and $n=3$ for compatibility with those obtained for $n=3$ and $n=2$, respectively. On the other hand, on the basis of the above observation we will extrapolate some results obtained for $n=2,3,4$ to arbitrary n which can be used later in the future as working hypothesis for further studies.

Having explicitly worked out the action map f between G_0 and G for low-dimensional Grassmann algebras, we will not stop our investigation at this point but pursue our study still one step further. In the Refs. 28, 73, and 74 it has been argued (in a quantum field theoretic context), that it might be physically sensible and interesting to look for actions $\Gamma_0[\phi]$ which are not quadratic functionals of the field ϕ (i.e., which do not describe free fields) but for which Eq. (7) also applies. For the purpose of the present investigation we will slightly extend our search. We will look for solutions to the equation $(0<\lambda\in\mathbf{R})$

$$G[\{\overline{\Psi}\},\{\Psi\}]=G_0[\{\lambda\overline{\Psi}\},\{\lambda\Psi\}]+\Delta_f(\lambda). \tag{21}$$

λ can be considered here as a finite-dimensional analogue of a wave function renormalization constant in continuous space–time quantum field theory. $\Delta_f(\lambda)$ is some constant which is allowed to depend on λ. Equation (21) turns the implicit representation of the map f given by the Eqs. (15), (16) into a Grassmann integral equation for $G_0[\{\overline{\Psi}\},\{\Psi\}]$ (more precisely, into a nonlinear Grassmann integro-differential equation). As we will see, this Grassmann integral equation is equivalent to a coupled system of nonlinear matrix equations whose solution in turn is equivalent to the solution of the considered Grassmann integral equation. In the present study, to us Eq. (21) is just a mathematical problem to be studied. The possible relevance of any solution of Eq. (21) to physical problems will remain beyond the scope of the present paper. Some comments in this respect can be found in Refs. 28 and 73.

The plan of the paper is as follows. In Sec. II we work out explicitly the action map f between G_0 and G. Section II A contains some mathematical preliminaries while the following three sections are devoted to the cases $n=2,3,4$, respectively. Section II E finally studies the extrapolation of some of the results obtained to Grassmann algebras $\mathcal{G}_{2n}$ with arbitrarily chosen n. Section III is concerned with the study of the Grassmann integral equation (21). On the basis of the results obtained in Sec. II, in Secs. III A–III C it is solved for $n=2,3,4$, respectively. Then, Sec. III D contains an analysis of certain aspects of the solutions of the Grassmann integral equation found for $n=4$. In Sec. IV the discussion of the results and conclusions can be found. The paper is supplemented by three Appendixes.

II. THE ACTION MAP FOR LOW-DIMENSIONAL GRASSMANN ALGEBRAS

A. Some definitions

To simplify the further considerations we introduce a set of $\binom{n}{k}\times\binom{n}{k}$ matrices $\mathsf{A}^{(2k)}$ $(k=1,\ldots,n)$ by writing (choose $l_1<l_2<\cdots<l_k$, $m_1<m_2<\cdots<m_k$),

$$\mathsf{A}^{(2k)}_{LM}=A^{(2k)}_{l_1\cdots l_k,m_1\cdots m_k} \tag{22}$$

(we identify the indices L, M with the ordered strings $l_1\cdots l_k$, $m_1\cdots m_k$) or, more generally (not requesting $l_1<l_2<\cdots<l_k$, $m_1<m_2<\cdots<m_k$)

$$\mathsf{A}^{(2k)}_{LM} = \operatorname{sgn}[\sigma_a(l_1,\ldots,l_k)]\operatorname{sgn}[\sigma_b(m_1,\ldots,m_k)]A^{(2k)}_{l_1\cdots l_k,m_1\cdots m_k}. \tag{23}$$

The indices L,M label the equivalence classes of all permutations of the indices $l_1,\ldots,l_k$ and $m_1,\ldots,m_k$, respectively, and σ_a, σ_b are the permutations which bring the indices l_i,m_i $(i=1,\ldots,k)$ into order with respect to the $<$ relation [i.e., $\sigma_a(l_1)<\sigma_a(l_2)<\cdots<\sigma_a(l_k)$, $\sigma_b(m_1)<\sigma_b(m_2)<\cdots<\sigma_b(m_k)$]. The matrix elements of the matrix $\mathsf{A}^{(2k)}$ are arranged according to the lexicographical order of the row and column indices L, M. [We identify the indices L, M with the ordered strings $\sigma_a(l_1)\cdots\sigma_a(l_k)$, $\sigma_b(m_1)\cdots\sigma_b(m_k)$, respectively.]

We also define a set of (dual) $\binom{n}{k}\times\binom{n}{k}$ matrices $\mathsf{A}^{(2k)\star}$ $(k=1,\ldots,n)$ by writing

$$\mathsf{A}^{(2k)\star}=\mathcal{E}^{(k)}\mathsf{A}^{(2k)T}\mathcal{E}^{(k)T}, \tag{24}$$

where the $\binom{n}{k}\times\binom{n}{k}$ matrix $\mathcal{E}^{(k)}$ is defined by

$$\mathcal{E}^{(k)}_{LM}=\epsilon_{l_1\cdots l_{n-k}m_1\cdots m_k}, \tag{25}$$

consequently,

$$\mathcal{E}^{(k)T}=(-1)^{(n-k)k}\mathcal{E}^{(n-k)}. \tag{26}$$

[Quite generally, for any $\binom{n}{k}\times\binom{n}{k}$ matrix B we define $\mathsf{B}^\star$ by $\mathsf{B}^\star=\mathcal{E}^{(k)}\mathsf{B}^T\mathcal{E}^{(k)T}$.] It holds ($\mathbf{1}_r$ is the $r\times r$ unit matrix)

$$\mathcal{E}^{(k)}\mathcal{E}^{(k)T}=\mathbf{1}_{\binom{n}{k}}, \tag{27}$$

$$\mathcal{E}^{(k)T}\mathcal{E}^{(k)}=\mathbf{1}_{\binom{n}{k}}. \tag{28}$$

The transition from a matrix B to the matrix $\mathsf{B}^\star$ corresponds to applying the Hodge star operation to the two subspaces of the Grassmann algebra $\mathcal{G}_{2n}$ generated by the two sets of Grassmann variables $\{\overline{\Psi}\}$ and $\{\Psi\}$ and interchanging them (cf., e.g., Ref. 75, Part II, Chap. 4, p. 50). This operation on the matrix B is an involution as $(\mathsf{B}^\star)^\star=\mathsf{B}$.

Furthermore, it turns out to be convenient to define arrays of partition functions (i.e., their finite-dimensional analogues). First, we choose

$$C=e^{-G_0[\{0\},\{0\}]} = e^{-A^{(0)}} . \tag{29}$$

This choice in effect cancels any constant term in Eq. (18) (in this respect also see Ref. 73, p. 288). Now, we define[76] (we apply the convention $\int \mathrm{d}\chi_i\,\chi_j=\delta_{ij}$)

$$P=\mathsf{P}^{(2n)\star}=C\int\prod_{l=1}^{n}(\mathrm{d}\chi_l\,\mathrm{d}\overline{\chi_l})\,e^{\,G_0[\{\overline{\chi}\},\{\chi\}]}. \tag{30}$$

We then define arrays of partition functions $\mathsf{P}^{(2n-2k)\star}$ [these are $\binom{n}{k}\times\binom{n}{k}$ matrices] for subsystems of Grassmann variables where k degrees of freedom have been omitted (in slight misuse of physics terminology we denote a pair of Grassmann variables $\overline{\Psi}_l$, Ψ_m by the term degree of freedom; $l_1<l_2<\cdots<l_k$, $m_1<m_2<\cdots<m_k$ in the following):

$$\mathsf{P}_{LM}^{(2n-2k)\star}=\frac{\partial}{\partial A^{(2)}_{l_1,m_1}}\cdots\frac{\partial}{\partial A^{(2)}_{l_k,m_k}}P \tag{31}$$

$$=(-1)^k\frac{\partial}{\partial\eta_{l_1}}\frac{\partial}{\partial\overline{\eta}_{m_1}}\cdots\frac{\partial}{\partial\eta_{l_k}}\frac{\partial}{\partial\overline{\eta}_{m_k}}Z[\{\overline{\eta}\},\{\eta\}]\bigg|_{\overline{\eta}=\eta=0}. \tag{32}$$

Recursively, Eq. (31) can be written as follows [$l_k>l_{k-1}$, $m_k>m_{k-1}$; note the different meaning of the indices L, M on the left-hand side (lhs) and on the right-hand side (rhs) of the equation]:

$$\mathsf{P}_{LM}^{(2n-2k)\star}=\frac{\partial\ \mathsf{P}_{LM}^{(2n-2k+2)\star}}{\partial A^{(2)}_{l_k,m_k}}. \tag{33}$$

Let us illustrate the above definitions by means of a simple example. Choose

$$G_0[\{\overline{\chi}\},\{\chi\}]=\sum_{l,m=1}^{n}A^{(2)}_{l,m}\overline{\chi}_l\chi_m. \tag{34}$$

Then

$$Z[\{\overline{\eta}\},\{\eta\}]=\det\mathsf{A}^{(2)}e^{-\overline{\eta}[\mathsf{A}^{(2)}]^{-1}\eta} \tag{35}$$

and

$$\mathsf{P}^{(2n-2k)\star}=C^{\ n-k}(\mathsf{A}^{(2)}) \tag{36}$$

[cf. the references cited in relation to Eq. (A2) of Appendix A and Ref. 77, Sec. 2, Ref. 78, also see Ref. 17, Chap. 1, Sec. 1.9]. Here, $C^{n-k}(\mathsf{A}^{(2)})$ is the $(n-k)$th supplementary compound matrix of the matrix $\mathsf{A}^{(2)}$ (for a definition and some properties of compound matrices see Appendix A). By virtue of Eq. (A6) (see Appendix A) it holds

$$\mathsf{P}^{(2n-2k)\star}\mathsf{P}^{(2k)}=\mathsf{P}^{(2k)}\mathsf{P}^{(2n-2k)\star}=\det\mathsf{A}^{(2)}\mathbf{1}_{\binom{n}{k}}. \tag{37}$$

B. Explicit calculation: $n=2$

The case of the Grassmann algebra $\mathcal{G}_4$ $(n=2)$ to be treated in the present section is still algebraically fairly simple but already exhibits many of the features which we will meet in considering the larger Grassmann algebras $\mathcal{G}_6$, $\mathcal{G}_8$. Therefore, to some extent this section serves a didactical purpose in order to give the reader a precise idea of the calculations to be performed in the following two sections. These calculations will proceed exactly by the same steps as in this section but the algebraic complexity of the expressions will grow considerably. Also from a practical, calculational point of view it is advisable to choose an approach which proceeds stepwise from the most simple case $(n=2)$ to the more involved ones $(n=3,4)$ in order to accumulate experience in dealing with this growing complexity. On the other hand, the case $n=2$ is special in some respect and deserves attention in its own right.

According to our general ansatz (18) we put

$$G_0[\{\overline{\Psi}\},\{\Psi\}]=A^{(0)}+\sum_{l,m=1}^{2}A^{(2)}_{l,m}\overline{\Psi}_l\Psi_m+A^{(4)}_{12,12}\overline{\Psi}_1\overline{\Psi}_2\Psi_1\Psi_2 \tag{38}$$

and $G[\{\overline{\Psi}\},\{\Psi\}]$ can be written in the same way

$$G[\{\overline{\Psi}\},\{\Psi\}]=A^{(0)\prime}+\sum_{l,m=1}^{2} A^{(2)\prime}_{l,m}\overline{\Psi}_l\Psi_m+A^{(4)\prime}_{12,12}\overline{\Psi}_1\overline{\Psi}_2\Psi_1\Psi_2\,. \tag{39}$$

No other terms will occur for symmetry reasons. One quickly finds for the partition function [cf. Eq. (30)]

$$P=e^{\,A^{(0)\prime}}=\mathsf{P}^{(4)\star}=\det\mathsf{A}^{(2)}-\mathsf{A}^{(4)\star}. \tag{40}$$

Of course, here $\mathsf{A}^{(4)\star}=A^{(4)}_{12,12}$ applies—again ignoring the fact that (very formally) these constants live in different spaces, cf. Eq. (24). The notation $\mathsf{P}^{(4)\star}$ is introduced in order to indicate how in larger Grassmann algebras this partition function transforms under linear (unitary) transformations of the two subsets $\{\overline{\Psi}\}$, $\{\Psi\}$ of the generators of the Grassmann algebra. Clearly, $\mathsf{P}^{(4)\star}$ then transforms exactly the same way as $\mathsf{A}^{(4)\star}$ does and this fact suggests the chosen notation. (The same will apply to any other partition function $\mathsf{P}^{(2n)\star}$ for larger Grassmann algebras $\mathcal{G}_{2n}$.) The result of the map $g_2\circ g_1$ reads (adj B denotes here the adjoint matrix of the matrix B)

$$W[\{\overline{\eta}\},\{\eta\}]=\ln\mathsf{P}^{(4)\star}-\sum_{l,m=1}^{2}\frac{(\mathrm{adj}\mathsf{A}^{(2)})_{lm}}{\mathsf{P}^{(4)\star}}\overline{\eta}_l\eta_m+\frac{A^{(4)}_{12,12}}{(\mathsf{P}^{(4)\star})^2}\overline{\eta}_1\overline{\eta}_2\eta_1\eta_2\,. \tag{41}$$

The only assumption made to arrive at this result is that $\mathsf{P}^{(4)\star}\neq 0$. We can now proceed on the basis of the general Eq. (13) specified to $n=2$,

$$G[\{\overline{\Psi}\},\{\Psi\}]=W[\{\overline{\eta}\},\{\eta\}]-\sum_{l=1}^{2}(\overline{\eta}_l\Psi_l+\overline{\Psi}_l\eta_l). \tag{42}$$

We insert Eq. (39) onto the lhs of Eq. (42) and the explicit expressions for $\overline{\eta}$, η found from Eq. (39) according to Eq. (16) on its rhs. Comparing coefficients on both sides we find the following two coupled equations:

$$\mathsf{A}^{(2)\prime}=2\mathsf{A}^{(2)\prime}-\mathsf{A}^{(2)\prime}\frac{\mathrm{adj}\mathsf{A}^{(2)}}{\mathsf{P}^{(4)\star}}\mathsf{A}^{(2)\prime}, \tag{43}$$

$$A^{(4)\prime}_{12,12}=4A^{(4)\prime}_{12,12}-2\,\frac{\mathrm{tr}[\mathsf{A}^{(2)\prime}\;\;\mathrm{adj}\mathsf{A}^{(2)}]}{\mathsf{P}^{(4)\star}}A^{(4)\prime}_{12,12}+\left(\frac{\det\mathsf{A}^{(2)\prime}}{\mathsf{P}^{(4)\star}}\right)^2A^{(4)}_{12,12}\,. \tag{44}$$

Equation (43) can immediately be simplified to read

$$\mathsf{A}^{(2)\prime}=\mathsf{A}^{(2)\prime}\frac{\mathrm{adj}\,\mathsf{A}^{(2)}}{\mathsf{P}^{(4)\star}}\mathsf{A}^{(2)\prime}. \tag{45}$$

From Eq. (45) one recognizes that the matrix $\mathsf{A}^{(2)\prime}$ is the generalized {2}-inverse of the matrix adj $\mathsf{A}^{(2)}/\mathsf{P}^{(4)\star}$ (cf., e.g., Ref. 79, Chap. 1, p. 7).

We can now successively solve the Eqs. (43), (44). Choosing $\det\mathsf{A}^{(2)\prime}\neq 0$ [By virtue of Eq. (45) this entails $\det\mathsf{A}^{(2)}\neq 0$.], we immediately find from Eq. (45)

$$\mathsf{A}^{(2)\prime}=\left(\frac{\mathsf{P}^{(4)\star}}{\det\mathsf{A}^{(2)}}\right)\mathsf{A}^{(2)}. \tag{46}$$

Inserting this expression for $\mathsf{A}^{(2)\prime}$ into Eq. (44) yields the following solution:

$$A^{(4)\prime}_{12,12} = \left(\frac{\mathsf{P}^{(4)\star}}{\det \mathsf{A}^{(2)}} \right)^2 A^{(4)}_{12,12} . \tag{47}$$

In analogy to Eq. (40), we can now define a quantity

$$\mathsf{P}^{(4)\star\prime} = \det \mathsf{A}^{(2)\prime} - \mathsf{A}^{(4)\star\prime} \tag{48}$$

and from Eqs. (46), (47) we find [taking into account Eq. (40)]

$$\mathsf{P}^{(4)\star\prime} = \frac{(\mathsf{P}^{(4)\star})^3}{(\det \mathsf{A}^{(2)})^2} = \left(\frac{\mathsf{P}^{(4)\star}}{\det \mathsf{A}^{(2)}} \right)^2 \mathsf{P}^{(4)\star}. \tag{49}$$

Taking the determinant on both sides of Eq. (46) provides us with the following useful relation:

$$\det \mathsf{A}^{(2)\prime} = \frac{(\mathsf{P}^{(4)\star})^2}{\det \mathsf{A}^{(2)}} . \tag{50}$$

Up to this point, very little is special to the case $n=2$ and we will meet the analogous equations in the next sections.

We turn now to some features which are closely related to the algebraic simplicity of the case $n=2$ and which cannot easily be identified in larger Grassmann algebras. The Eqs. (49) and (50) can now be combined to yield the equation

$$\mathsf{P}^{(4)\star\prime} = \frac{\det \mathsf{A}^{(2)\prime}}{\det \mathsf{A}^{(2)}} \mathsf{P}^{(4)\star}, \tag{51}$$

which is converted ($\mathsf{P}^{(4)\star}, \det \mathsf{A}^{(2)\prime} \neq 0$ entail $\mathsf{P}^{(4)\star\prime} \neq 0$) into

$$\frac{\det \mathsf{A}^{(2)\prime}}{\mathsf{P}^{(4)\star\prime}} = \frac{\det \mathsf{A}^{(2)}}{\mathsf{P}^{(4)\star}} . \tag{52}$$

An equivalent form of Eq. (52) is

$$\frac{A^{(4)\prime}_{12,12}}{\det \mathsf{A}^{(2)\prime}} = \frac{A^{(4)}_{12,12}}{\det \mathsf{A}^{(2)}}. \tag{53}$$

From Eqs. (52) and (53) we recognize that for $n=2$ the action map f has an invariant which can be calculated from the left- or right-hand sides of these equations.

We are now going to invert the action map f.[80] From Eqs. (49) and (52) we easily find

$$\mathsf{P}^{(4)\star} = \frac{(\det \mathsf{A}^{(2)\prime})^2}{\mathsf{P}^{(4)\star\prime}} = \left(\frac{\det \mathsf{A}^{(2)\prime}}{\mathsf{P}^{(4)\star\prime}} \right)^2 \mathsf{P}^{(4)\star\prime}. \tag{54}$$

Equation (52) also allows us to find the following inversion formulas for the map f from Eqs. (46), (47):

$$\mathsf{A}^{(2)} = \left(\frac{\det \mathsf{A}^{(2)\prime}}{\mathsf{P}^{(4)\star\prime}} \right) \mathsf{A}^{(2)\prime}, \tag{55}$$

$$A^{(4)}_{12,12} = \left(\frac{\det \mathsf{A}^{(2)\prime}}{\mathsf{P}^{(4)\star\prime}} \right)^2 A^{(4)\prime}_{12,12} . \tag{56}$$

From the above equations we see that for $n=2$ the action map f can easily be inverted (once one assumes $\mathsf{P}^{(4)\star} \neq 0$, $\det \mathsf{A}^{(2)} \neq 0$, $\mathsf{P}^{(4)\star\prime} \neq 0$, $\det \mathsf{A}^{(2)\prime} \neq 0$).

C. Explicit calculation: $n=3$

The case $n=3$ is already considerably more involved in comparison with the case $n=2$ treated in the preceding section. In the present and the next sections, as far as possible and appropriate we will apply the same wording as in Sec. II B in order to emphasize their close relation.

We start by parametrizing G_0 and G according to our general ansatz [cf. Eq. (18) and Eqs. (38), (39)].

$$G_0[\{\overline{\Psi}\},\{\Psi\}]=A^{(0)}+\sum_{l,m=1}^{3} A^{(2)}_{l,m}\overline{\Psi}_l\Psi_m+\frac{1}{4}\sum_{l_1,l_2,m_1,m_2=1}^{3} A^{(4)}_{l_1l_2,m_1m_2}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\Psi_{m_1}\Psi_{m_2} + A^{(6)}_{123,123}\overline{\Psi}_1\overline{\Psi}_2\overline{\Psi}_3\Psi_1\Psi_2\Psi_3, \tag{57}$$

$$G[\{\overline{\Psi}\},\{\Psi\}]=A^{(0)\prime}+\sum_{l,m=1}^{3} A^{(2)\prime}_{l,m}\overline{\Psi}_l\Psi_m+\frac{1}{4}\sum_{l_1,l_2,m_1,m_2=1}^{3} A^{(4)\prime}_{l_1l_2,m_1m_2}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\Psi_{m_1}\Psi_{m_2} + A^{(6)\prime}_{123,123}\overline{\Psi}_1\overline{\Psi}_2\overline{\Psi}_3\Psi_1\Psi_2\Psi_3. \tag{58}$$

For the partition function we find [cf. Eq. (30)]

$$P=e^{A^{(0)\prime}}=\mathsf{P}^{(6)\star}=\det \mathsf{A}^{(2)}-\mathrm{tr}(\mathsf{A}^{(4)\star}\mathsf{A}^{(2)})-\mathsf{A}^{(6)\star} \tag{59}$$

$$=-2\det \mathsf{A}^{(2)}+\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)})-\mathsf{A}^{(6)\star}. \tag{60}$$

In analogy to Eq. (40), here $\mathsf{A}^{(6)\star}=A^{(6)}_{123,123}$ applies. In the lower line [Eq. (60)], we use the notation [cf. Eqs. (31) and (40)]

$$\mathsf{P}^{(4)}=C_2(\mathsf{A}^{(2)})-\mathsf{A}^{(4)},\quad \mathsf{P}^{(4)\star}=\mathrm{adj}\,\mathsf{A}^{(2)}-\mathsf{A}^{(4)\star} \tag{61}$$

[$\mathrm{adj}\,\mathsf{A}^{(2)}=C_2(\mathsf{A}^{(2)})^\star$].

After some calculation we obtain the following expression for $W[\{\overline{\eta}\},\{\eta\}]$ (to arrive at it we only assume $\mathsf{P}^{(6)\star}\neq 0$):

$$W[\{\overline{\eta}\},\{\eta\}]=\ln \mathsf{P}^{(6)\star}-\frac{\mathsf{P}^{(4)\star}_{lm}}{\mathsf{P}^{(6)\star}}\overline{\eta}_l\eta_m-\frac{\mathsf{A}^{(2)\star}_{ML}}{\mathsf{P}^{(6)\star}}\overline{\eta}_{l_1}\overline{\eta}_{l_2}\eta_{m_1}\eta_{m_2}-\frac{1}{2}\left(\frac{\mathsf{P}^{(4)\star}_{lm}}{\mathsf{P}^{(6)\star}}\overline{\eta}_l\eta_m\right)^2 + \frac{1}{\mathsf{P}^{(6)\star}}\left[1-\frac{\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)})}{\mathsf{P}^{(6)\star}}+\frac{2\det \mathsf{P}^{(4)\star}}{(\mathsf{P}^{(6)\star})^2}\right]\overline{\eta}_1\overline{\eta}_2\overline{\eta}_3\eta_1\eta_2\eta_3. \tag{62}$$

Here and in the following we use the notation $B_{ML}\overline{\eta}_{l_1}\overline{\eta}_{l_2}\eta_{m_1}\eta_{m_2}$ for a multiple sum over l_1, l_2, m_1, m_2 with the restrictions $l_1<l_2$, $m_1<m_2$ applied; $L=\{l_1,l_2\}$, $M=\{m_1,m_2\}$. The analogous convention is also applied to multiple sums over more indices. To arrive at the further results it is useful to take note of the equation

$$(\mathsf{P}^{(4)\star}_{lm}\overline{\eta}_l\eta_m)^2=-2\,C_2(\mathsf{P}^{(4)\star})_{LM}\ \overline{\eta}_{l_1}\overline{\eta}_{l_2}\eta_{m_1}\eta_{m_2}\ . \tag{63}$$

We proceed now exactly the same way as in Sec. II B. We insert Eq. (58) onto the lhs of Eq. (13) and the explicit expressions for $\overline{\eta}$, η found from Eq. (58) according to Eq. (16) on its rhs. Again, comparing coefficients on both sides we find the following three coupled nonlinear matrix equations:

$$\mathsf{A}^{(2)\prime}=2\mathsf{A}^{(2)\prime}-\mathsf{A}^{(2)\prime}\frac{\mathsf{P}^{(4)\star}}{\mathsf{P}^{(6)\star}}\mathsf{A}^{(2)\prime}, \tag{64}$$

$$\mathsf{A}^{(4)\star\prime}=4\mathsf{A}^{(4)\star\prime}+\mathsf{A}^{(4)\star\prime}\,\frac{\mathsf{A}^{(2)\prime}\mathsf{P}^{(4)\star}-\mathrm{tr}(\mathsf{A}^{(2)\prime}\mathsf{P}^{(4)\star})\mathbf{1}_3}{\mathsf{P}^{(6)\star}}+\frac{\mathsf{P}^{(4)\star}\mathsf{A}^{(2)\prime}-\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)\prime})\mathbf{1}_3}{\mathsf{P}^{(6)\star}}\,\mathsf{A}^{(4)\star\prime}-\mathrm{adj}\,\mathsf{A}^{(2)\prime}\,\frac{\mathsf{P}^{(6)\star}\mathsf{A}^{(2)}-\mathrm{adj}\,\mathsf{P}^{(4)\star}}{(\mathsf{P}^{(6)\star})^2}\,\mathrm{adj}\,\mathsf{A}^{(2)\prime}, \tag{65}$$

$$\begin{aligned}A^{(6)\prime}_{123,123}=6A^{(6)\prime}_{123,123}+\frac{2}{\mathsf{P}^{(6)\star}}\Big\{&-A^{(6)\prime}_{123,123}\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)\prime})+\mathrm{tr}(\mathsf{P}^{(4)\star}\mathrm{adj}\mathsf{A}^{(4)\star\prime})\\&+\mathrm{tr}(\mathsf{A}^{(2)}\mathrm{adj}\mathsf{A}^{(2)\prime})\mathrm{tr}(\mathsf{A}^{(2)\prime}\mathsf{A}^{(4)\star\prime})-\det\mathsf{A}^{(2)\prime}\mathrm{tr}(\mathsf{A}^{(2)}\mathsf{A}^{(4)\star\prime})+\frac{(\det\mathsf{A}^{(2)\prime})^2}{2}\Big\}\\&+\frac{2}{(\mathsf{P}^{(6)\star})^2}\Big\{\det\mathsf{A}^{(2)\prime}\mathrm{tr}(\mathsf{A}^{(4)\star\prime}\mathrm{adj}\;\mathsf{P}^{(4)\star})-\mathrm{tr}[\mathrm{adj}(\mathsf{A}^{(2)\prime}\mathsf{P}^{(4)\star})]\,\mathrm{tr}(\mathsf{A}^{(2)\prime}\mathsf{A}^{(4)\star\prime})\\&-\frac{1}{2}(\det\mathsf{A}^{(2)\prime})^2\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)})\Big\}+\frac{2}{(\mathsf{P}^{(6)\star})^3}(\det\mathsf{A}^{(2)\prime})^2\det\mathsf{P}^{(4)\star}.\end{aligned} \tag{66}$$

Equation (64) is equivalent to the equation

$$\mathsf{A}^{(2)\prime}=\mathsf{A}^{(2)\prime}\,\frac{\mathsf{P}^{(4)\star}}{\mathsf{P}^{(6)\star}}\,\mathsf{A}^{(2)\prime}. \tag{67}$$

The matrix $\mathsf{A}^{(2)\prime}$ is the generalized {2}-inverse of the matrix $\mathsf{P}^{(4)\star}/\mathsf{P}^{(6)\star}$ (cf., e.g., Ref. 79, Chap. 1, p. 7).

In analogy to the procedure applied in Sec. II B, we can now successively solve the Eqs. (64)–(66). Choosing $\det\mathsf{A}^{(2)\prime}\neq 0$ [by virtue of Eq. (67) this entails $\det\mathsf{P}^{(4)\star}\neq 0$], we immediately find from Eq. (67) an explicit expression for $\mathsf{A}^{(2)\prime}$. This can be inserted into Eq. (65) to also find an explicit expression for $\mathsf{A}^{(4)\star\prime}$. Finally, both these explicit expressions for $\mathsf{A}^{(2)\prime}$ and $\mathsf{A}^{(4)\star\prime}$ can now be inserted into Eq. (66) to solve it for $A^{(6)\prime}_{123,123}$. The results obtained read as follows:

$$\mathsf{A}^{(2)\prime}=\mathsf{P}^{(6)\star}[\mathsf{P}^{(4)\star}]^{-1}=\frac{\mathsf{P}^{(6)\star}}{\det\mathsf{P}^{(4)\star}}\,\mathrm{adj}\,\mathsf{P}^{(4)\star}, \tag{68}$$

$$\mathsf{A}^{(4)\star\prime}=-\frac{(\mathsf{P}^{(6)\star})^2}{\det\mathsf{P}^{(4)\star}}\left[\frac{\mathsf{P}^{(6)\star}}{\det\mathsf{P}^{(4)\star}}\,\mathsf{P}^{(4)\star}\mathsf{A}^{(2)}\mathsf{P}^{(4)\star}-\mathsf{P}^{(4)\star}\right], \tag{69}$$

$$A^{(6)\prime}_{123,123}=\frac{(\mathsf{P}^{(6)\star})^5}{(\det\mathsf{P}^{(4)\star})^2}\left\{1-\frac{2}{\det\mathsf{P}^{(4)\star}}\,\mathrm{tr}[\mathrm{adj}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)})]\right\}+\frac{3(\mathsf{P}^{(6)\star})^4}{(\det\mathsf{P}^{(4)\star})^2}\,\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)})-\frac{4(\mathsf{P}^{(6)\star})^3}{\det\mathsf{P}^{(4)\star}}. \tag{70}$$

In deriving Eq. (70) we have made use of the identity (B2) given in Appendix B. In analogy to the Eqs. (61) and (59), we can now define

$$\mathsf{P}^{(4)\star\prime}=\mathrm{adj}\,\mathsf{A}^{(2)\prime}-\mathsf{A}^{(4)\star\prime}, \tag{71}$$

$$\mathsf{P}^{(6)\star\prime}=\det\mathsf{A}^{(2)\prime}-\mathrm{tr}(\mathsf{A}^{(4)\star\prime}\mathsf{A}^{(2)\prime})-\mathsf{A}^{(6)\star\prime}, \tag{72}$$

and from the Eqs. (68)–(70) we find

$$\mathsf{P}^{(4)\star\prime}=\frac{(\mathsf{P}^{(6)\star})^3}{(\det\mathsf{P}^{(4)\star})^2}\,\mathsf{P}^{(4)\star}\mathsf{A}^{(2)}\mathsf{P}^{(4)\star}, \tag{73}$$

$$\mathsf{P}^{(6)\star\prime}=-\frac{(\mathsf{P}^{(6)\star})^5}{(\det \mathsf{P}^{(4)\star})^2}\left\{1-\frac{2}{\det \mathsf{P}^{(4)\star}}\,\mathrm{tr}[\mathrm{adj}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)})]\right\}-\frac{2(\mathsf{P}^{(6)\star})^4}{(\det \mathsf{P}^{(4)\star})^2}\,\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{A}^{(2)})+\frac{2(\mathsf{P}^{(6)\star})^3}{\det \mathsf{P}^{(4)\star}}. \tag{74}$$

Taking the determinant on both sides of the Eqs. (68) and (73) provides us with the following useful relations:

$$\det \mathsf{A}^{(2)\prime}=\frac{(\mathsf{P}^{(6)\star})^3}{\det \mathsf{P}^{(4)\star}}, \tag{75}$$

$$\det \mathsf{P}^{(4)\star\prime}=\frac{(\mathsf{P}^{(6)\star})^9}{(\det \mathsf{P}^{(4)\star})^4}\det \mathsf{A}^{(2)}. \tag{76}$$

Finally, also for the case $n=3$ we derive equations which describe the inverse of the action map f (the comment made in Ref. 80 of Sec. II B also applies here). From Eqs. (68), (69), (73), we find

$$\mathsf{P}^{(4)\star}=\mathsf{P}^{(6)\star}[\mathsf{A}^{(2)\prime}]^{-1}=\frac{\mathsf{P}^{(6)\star}}{\det \mathsf{A}^{(2)\prime}}\,\mathrm{adj}\,\mathsf{A}^{(2)\prime}, \tag{77}$$

$$\mathsf{A}^{(4)\star}=\frac{\mathsf{P}^{(6)\star}}{\det \mathsf{A}^{(2)\prime}}\left\{\frac{\mathsf{P}^{(6)\star}}{(\det \mathsf{A}^{(2)\prime})^3}\,\mathrm{adj}(\mathsf{A}^{(2)\prime}\mathsf{P}^{(4)\star\prime}\mathsf{A}^{(2)\prime})-\mathrm{adj}\,\mathsf{A}^{(2)\prime}\right\}, \tag{78}$$

$$\mathsf{A}^{(2)}=\frac{\mathsf{P}^{(6)\star}}{(\det \mathsf{A}^{(2)\prime})^2}\,\mathsf{A}^{(2)\prime}\mathsf{P}^{(4)\star\prime}\mathsf{A}^{(2)\prime}, \tag{79}$$

where now $\mathsf{P}^{(6)\star}$ is being understood as a function of the primed quantities whose explicit shape remains to be determined. Inserting Eqs. (77), (78) into Eq. (74) allows us to derive the following explicit representation of the partition function $\mathsf{P}^{(6)\star}$ in terms of the primed quantities:

$$\mathsf{P}^{(6)\star}=(\det \mathsf{A}^{(2)\prime})^2\left\{2\det \mathsf{A}^{(2)\prime}-2\,\mathrm{tr}(\mathsf{P}^{(4)\star\prime}\mathsf{A}^{(2)\prime})+\frac{2}{\det \mathsf{A}^{(2)\prime}}\,\mathrm{tr}[\mathrm{adj}(\mathsf{P}^{(4)\star\prime}\mathsf{A}^{(2)\prime})]-\mathsf{P}^{(6)\star\prime}\right\}^{-1}. \tag{80}$$

In principle, on the basis of this result also an explicit representation of $A^{(6)}_{123,123}$ in terms of the primed quantities can be established [relying on Eq. (59)] but we refrain from also writing it down here. As one recognizes from Eq. (80), in the case $n=3$ the description of the inverse of the action map f already involves fairly complicated expressions and we will not attempt to generalize these in the next section to the case $n=4$.

The results obtained in the present section can be checked for consistency in two ways. First, based on the procedure described in the Introduction in the context of Eqs. (19) and (20) one can convince oneself that the results—wherever appropriate—are consistent with the results obtained in Sec. II B for the case of the Grassmann algebra $\mathcal{G}_4$ $(n=2)$. Second, choosing for $G_0[\{\bar{\Psi}\},\{\Psi\}]$ the form (34) one can also convince oneself that then $\mathsf{A}^{(2)\prime}=\mathsf{A}^{(2)}$ and $\mathsf{A}^{(4)\star\prime}$, $A^{(6)\prime}_{123,123}$ vanish as expected.

D. Explicit calculation: $n=4$

We are now prepared to study the algebraically most involved case to be treated in the present paper—the case of the Grassmann algebra $\mathcal{G}_8$ $(n=4)$. The calculational experience collected in the last two sections allows us to manage the fairly involved expressions.

We start again by parametrizing G_0 and G according to our general ansatz [cf. Eq. (18)],

$$G_0[\{\overline{\Psi}\},\{\Psi\}]=A^{(0)}+\sum_{l,m=1}^{4} A^{(2)}_{l,m}\overline{\Psi}_l\Psi_m+\frac{1}{4}\sum_{l_1,l_2,m_1,m_2=1}^{4} A^{(4)}_{l_1l_2,m_1m_2}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\Psi_{m_1}\Psi_{m_2}$$
$$+\frac{1}{36}\sum_{l_1,l_2,l_3,m_1,m_2,m_3=1}^{4} A^{(6)}_{l_1l_2l_3,m_1m_2m_3}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\overline{\Psi}_{l_3}\Psi_{m_1}\Psi_{m_2}\Psi_{m_3}$$
$$+A^{(8)}_{1234,1234}\overline{\Psi}_1\overline{\Psi}_2\overline{\Psi}_3\overline{\Psi}_4\Psi_1\Psi_2\Psi_3\Psi_4. \tag{81}$$

For G the analogous representation can be used,

$$G[\{\overline{\Psi}\},\{\Psi\}]=A^{(0)\prime}+\sum_{l,m=1}^{4} A^{(2)\prime}_{l,m}\overline{\Psi}_l\Psi_m+\frac{1}{4}\sum_{l_1,l_2,m_1,m_2=1}^{4} A^{(4)\prime}_{l_1l_2,m_1m_2}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\Psi_{m_1}\Psi_{m_2}$$
$$+\frac{1}{36}\sum_{l_1,l_2,l_3,m_1,m_2,m_3=1}^{4} A^{(6)\prime}_{l_1l_2l_3,m_1m_2m_3}\overline{\Psi}_{l_1}\overline{\Psi}_{l_2}\overline{\Psi}_{l_3}\Psi_{m_1}\Psi_{m_2}\Psi_{m_3}$$
$$+A^{(8)\prime}_{1234,1234}\overline{\Psi}_1\overline{\Psi}_2\overline{\Psi}_3\overline{\Psi}_4\Psi_1\Psi_2\Psi_3\Psi_4. \tag{82}$$

The partition function reads [cf. Eq. (30)]

$$P=e^{A^{(0)\prime}}=\mathsf{P}^{(8)\star}=\det \mathsf{A}^{(2)}-\mathrm{tr}[\mathsf{A}^{(4)\star}C_2(\mathsf{A}^{(2)})]+\tfrac{1}{2}\mathrm{tr}(\mathsf{A}^{(4)\star}\mathsf{A}^{(4)})-\mathrm{tr}(\mathsf{A}^{(6)\star}\mathsf{A}^{(2)})+\mathsf{A}^{(8)\star} \tag{83}$$
$$=6\det \mathsf{A}^{(2)}-2\,\mathrm{tr}[\mathsf{P}^{(4)\star}C_2(\mathsf{A}^{(2)})]+\tfrac{1}{2}\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{P}^{(4)})+\mathrm{tr}(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)})+\mathsf{A}^{(8)\star}. \tag{84}$$

In analogy to Eq. (40), here $\mathsf{A}^{(8)\star}=A^{(8)}_{1234,1234}$ applies. In the lower line [Eq. (84)], we have made use of the expressions [cf. Eqs. (61), (59)]

$$\mathsf{P}^{(4)\star}=C_2(\mathsf{A}^{(2)})^\star-\mathsf{A}^{(4)\star}, \tag{85}$$

$$\mathsf{P}^{(6)\star}=\mathrm{adj}\,\mathsf{A}^{(2)}-\mathsf{F}_a(\mathsf{A}^{(2)},\mathsf{A}^{(4)})-\mathsf{A}^{(6)\star} \tag{86}$$

($\mathrm{adj}\,\mathsf{A}^{(2)}=C_3(\mathsf{A}^{(2)})^\star$). The form F_a is defined as follows:

$$\mathsf{F}_a(\mathsf{A}^{(2)},\mathsf{A}^{(4)})_{lm}=\epsilon_{lrK}\epsilon_{msN}\mathsf{A}^{(2)}_{sr}\mathsf{A}^{(4)}_{NK}. \tag{87}$$

In making the transition from Eq. (83) to Eq. (84) we have used the relations

$$2\,\mathrm{tr}[C_2(\mathsf{A}^{(2)})\mathsf{A}^{(4)\star}]=\mathrm{tr}[\mathsf{F}_a(\mathsf{A}^{(2)},\mathsf{A}^{(4)})\mathsf{A}^{(2)}], \tag{88}$$

$$C_2(\mathsf{A}^{(2)})C_2(\mathsf{A}^{(2)})^\star=C_2(\mathsf{A}^{(2)})^\star C_2(\mathsf{A}^{(2)})=\det \mathsf{A}^{(2)}\mathbf{1}_6 \tag{89}$$

[Eq. (89) is a special case of Eq. (A6), see Appendix A]. As next step, we can calculate $W[\{\overline{\eta}\},\{\eta\}]$ which reads (to arrive at it we only assume $\mathsf{P}^{(8)\star}\neq 0$)

$$W[\{\overline{\eta}\},\{\eta\}]=\ln \mathsf{P}^{(8)\star}-\frac{\mathsf{P}^{(6)\star}_{lm}}{\mathsf{P}^{(8)\star}}\overline{\eta}_l\eta_m-\frac{\mathsf{P}^{(4)\star}_{LM}}{\mathsf{P}^{(8)\star}}\overline{\eta}_{l_1}\overline{\eta}_{l_2}\eta_{m_1}\eta_{m_2}-\frac{1}{2}\left(\frac{\mathsf{P}^{(6)\star}_{lm}}{\mathsf{P}^{(8)\star}}\overline{\eta}_l\eta_m\right)^2$$
$$+\frac{1}{\mathsf{P}^{(8)\star}}\left[\mathsf{A}^{(2)\star}-\frac{\mathsf{F}_a(\mathsf{P}^{(6)\star},\mathsf{P}^{(4)\star})^\star}{\mathsf{P}^{(8)\star}}+\frac{2\,C_3(\mathsf{P}^{(6)\star})}{(\mathsf{P}^{(8)\star})^2}\right]_{LM}\overline{\eta}_{l_1}\overline{\eta}_{l_2}\overline{\eta}_{l_3}\eta_{m_1}\eta_{m_2}\eta_{m_3}$$
$$+\frac{1}{\mathsf{P}^{(8)\star}}\left\{1-\frac{\mathrm{tr}(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)})}{\mathsf{P}^{(8)\star}}-\frac{\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{P}^{(4)})}{2\,\mathsf{P}^{(8)\star}}+\frac{2\,\mathrm{tr}[\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star})]}{(\mathsf{P}^{(8)\star})^2}\right.$$
$$\left.-\frac{6\det \mathsf{P}^{(6)\star}}{(\mathsf{P}^{(8)\star})^3}\right\}\overline{\eta}_1\overline{\eta}_2\overline{\eta}_3\overline{\eta}_4\eta_1\eta_2\eta_3\eta_4. \tag{90}$$

In the following, we need a number of forms which we list here for further reference. The index convention applied here requires some explanation. For example, $(\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{ltur}$ up to the sign denotes elements of the 6×6 matrix $\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)}$. If $l<t$, $u<r$, it denotes the matrix element $(\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{\{l,t\}\{u,r\}}$. If $l>t$, $u<r$, it denotes the matrix element $(-\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{\{t,l\}\{u,r\}}$ and if $l<t$, $u>r$, it denotes the matrix element $(-\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{\{l,t\}\{r,u\}}$. And finally, if $l>t$, $u>r$, it denotes the matrix element $(\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{\{t,l\}\{r,u\}}$. Of course, $(\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{\{l,t\}\{u,r\}}=0$ if $l=t$ or $u=r$. In the following, summation is understood over repeated indices:

$$\mathsf{F}_b(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star})_{LM}=\epsilon_{Lrk}\,(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star})_{sr}\,\epsilon_{skM}\,, \tag{91}$$

$$\mathsf{F}_c(\mathsf{A}^{(4)\star\prime},\mathsf{P}^{(6)\star},\mathsf{A}^{(4)\star\prime})_{lm}=(\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{ltur}\mathsf{P}^{(6)\star}_{sr}(\mathcal{E}^{(2)}\mathsf{A}^{(4)\star\prime})_{stum}\,, \tag{92}$$

$$\mathsf{F}_{d1}(\mathsf{A}^{(4)\star\prime},\mathsf{A}^{(2)\prime},\mathsf{P}^{(4)}C_2(\mathsf{A}^{(2)\prime})^\star)_{lm}=(\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{lrtu}\mathsf{A}^{(2)\prime}_{sr}[\mathcal{E}^{(2)}\mathsf{P}^{(4)}C_2(\mathsf{A}^{(2)\prime})^\star]_{tsum}\,, \tag{93}$$

$$\mathsf{F}_{d2}(C_2(\mathsf{A}^{(2)\prime})^\star\mathsf{P}^{(4)},\mathsf{A}^{(2)\prime},\mathsf{A}^{(4)\star\prime})_{lm}=[C_2(\mathsf{A}^{(2)\prime})^\star\mathsf{P}^{(4)}\mathcal{E}^{(2)}]_{lutr}\mathsf{A}^{(2)\prime}_{sr}(\mathcal{E}^{(2)}\mathsf{A}^{(4)\star\prime})_{tusm}\,, \tag{94}$$

$$\mathsf{F}_e(\mathsf{A}^{(2)\prime},\mathsf{A}^{(4)\prime},\mathsf{A}^{(4)\prime},\mathsf{A}^{(2)\prime})_{LM}=\mathcal{E}^{(2)}_{Lab}\mathsf{A}^{(2)\prime}_{ra}(\mathcal{E}^{(2)}\mathsf{A}^{(4)\prime})_{rtbu}(\mathsf{A}^{(4)\prime}\mathcal{E}^{(2)})_{dtsu}\mathsf{A}^{(2)\prime}_{cs}\mathcal{E}^{(2)}_{cdM}\,, \tag{95}$$

$$\mathsf{F}_f(\mathsf{A}^{(4)\star\prime},\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star},\mathsf{A}^{(4)\prime})_{lm}=(\mathsf{A}^{(4)\star\prime}\mathcal{E}^{(2)})_{lcda}(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star})_{ba}(\mathcal{E}^{(2)}\mathsf{A}^{(4)\prime})_{bdcm}\,, \tag{96}$$

$$\begin{aligned}&\mathsf{F}_g(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)},\mathsf{P}^{(6)\star},\mathsf{P}^{(6)\star},\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}))_{LM}\\&\quad=\mathcal{E}^{(2)}_{Lab}[\mathcal{E}^{(2)}C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}]_{arbt}\mathsf{P}^{(6)\star}_{rs}\mathsf{P}^{(6)\star}_{tu}[\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star})\mathcal{E}^{(2)}]_{csdu}\mathcal{E}^{(2)}_{cdM}\,.\end{aligned} \tag{97}$$

To arrive at the further results it is useful to take note of the equation

$$(\mathsf{P}^{(6)\star}_{lm}\overline{\eta}_l\eta_m)^2=-2C_2(\mathsf{P}^{(6)\star})_{LM}\overline{\eta}_{l_1}\overline{\eta}_{l_2}\eta_{m_1}\eta_{m_2}. \tag{98}$$

We now apply exactly the same procedure as in Secs. II B, II C. We insert Eq. (82) onto the lhs of Eq. (13) and the explicit expressions for $\overline{\eta}$, η found from Eq. (82) according to Eq. (16) on its rhs. Again, comparing coefficients on both sides we find the following four coupled nonlinear matrix equations:

$$\mathsf{A}^{(2)\prime}=2\mathsf{A}^{(2)\prime}-\mathsf{A}^{(2)\prime}\frac{\mathsf{P}^{(6)\star}}{\mathsf{P}^{(8)\star}}\mathsf{A}^{(2)\prime}, \tag{99}$$

$$\begin{aligned}\mathsf{A}^{(4)\star\prime}=4\mathsf{A}^{(4)\star\prime}-\frac{\mathsf{F}_b(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star})}{\mathsf{P}^{(8)\star}}\mathsf{A}^{(4)\star\prime}-\mathsf{A}^{(4)\star\prime}\frac{\mathsf{F}_b(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime})}{\mathsf{P}^{(8)\star}}\\-C_2(\mathsf{A}^{(2)\prime})^\star\frac{\mathsf{P}^{(8)\star}\mathsf{P}^{(4)}-C_2(\mathsf{P}^{(6)\star})^\star}{(\mathsf{P}^{(8)\star})^2}C_2(\mathsf{A}^{(2)\prime})^\star,\end{aligned} \tag{100}$$

$$\begin{aligned}\mathsf{A}^{(6)\star\prime}=6\mathsf{A}^{(6)\star\prime}&+\frac{1}{\mathsf{P}^{(8)\star}}\{\mathsf{A}^{(6)\star\prime}[\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}-\mathrm{tr}(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star})\mathbf{1}_4]\\&+[\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}-\mathrm{tr}(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime})\mathbf{1}_4]\mathsf{A}^{(6)\star\prime}-\mathsf{F}_c(\mathsf{A}^{(4)\star\prime},\mathsf{P}^{(6)\star},\mathsf{A}^{(4)\star\prime})\\&+\mathsf{F}_{d1}(\mathsf{A}^{(4)\star\prime},\mathsf{A}^{(2)\prime},\mathsf{P}^{(4)}C_2(\mathsf{A}^{(2)\prime})^\star)+\mathsf{F}_{d2}(C_2(\mathsf{A}^{(2)\prime})^\star\mathsf{P}^{(4)},\mathsf{A}^{(2)\prime},\mathsf{A}^{(4)\star\prime})\\&+\mathrm{adj}(\mathsf{A}^{(2)\prime})\mathsf{A}^{(2)}\,\mathrm{adj}(\mathsf{A}^{(2)\prime})\}-\frac{1}{(\mathsf{P}^{(8)\star})^2}\{\mathsf{F}_{d1}(\mathsf{A}^{(4)\star\prime},\mathsf{A}^{(2)\prime},C_2(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star})^\star)\\&+\mathsf{F}_{d2}(C_2(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime})^\star,\mathsf{A}^{(2)\prime},\mathsf{A}^{(4)\star\prime})+\mathrm{adj}(\mathsf{A}^{(2)\prime})\mathsf{F}_a(\mathsf{P}^{(6)\star},\mathsf{P}^{(4)\star})\mathrm{adj}(\mathsf{A}^{(2)\prime})\}\\&+\frac{2}{(\mathsf{P}^{(8)\star})^3}\mathrm{adj}(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}),\end{aligned} \tag{101}$$

$$
\begin{aligned}
A^{(8)\prime}_{1234,1234} = {} & 8A^{(8)\prime}_{1234,1234} + \frac{1}{\mathsf{P}^{(8)\star}}\{-2A^{(8)\prime}_{1234,1234}\mathrm{tr}(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}) - 2\,\mathrm{tr}[\mathsf{P}^{(6)\star}\mathsf{F}_a(\mathsf{A}^{(6)\star\prime},\mathsf{A}^{(4)\star\prime})] \\
& - 2\,\mathrm{tr}[\mathsf{A}^{(2)\prime}\mathsf{A}^{(6)\star\prime}\mathsf{A}^{(2)\prime}\mathsf{F}_a(\mathsf{A}^{(2)\prime},\mathsf{P}^{(4)})] - \mathrm{tr}[\mathsf{P}^{(4)}\mathsf{F}_e(\mathsf{A}^{(2)\prime},\mathsf{A}^{(4)\prime},\mathsf{A}^{(4)\prime},\mathsf{A}^{(2)\prime})] \\
& - \tfrac{1}{2}\mathrm{tr}[\mathsf{F}_c(\mathsf{P}^{(4)\star}C_2(\mathsf{A}^{(2)\prime}),\mathbf{1}_4,\mathsf{A}^{(4)\star\prime}\mathsf{A}^{(4)\prime})] - \tfrac{1}{2}\mathrm{tr}[\mathsf{F}_c(\mathsf{A}^{(4)\prime}\mathsf{A}^{(4)\star\prime},\mathbf{1}_4,C_2(\mathsf{A}^{(2)\prime})\mathsf{P}^{(4)\star})] \\
& + \mathrm{tr}[\mathsf{A}^{(4)\star\prime}C_2(\mathsf{A}^{(2)\prime})\mathsf{F}_b((\mathrm{adj}\,\mathsf{A}^{(2)\prime})\mathsf{A}^{(2)})] + \mathrm{tr}[C_2(\mathsf{A}^{(2)\prime})\mathsf{A}^{(4)\star\prime}\mathsf{F}_b(\mathsf{A}^{(2)}\,\mathrm{adj}\,\mathsf{A}^{(2)\prime})] \\
& + (\det \mathsf{A}^{(2)\prime})^2\} + \frac{1}{(\mathsf{P}^{(8)\star})^2}\{2\,\mathrm{tr}(\mathsf{A}^{(2)\prime}\mathsf{A}^{(6)\star\prime}\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star})\mathrm{tr}(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}) \\
& - 2\,\mathrm{tr}(\mathsf{A}^{(2)\prime}\mathsf{A}^{(6)\star\prime}\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}) + \mathrm{tr}[\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}\mathsf{F}_c(\mathsf{A}^{(4)\star\prime},\mathsf{P}^{(6)\star},\mathsf{A}^{(4)\star\prime})] \\
& - \mathrm{tr}[\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}\mathsf{F}_f(\mathsf{A}^{(4)\star\prime},\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star},\mathsf{A}^{(4)\prime})] + \tfrac{1}{2}\mathrm{tr}[\mathsf{F}_c(C_2(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}),\mathbf{1}_4,\mathsf{A}^{(4)\star\prime}\mathsf{A}^{(4)\prime})] \\
& + \tfrac{1}{2}\mathrm{tr}[\mathsf{F}_c(\mathsf{A}^{(4)\prime}\mathsf{A}^{(4)\star\prime},\mathbf{1}_4,C_2(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}))] \\
& - \mathrm{tr}[\mathsf{F}_a(\mathsf{P}^{(6)\star},\mathsf{P}^{(4)\star})\mathsf{F}_a(\mathbf{1}_4,\mathsf{A}^{(4)\star\prime}C_2(\mathsf{A}^{(2)\prime}))\mathrm{adj}\,\mathsf{A}^{(2)\prime}] \\
& - \mathrm{tr}[\mathsf{F}_a(\mathbf{1}_4,C_2(\mathsf{A}^{(2)\prime})\mathsf{A}^{(4)\star\prime})\mathsf{F}_a(\mathsf{P}^{(6)\star},\mathsf{P}^{(4)\star})\mathrm{adj}\,\mathsf{A}^{(2)\prime}] \\
& - (\det \mathsf{A}^{(2)\prime})^2[\mathrm{tr}(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)}) + \tfrac{1}{2}\mathrm{tr}(\mathsf{P}^{(4)\star}\mathsf{P}^{(4)})]\} \\
& + \frac{2}{(\mathsf{P}^{(8)\star})^3}\{\mathrm{tr}[\mathsf{A}^{(4)\star\prime}C_2(\mathsf{A}^{(2)\prime})\mathsf{F}_b(\mathrm{adj}\,(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)\prime}))] \\
& + \mathrm{tr}[C_2(\mathsf{A}^{(2)\prime})\mathsf{A}^{(4)\star\prime}\mathsf{F}_b(\mathrm{adj}\,(\mathsf{A}^{(2)\prime}\mathsf{P}^{(6)\star}))] \\
& + (\det \mathsf{A}^{(2)\prime})^2\mathrm{tr}[\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star})]\} - \frac{6}{(\mathsf{P}^{(8)\star})^4}(\det \mathsf{A}^{(2)\prime})^2\,\det \mathsf{P}^{(6)\star}.
\end{aligned}
\tag{102}
$$

Equation (99) is equivalent to the equation

$$
\mathsf{A}^{(2)\prime} = \mathsf{A}^{(2)\prime}\frac{\mathsf{P}^{(6)\star}}{\mathsf{P}^{(8)\star}}\mathsf{A}^{(2)\prime}. \tag{103}
$$

The matrix $\mathsf{A}^{(2)\prime}$ is the generalized {2}-inverse of the matrix $\mathsf{P}^{(6)\star}/\mathsf{P}^{(8)\star}$ (cf., e.g., Ref. 79, Chap. 1, p. 7).

For solving the Eqs. (99)–(102) we apply again the same method as in Secs. II B and II C. Choosing $\det \mathsf{A}^{(2)\prime} \neq 0$ [by virtue of Eq. (103) this entails $\det \mathsf{P}^{(6)\star} \neq 0$], we immediately find from Eq. (103) an explicit expression for $\mathsf{A}^{(2)\prime}$. This can be inserted into Eq. (100) to also find an explicit expression for $\mathsf{A}^{(4)\star\prime}$.

$$
\mathsf{A}^{(2)\prime} = \mathsf{P}^{(8)\star}\,[\mathsf{P}^{(6)\star}]^{-1} = \frac{\mathsf{P}^{(8)\star}}{\det \mathsf{P}^{(6)\star}}\mathrm{adj}\,\mathsf{P}^{(6)\star}, \tag{104}
$$

$$
\mathsf{A}^{(4)\star\prime} = -\frac{(\mathsf{P}^{(8)\star})^2}{\det\,\mathsf{P}^{(6)\star}}\left[\frac{\mathsf{P}^{(8)\star}}{\det \mathsf{P}^{(6)\star}}C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}) - C_2(\mathsf{P}^{(6)\star})\right]. \tag{105}
$$

To arrive at Eq. (105) we have relied on the following calculation [cf. Appendix A, Eqs. (A6), (A5)]:

$$C_2(\mathsf{A}^{(2)\prime})^\star = (\mathsf{P}^{(8)\star})^2 C_2([\mathsf{P}^{(6)\star}]^{-1})^\star = \frac{(\mathsf{P}^{(8)\star})^2}{\det \mathsf{P}^{(6)\star}} C_2([\mathsf{P}^{(6)\star}]^{-1})^{-1} = \frac{(\mathsf{P}^{(8)\star})^2}{\det \mathsf{P}^{(6)\star}} C_2(\mathsf{P}^{(6)\star}). \tag{106}$$

Having obtained explicit expressions for $\mathsf{A}^{(2)\prime}$ and $\mathsf{A}^{(4)\star\prime}$ we can now insert them into Eq. (101) to solve it. We find

$$\begin{aligned}\mathsf{A}^{(6)\star\prime} = {} & \frac{(\mathsf{P}^{(8)\star})^5}{(\det \mathsf{P}^{(6)\star})^2} \mathsf{P}^{(6)\star} \Bigg\{ \mathsf{A}^{(2)} - \frac{1}{2 \det \mathsf{P}^{(6)\star}} \mathsf{F}_{d1}(\mathsf{P}^{(4)}, \mathsf{P}^{(6)\star}, C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}) \\ & - \frac{1}{2 \det \mathsf{P}^{(6)\star}} \mathsf{F}_{d2}(\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star}), \mathsf{P}^{(6)\star}, \mathsf{P}^{(4)}) \Bigg\} \mathsf{P}^{(6)\star} \\ & + \frac{3(\mathsf{P}^{(8)\star})^4}{(\det \mathsf{P}^{(6)\star})^2} \mathsf{P}^{(6)\star} \mathsf{F}_a(\mathsf{P}^{(6)\star}, \mathsf{P}^{(4)\star}) \mathsf{P}^{(6)\star} - \frac{4(\mathsf{P}^{(8)\star})^3}{\det \mathsf{P}^{(6)\star}} \mathsf{P}^{(6)\star} \end{aligned} \tag{107}$$

$$\begin{aligned} = {} & \frac{(\mathsf{P}^{(8)\star})^5}{(\det \mathsf{P}^{(6)\star})^2} \mathsf{P}^{(6)\star} \mathsf{A}^{(2)} \mathsf{P}^{(6)\star} \\ & + \frac{(\mathsf{P}^{(8)\star})^5}{(\det \mathsf{P}^{(6)\star})^4} \mathsf{F}_c(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star}), \mathsf{P}^{(6)\star}, C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star})) \\ & + \frac{3(\mathsf{P}^{(8)\star})^4}{(\det \mathsf{P}^{(6)\star})^2} \mathsf{P}^{(6)\star} \mathsf{F}_a(\mathsf{P}^{(6)\star}, \mathsf{P}^{(4)\star}) \mathsf{P}^{(6)\star} - \frac{4(\mathsf{P}^{(8)\star})^3}{\det \mathsf{P}^{(6)\star}} \mathsf{P}^{(6)\star}. \end{aligned} \tag{108}$$

The equivalence of Eqs. (107) and (108) is based on the relation

$$\begin{aligned} & (\det \mathsf{P}^{(6)\star}) \mathsf{P}^{(6)\star} \mathsf{F}_{d1}(\mathsf{P}^{(4)}, \mathsf{P}^{(6)\star}, C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}) \mathsf{P}^{(6)\star} \\ & \quad = (\det \mathsf{P}^{(6)\star}) \mathsf{P}^{(6)\star} \mathsf{F}_{d2}(\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star}), \mathsf{P}^{(6)\star}, \mathsf{P}^{(4)}) \mathsf{P}^{(6)\star} \\ & \quad = -\mathsf{F}_c(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star}), \mathsf{P}^{(6)\star}, C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star})). \end{aligned} \tag{109}$$

Finally, inserting Eqs. (104), (105), (108) into (102) allows us to find the following explicit solution for $A^{(8)\prime}_{1234,1234}$:

$$\begin{aligned} A^{(8)\prime}_{1234,1234} = {} & \frac{(\mathsf{P}^{(8)\star})^7}{(\det \mathsf{P}^{(6)\star})^2} \{1 - 2\,\mathrm{tr}[\mathsf{A}^{(2)} \mathsf{F}_a((\mathsf{P}^{(6)\star})^{-1}, \mathsf{P}^{(4)})]\} \\ & + \frac{(\mathsf{P}^{(8)\star})^7}{(\det \mathsf{P}^{(6)\star})^4} \{\mathrm{tr}[\mathsf{P}^{(4)} \mathsf{F}_g(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}, \mathsf{P}^{(6)\star}, \mathsf{P}^{(6)\star}, \mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star}))] \\ & - \tfrac{1}{2}\,\mathrm{tr}[\mathsf{F}_c(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}\mathsf{P}^{(4)\star} C_2(\mathsf{P}^{(6)\star})^\star, \mathbf{1}_4, C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)})] \\ & - \tfrac{1}{2}\,\mathrm{tr}[\mathsf{F}_c(C_2(\mathsf{P}^{(6)\star})^\star \mathsf{P}^{(4)\star}\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star}), \mathbf{1}_4, \mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star}))]\} \\ & + \frac{(\mathsf{P}^{(8)\star})^6}{(\det \mathsf{P}^{(6)\star})^2} \Bigg\{ \frac{11}{2}\,\mathrm{tr}(\mathsf{P}^{(4)}\mathsf{P}^{(4)\star}) + 5\,\mathrm{tr}(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)}) \\ & - \frac{5}{\det \mathsf{P}^{(6)\star}}\,\mathrm{tr}[\mathsf{F}_a(\mathbf{1}_4, C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}) \mathsf{F}_a(\mathbf{1}_4, \mathsf{P}^{(4)\star} C_2(\mathsf{P}^{(6)\star})^\star)] \Bigg\} \\ & + 18 \frac{(\mathsf{P}^{(8)\star})^5}{(\det \mathsf{P}^{(6)\star})^2}\,\mathrm{tr}[\mathsf{P}^{(4)} C_2(\mathsf{P}^{(6)\star})] - 30 \frac{(\mathsf{P}^{(8)\star})^4}{\det \mathsf{P}^{(6)\star}}. \end{aligned} \tag{110}$$

In analogy to the Eqs. (85), (86) and (83), we can now define

$$\mathsf{P}^{(4)\star\prime}=C_2(\mathsf{A}^{(2)\prime})^\star-\mathsf{A}^{(4)\star\prime}, \tag{111}$$

$$\mathsf{P}^{(6)\star\prime}=\operatorname{adj}\mathsf{A}^{(2)\prime}-\mathsf{F}_a(\mathsf{A}^{(2)\prime},\mathsf{A}^{(4)\prime})-\mathsf{A}^{(6)\star\prime}, \tag{112}$$

$$\mathsf{P}^{(8)\star\prime}=\det\mathsf{A}^{(2)\prime}-\operatorname{tr}[\mathsf{A}^{(4)\star\prime}C_2(\mathsf{A}^{(2)\prime})]+\tfrac{1}{2}\operatorname{tr}(\mathsf{A}^{(4)\star\prime}\mathsf{A}^{(4)\prime})-\operatorname{tr}(\mathsf{A}^{(6)\star\prime}\mathsf{A}^{(2)\prime})+\mathsf{A}^{(8)\star\prime}, \tag{113}$$

and from Eqs. (104), (105), (108), (110), we find

$$\mathsf{P}^{(4)\star\prime}=\frac{(\mathsf{P}^{(8)\star})^3}{(\det\mathsf{P}^{(6)\star})^2}C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}), \tag{114}$$

$$\begin{aligned}\mathsf{P}^{(6)\star\prime}=&-\frac{(\mathsf{P}^{(8)\star})^5}{(\det\mathsf{P}^{(6)\star})^2}\mathsf{P}^{(6)\star}\bigg\{\mathsf{A}^{(2)}-\frac{1}{2\det\mathsf{P}^{(6)\star}}\mathsf{F}_{d1}(\mathsf{P}^{(4)},\mathsf{P}^{(6)\star},C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)})\\&-\frac{1}{2\det\mathsf{P}^{(6)\star}}\mathsf{F}_{d2}(\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}),\mathsf{P}^{(6)\star},\mathsf{P}^{(4)})\bigg\}\mathsf{P}^{(6)\star}\\&-\frac{2(\mathsf{P}^{(8)\star})^4}{(\det\mathsf{P}^{(6)\star})^2}\mathsf{P}^{(6)\star}\mathsf{F}_a(\mathsf{P}^{(6)\star},\mathsf{P}^{(4)\star})\mathsf{P}^{(6)\star}+\frac{2(\mathsf{P}^{(8)\star})^3}{\det\mathsf{P}^{(6)\star}}\mathsf{P}^{(6)\star}\end{aligned} \tag{115}$$

$$\begin{aligned}=&-\frac{(\mathsf{P}^{(8)\star})^5}{(\det\mathsf{P}^{(6)\star})^2}\mathsf{P}^{(6)\star}\mathsf{A}^{(2)}\mathsf{P}^{(6)\star}\\&-\frac{(\mathsf{P}^{(8)\star})^5}{(\det\mathsf{P}^{(6)\star})^4}\mathsf{F}_c(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}),\mathsf{P}^{(6)\star},C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}))\\&-\frac{2(\mathsf{P}^{(8)\star})^4}{(\det\mathsf{P}^{(6)\star})^2}\mathsf{P}^{(6)\star}\mathsf{F}_a(\mathsf{P}^{(6)\star},\mathsf{P}^{(4)\star})\mathsf{P}^{(6)\star}+\frac{2(\mathsf{P}^{(8)\star})^3}{\det\mathsf{P}^{(6)\star}}\mathsf{P}^{(6)\star},\end{aligned} \tag{116}$$

$$\begin{aligned}\mathsf{P}^{(8)\star\prime}=&\frac{(\mathsf{P}^{(8)\star})^7}{(\det\mathsf{P}^{(6)\star})^2}\{1-2\operatorname{tr}[\mathsf{A}^{(2)}\mathsf{F}_a((\mathsf{P}^{(6)\star})^{-1},\mathsf{P}^{(4)})]\}\\&+\frac{(\mathsf{P}^{(8)\star})^7}{(\det\mathsf{P}^{(6)\star})^4}\{\operatorname{tr}[\mathsf{P}^{(4)}\mathsf{F}_g(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)},\mathsf{P}^{(6)\star},\mathsf{P}^{(6)\star},\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}))]\\&-\tfrac{1}{2}\operatorname{tr}[\mathsf{F}_c(C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)}\mathsf{P}^{(4)\star}C_2(\mathsf{P}^{(6)\star})^\star,\mathbf{1}_4,C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)})]\\&-\tfrac{1}{2}\operatorname{tr}[\mathsf{F}_c(C_2(\mathsf{P}^{(6)\star})^\star\mathsf{P}^{(4)\star}\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}),\mathbf{1}_4,\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star}))]\}\\&+4\frac{(\mathsf{P}^{(8)\star})^6}{(\det\mathsf{P}^{(6)\star})^2}\bigg\{\operatorname{tr}(\mathsf{P}^{(4)}\mathsf{P}^{(4)\star})+\operatorname{tr}(\mathsf{P}^{(6)\star}\mathsf{A}^{(2)})\\&-\frac{1}{\det\mathsf{P}^{(6)\star}}\operatorname{tr}[\mathsf{F}_a(\mathbf{1}_4,C_2(\mathsf{P}^{(6)\star})\mathsf{P}^{(4)})\mathsf{F}_a(\mathbf{1}_4,\mathsf{P}^{(4)\star}C_2(\mathsf{P}^{(6)\star})^\star)]\bigg\}\\&+12\frac{(\mathsf{P}^{(8)\star})^5}{(\det\mathsf{P}^{(6)\star})^2}\operatorname{tr}[\mathsf{P}^{(4)}C_2(\mathsf{P}^{(6)\star})]-16\frac{(\mathsf{P}^{(8)\star})^4}{\det\mathsf{P}^{(6)\star}}.\end{aligned} \tag{117}$$

Taking the determinant on both sides of the Eqs. (104) and (114) provides us with the following useful relations:

$$\det\mathsf{A}^{(2)\prime}=\frac{(\mathsf{P}^{(8)\star})^4}{\det\mathsf{P}^{(6)\star}}, \tag{118}$$

$$\det \mathsf{P}^{(4)\star\prime} = \frac{(\mathsf{P}^{(8)\star})^{18}}{(\det \mathsf{P}^{(6)\star})^6} \det \mathsf{P}^{(4)\star}. \tag{119}$$

In deriving Eq. (119) we have relied on the following (Sylvester–Franke) identity [cf. Appendix A, Eq. (A8)].

$$\det C_2(\mathsf{P}^{(6)\star}) = (\det \mathsf{P}^{(6)\star})^3. \tag{120}$$

We can finally check the obtained results for consistency in the same way as done at the end of the preceding section for $n=3$. First, based on the procedure described in the Introduction in the context of Eqs. (19), (20) one can convince oneself again that the results—wherever appropriate—are consistent with the results obtained in Sec. II C for the case of the Grassmann algebra $\mathcal{G}_6$ ($n=3$). Second, choosing for $G_0[\{\overline{\Psi}\},\{\Psi\}]$ the form (34) one can also convince oneself that then $\mathsf{A}^{(2)\prime} = \mathsf{A}^{(2)}$ and $\mathsf{A}^{(4)\star\prime}$, $\mathsf{A}^{(6)\star\prime}$, $A^{(8)\prime}_{1234,1234}$ vanish as expected. Given the combinatorial factors involved, this represents a fairly sensitive check of the present results.

E. Some heuristics for arbitrary *n*

Having gained a fairly broad calculational and structural experience in the preceding sections in considering the present formalism for the case of the Grassmann algebras $\mathcal{G}_{2n}$, $n=2,3,4$, we are going to generalize now some of the obtained results to arbitrary values of n. This analytic extrapolation is a heuristic procedure with heuristic purposes. No proof is being attempted here which would need to be the subject of a separate study.

From Eqs. (46), (68), (104) and (47), (69), (105) we infer the following general (for arbitrary values of n) form of the matrices $\mathsf{A}^{(2)\prime}$, $\mathsf{A}^{(4)\prime}$ (of course, the result for $\mathsf{A}^{(2)\prime}$ is elementary),

$$\mathsf{A}^{(2)\prime} = \mathsf{P}^{(2n)\star}[\mathsf{P}^{(2n-2)\star}]^{-1} = \frac{\mathsf{P}^{(2n)\star}}{\det \mathsf{P}^{(2n-2)\star}} \operatorname{adj} \mathsf{P}^{(2n-2)\star}, \tag{121}$$

$$\mathsf{A}^{(4)\prime} = -\frac{(\mathsf{P}^{(2n)\star})^2}{\det \mathsf{P}^{(2n-2)\star}} \left[\frac{\mathsf{P}^{(2n)\star}}{\det \mathsf{P}^{(2n-2)\star}} C_{n-2}(\mathsf{P}^{(2n-2)\star})^\star \mathsf{P}^{(2n-4)\star} C_{n-2}(\mathsf{P}^{(2n-2)\star})^\star - C_{n-2}(\mathsf{P}^{(2n-2)\star})^\star \right]. \tag{122}$$

Emphasizing the role of the effective propagator $\mathsf{P}^{(2n-2)\star}/\mathsf{P}^{(2n)\star}$ [cf. Eq. (121)] we can rewrite Eq. (122) in the following form:

$$\mathsf{A}^{(4)\prime} = -\frac{C_{n-2}\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)^\star}{\det\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)} \frac{\mathsf{P}^{(2n-4)\star}}{\mathsf{P}^{(2n)\star}} \frac{C_{n-2}\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)^\star}{\det\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)} + \frac{C_{n-2}\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)^\star}{\det\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)}. \tag{123}$$

Unfortunately, the results obtained in the preceding sections do not yet admit any reliable analytical (heuristic) extrapolation to arbitrary values of n for further quantities beyond $\mathsf{A}^{(2)\prime}$, $\mathsf{A}^{(4)\prime}$. For example, to heuristically derive an analogous expression for $\mathsf{A}^{(6)\prime}$ one would have to perform a calculation for $n=5$ first in order to approach this task. However, in analogy to the preceding sections it is still possible to derive one further result for arbitrary n. Again, writing [cf. Eqs. (48), (71), (111)]

$$\mathsf{P}^{(4)\star\prime} = C_2(\mathsf{A}^{(2)\prime})^\star - \mathsf{A}^{(4)\star\prime} \tag{124}$$

we find from Eqs. (121), (122) [cf. Eqs. (49), (73), (114)]

$$\mathsf{P}^{(4)\prime}=\frac{(\mathsf{P}^{(2n)\star})^3}{(\det \mathsf{P}^{(2n-2)\star})^2}C_{n-2}(\mathsf{P}^{(2n-2)\star})^{\star}\mathsf{P}^{(2n-4)\star}C_{n-2}(\mathsf{P}^{(2n-2)\star})^{\star}. \tag{125}$$

In analogy to Eq. (123), this can equivalently be written as

$$\mathsf{P}^{(4)\prime}=\frac{C_{n-2}\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)^{\star}}{\det\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)}\frac{\mathsf{P}^{(2n-4)\star}}{\mathsf{P}^{(2n)\star}}\frac{C_{n-2}\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)^{\star}}{\det\left(\dfrac{\mathsf{P}^{(2n-2)\star}}{\mathsf{P}^{(2n)\star}}\right)}. \tag{126}$$

To arrive at Eq. (125) we have relied on the following calculation [cf. Appendix A, Eqs. (A6), (A5)]:

$$\begin{aligned} C_2(\mathsf{A}^{(2)\prime}) &= (\mathsf{P}^{(2n)\star})^2 C_2([\mathsf{P}^{(2n-2)\star}]^{-1}) \\ &= (\mathsf{P}^{(2n)\star})^2 C_2(\mathsf{P}^{(2n-2)\star})^{-1} \\ &= \frac{(\mathsf{P}^{(2n)\star})^2}{\det \mathsf{P}^{(2n-2)\star}} C_{n-2}(\mathsf{P}^{(2n-2)\star})^{\star}. \end{aligned} \tag{127}$$

Taking the determinant on both sides of the Eqs. (121) and (126) yields the relations [cf. Eqs. (50), (75), (118) and (76), (119)]

$$\det \mathsf{A}^{(2)\prime}=\frac{(\mathsf{P}^{(2n)\star})^n}{\det \mathsf{P}^{(2n-2)\star}}, \tag{128}$$

$$\det \mathsf{P}^{(4)\star\prime}=\frac{(\mathsf{P}^{(2n)\star})^{3\binom{n}{2}}}{(\det \mathsf{P}^{(2n-2)\star})^{2(n-1)}}\det \mathsf{P}^{(2n-4)}. \tag{129}$$

In deriving Eq. (129) we have relied on the (Sylvester–Franke) identity [cf. Appendix A, Eq. (A8)]

$$\det C_{n-2}(\mathsf{P}^{(2n-2)\star})=(\det \mathsf{P}^{(2n-2)\star})^{\binom{n-1}{n-3}}. \tag{130}$$

III. THE GRASSMANN INTEGRAL EQUATION

Having obtained in the preceding section explicit formulas for the action map f for the case of the Grassmann algebras $\mathcal{G}_{2n}$, $n=2,3,4$, we can now concentrate on the study of certain particular relations between $G_0[\{\bar{\Psi}\},\{\Psi\}]$ and $G[\{\bar{\Psi}\},\{\Psi\}]$. As explained in the Introduction we are interested in the equation ($0<\lambda\in\mathbf{R}$)

$$G[\{\bar{\Psi}\},\{\Psi\}]=G_0[\{\lambda\bar{\Psi}\},\{\lambda\Psi\}]+\Delta_f(\lambda). \tag{131}$$

$\Delta_f(\lambda)$ is some constant which is allowed to depend on λ and which we choose to obey [in view of Eq. (29), we have the freedom to do so]

$$\Delta_f(1)=0. \tag{132}$$

For $\lambda=1$, Eq. (131) is the fixed point equation for the action map f (cf. Ref. 73, p. 288). Equation (131) applied to Eq. (15), the latter reads ($\tilde{C}=\exp[-A^{(0)}-\Delta_f(\lambda)]$)

$$e^{G_0[\{\lambda\bar{\Psi}\},\{\lambda\Psi\}]}=\tilde{C}\int\prod_{l=1}^{n}(\mathrm{d}\chi_l\,\mathrm{d}\bar{\chi}_l)\,e^{G_0[\{\bar{\chi}+\bar{\Psi}\},\{\chi+\Psi\}]+\Sigma_{l=1}^{n}(\bar{\eta}_l\chi_l+\bar{\chi}_l\eta_l)}, \tag{133}$$

$$\overline{\eta}_l = \frac{\partial G_0[\{\lambda\overline{\Psi}\},\{\lambda\Psi\}]}{\partial \Psi_l}, \quad \eta_l = -\frac{\partial G_0[\{\lambda\overline{\Psi}\},\{\lambda\Psi\}]}{\partial \overline{\Psi}_l}. \tag{134}$$

Clearly, this a Grassmann integral equation for $G_0[\{\overline{\Psi}\},\{\Psi\}]$ (more precisely, a nonlinear Grassmann integro-differential equation). In view of Eq. (18), Eq. (131) is equivalent to

$$A^{(0)\prime} = A^{(0)} + \Delta_f(\lambda), \tag{135}$$

$$\mathsf{A}^{(2k)\prime} = \lambda^{2k}\mathsf{A}^{(2k)}, \quad k>0. \tag{136}$$

Equation (136) represents a coupled system of nonlinear matrix equations. We are now going to solve Eq. (133) [i.e., Eq. (131)] for $n=2,3,4$ by solving Eq. (136).

A. The case $n=2$

Applying Eq. (136) for $k=1$ to Eq. (46), we find

$$\mathsf{P}^{(4)\star} = \lambda^2 \det \mathsf{A}^{(2)}. \tag{137}$$

Equation (40) then immediately yields

$$A^{(4)}_{12,12} = (1-\lambda^2)\det \mathsf{A}^{(2)}. \tag{138}$$

$\mathsf{A}^{(2)}$ remains an arbitrary matrix with $\det \mathsf{A}^{(2)} \neq 0$. To determine $A^{(0)}$ imagine that the action $G_0[\{\overline{\Psi}\},\{\Psi\}]$ specified by Eq. (138) would have been induced by some action $G_{-1}[\{\overline{\Psi}\},\{\Psi\}] = G_0[\{\lambda^{-1}\overline{\Psi}\},\{\lambda^{-1}\Psi\}]$ [by means of Eq. (15)—replacing G by G_0 and G_0 by G_{-1}, respectively] with the partition function $P(G_{-1}) = \lambda^{-2}\det \mathsf{A}^{(2)}$ [cf. Eq. (40)]. Then (cf. Ref. 80 of Sec. II B)

$$A^{(0)} = \ln P(G_{-1}) = \ln \det \mathsf{A}^{(2)} - 2\ln\lambda \tag{139}$$

and, consequently,

$$\Delta_f(\lambda) = 4\ln\lambda. \tag{140}$$

From the above considerations we see that for $n=2$, Eq. (131) has always a solution for any value of λ $(0<\lambda\in\mathbf{R})$. For $\lambda=1$ the solution corresponds to a Gaussian integral while for $\lambda\neq 1$ it corresponds to some non-Gaussian integral [cf. Eq. (133)]. Consequently, for any even value of n Eq. (131) has always a solution for any value of λ $(0<\lambda\in\mathbf{R})$. This follows from the fact that these solutions can be constructed as a sum of $n=2$ solutions with a common value of λ.

B. The case $n=3$

Applying Eq. (136) for $k=1$ to Eq. (68), we find

$$\mathsf{P}^{(6)\star}\mathbf{1}_3 = \lambda^2\mathsf{P}^{(4)\star}\mathsf{A}^{(2)} = \lambda^2\mathsf{A}^{(2)}\mathsf{P}^{(4)\star} \tag{141}$$

$$= \lambda^2[\det \mathsf{A}^{(2)}\mathbf{1}_3 - \mathsf{A}^{(4)\star}\mathsf{A}^{(2)}] = \lambda^2[\det \mathsf{A}^{(2)}\mathbf{1}_3 - \mathsf{A}^{(2)}\mathsf{A}^{(4)\star}]. \tag{142}$$

Furthermore, combining Eqs. (75), (76), (71), (136) we obtain the relations

$$\lambda^6 \det \mathsf{P}^{(4)\star}\det \mathsf{A}^{(2)} = (\mathsf{P}^{(6)\star})^3, \tag{143}$$

$$\lambda^{12}\frac{(\det \mathsf{P}^{(4)\star})^5}{\det \mathsf{A}^{(2)}} = (\mathsf{P}^{(6)\star})^9. \tag{144}$$

From these two equations we can conclude that

$$\det \mathsf{P}^{(4)\star} = \pm\lambda^3 (\det \mathsf{A}^{(2)})^2, \tag{145}$$

$$\mathsf{P}^{(6)\star} = \pm\lambda^3 \det \mathsf{A}^{(2)}. \tag{146}$$

Taking into account the above equations, from Eq. (69) we find then

$$\mathsf{A}^{(4)\star} = -\left(1 \mp \frac{1}{\lambda}\right) \mathsf{P}^{(4)\star}. \tag{147}$$

By virtue of Eq. (61) this entails

$$\mathsf{A}^{(4)\star} = (1 \mp \lambda) \operatorname{adj} \mathsf{A}^{(2)}, \tag{148}$$

$$\mathsf{P}^{(4)\star} = \pm\lambda \operatorname{adj} \mathsf{A}^{(2)}. \tag{149}$$

One easily sees that Eq. (149) is in line with the result (145). Finally, applying Eqs. (136), (141), (145), (146) to Eq. (70) we calculate $A^{(6)}_{123,123}$. It reads

$$A^{(6)}_{123,123} = (\lambda \mp 1)^2 (\pm\lambda - 4) \det \mathsf{A}^{(2)}. \tag{150}$$

Applying the same procedure to Eq. (74), we find the consistency equation

$$(\lambda \mp 1)^3 = 0, \tag{151}$$

which has only one solution, namely $\lambda = 1$ (choose the upper sign). This solution is just the elementary one which corresponds to a Gaussian integral [cf. Eq. (133)].

C. The case $n=4$

Applying Eq. (136) for $k=1$ to Eq. (104), we find

$$\mathsf{P}^{(8)\star} \mathbf{1}_4 = \lambda^2 \mathsf{P}^{(6)\star} \mathsf{A}^{(2)} = \lambda^2 \mathsf{A}^{(2)} \mathsf{P}^{(6)\star} \tag{152}$$

$$= \lambda^2 [\det \mathsf{A}^{(2)} \mathbf{1}_4 - \mathsf{F}_a(\mathbf{1}_4, C_2(\mathsf{A}^{(2)})^\star \mathsf{A}^{(4)\star}) - \mathsf{A}^{(6)\star} \mathsf{A}^{(2)}]$$

$$= \lambda^2 [\det \mathsf{A}^{(2)} \mathbf{1}_4 - \mathsf{F}_a(\mathbf{1}_4, \mathsf{A}^{(4)\star} C_2(\mathsf{A}^{(2)})^\star) - \mathsf{A}^{(2)} \mathsf{A}^{(6)\star}]. \tag{153}$$

Furthermore, combining Eqs. (118), (119), (111), (136) we obtain the relations

$$\lambda^8 \det \mathsf{P}^{(6)\star} \det \mathsf{A}^{(2)} = (\mathsf{P}^{(8)\star})^4, \tag{154}$$

$$\lambda^{24} (\det \mathsf{P}^{(6)\star})^6 = (\mathsf{P}^{(8)\star})^{18}. \tag{155}$$

From these two equations we can conclude that

$$\det \mathsf{P}^{(6)\star} = \lambda^8 (\det \mathsf{A}^{(2)})^3, \tag{156}$$

$$\mathsf{P}^{(8)\star} = \pm\lambda^4 \det \mathsf{A}^{(2)}. \tag{157}$$

We can now apply Eq. (136) to the Eqs. (105) and (114). Taking into account Eqs. (85), (111), we can derive from these two equations the following compound matrix equation:

$$C_2(\mathsf{P}^{(6)\star} \mathsf{A}^{(2)}) = (\lambda^2 \det \mathsf{A}^{(2)})^2 \mathbf{1}_6. \tag{158}$$

Its solution reads [cf. Ref. 81, Sec. 3, p. 149, Eq. (11)]

$$\mathsf{P}^{(6)\star} = \pm \lambda^2 \operatorname{adj} \mathsf{A}^{(2)}. \tag{159}$$

Equation (159) is in line with Eq. (156) [the signs on the rhs are fixed by making reference to Eqs. (152), (157)]. We can now take into account Eq. (159) in considering Eq. (114) further. Eq. (114) then yields the following matrix equation:

$$\mathsf{P}^{(4)} C_2(\mathsf{A}^{(2)})^\star = \pm C_2(\mathsf{A}^{(2)}) \mathsf{P}^{(4)\star}. \tag{160}$$

By virtue of Eq. (85), Eq. (160) can equivalently be written as

$$\mathsf{A}^{(4)} C_2(\mathsf{A}^{(2)})^\star = \pm C_2(\mathsf{A}^{(2)}) \mathsf{A}^{(4)\star}. \tag{161}$$

We will not study here the complete set of solutions of Eq. (161). This would need to be the subject of a study in its own. Here, it suffices to mention that for the ansatz (α is some arbitrary constant, B some 4×4 matrix)

$$\mathsf{A}^{(4)} = \alpha C_2(\mathsf{B})^\star. \tag{162}$$

Equation (161) reads

$$C_2(\mathsf{A}^{(2)}\mathsf{B})^\star = \pm C_2(\mathsf{A}^{(2)}\mathsf{B}). \tag{163}$$

For the upper sign, this is exactly the type of compound matrix equation studied in Ref. 81. Of course, Eq. (161) has solutions which correspond to two $n=2$ solutions (with a common value of λ) discussed at the end of Sec. III A.[82] Here, we want to go beyond these solutions.

For the present purpose, we consider in Eq. (161) only the upper sign on the rhs and study the ansatz ($\kappa \in \mathbf{R}$)

$$\mathsf{P}^{(4)} = \kappa C_2(\mathsf{A}^{(2)}), \tag{164}$$

$$\mathsf{A}^{(4)} = (1-\kappa) C_2(\mathsf{A}^{(2)}), \tag{165}$$

which is a special version of Eq. (162). Inserting this ansatz into Eq. (107) and taking into account Eqs. (136), (157), (159), we find

$$\mathsf{A}^{(6)\star} = (\lambda^2 - 6\kappa^2 + 9\kappa - 4) \operatorname{adj} \mathsf{A}^{(2)}. \tag{166}$$

Applying the same procedure to Eq. (115), we obtain the following consistency condition:

$$\lambda^2 - 3\kappa^2 + 3\kappa - 1 = \lambda^2 - 3\kappa(\kappa - 1) - 1 = 0. \tag{167}$$

Furthermore, applying the ansatz (164) to Eq. (110) and taking into account Eqs. (136), (157), (159) yields

$$A^{(8)}_{1234,1234} = (\lambda^4 + 20\lambda^2 - 24\lambda^2\kappa + 72\kappa^3 - 147\kappa^2 + 108\kappa - 30) \det \mathsf{A}^{(2)}. \tag{168}$$

Again, subjecting Eq. (117) to the same procedure we obtain yet another consistency condition,

$$2\lambda^2 - 3\lambda^2\kappa + 9\kappa^3 - 15\kappa^2 + 9\kappa - 2 = (2 - 3\kappa)[\lambda^2 - 3\kappa(\kappa - 1) - 1] = 0. \tag{169}$$

Obviously, this equation is fulfilled if λ, κ obey Eq. (167). Consequently, we can restrict our attention to solutions of Eq. (167).

From Eq. (167) we conclude that the ansatz (164) provides us with solutions of Eq. (131) for any value of $\lambda \geqslant 1/2$ (if κ assumes real values only). Of particular interest to us are solutions of Eq. (167) for $\lambda = 1$ (see Refs. 28 and 73). In this case, Eq. (167) reads

$$\kappa(\kappa - 1) = 0. \tag{170}$$

Clearly, this equation has two solutions,

$$\kappa_{\mathrm{I}} = 1, \tag{171}$$

$$\kappa_{\mathrm{II}} = 0. \tag{172}$$

The corresponding expressions for the action $G_0 = G$ then read as follows:

$$G_{0\mathrm{I}}[\{\overline{\Psi}\},\{\Psi\}] = G_{0\mathrm{I}}(G_q) = \ln\det \mathsf{A}^{(2)} + G_q, \tag{173}$$

$$G_{0\mathrm{II}}[\{\overline{\Psi}\},\{\Psi\}] = G_{0\mathrm{II}}(G_q) = \ln\det \mathsf{A}^{(2)} + G_q - \tfrac{1}{2}G_q^2 + \tfrac{1}{2}G_q^3 - \tfrac{3}{8}G_q^4, \tag{174}$$

$$G_q = G_q[\{\overline{\Psi}\},\{\Psi\}] = \sum_{l,m=1}^{4} A_{l,m}^{(2)}\overline{\Psi}_l\Psi_m = \overline{\Psi}\mathsf{A}^{(2)}\Psi. \tag{175}$$

As one can see from Eq. (133) $G_{0\mathrm{I}}$ corresponds to a Gaussian integral while $G_{0\mathrm{II}}$ corresponds to some non-Gaussian integral. While it is well known that for the action $G_0 = G_{0\mathrm{I}}$ the equation $G = G_0$ applies it is indeed a remarkable fact that the same is true for $G_0 = G_{0\mathrm{II}}$. However, this is not yet the end of remarkable features of these actions. It is also instructive to work out for $\kappa_{\mathrm{I}} = 1$ and $\kappa_{\mathrm{II}} = 0$ the corresponding expressions for $W[\{\overline{\eta}\},\{\eta\}]$ on the basis of Eq. (90).

$$W_{\mathrm{I}}[\{\overline{\eta}\},\{\eta\}] = W_{\mathrm{I}}(W_q) = G_{0\mathrm{I}}(W_q) = \ln\det \mathsf{A}^{(2)} + W_q, \tag{176}$$

$$W_{\mathrm{II}}[\{\overline{\eta}\},\{\eta\}] = W_{\mathrm{II}}(W_q) = G_{0\mathrm{II}}(W_q) = \ln\det \mathsf{A}^{(2)} + W_q - \tfrac{1}{2}W_q^2 + \tfrac{1}{2}W_q^3 - \tfrac{3}{8}W_q^4, \tag{177}$$

$$W_q = W_q[\{\overline{\eta}\},\{\eta\}] = -\overline{\eta}[\mathsf{A}^{(2)}]^{-1}\eta. \tag{178}$$

Again, while the relation $W_{\mathrm{I}} = G_{0\mathrm{I}}$ is well known in the present context the equality $W_{\mathrm{II}} = G_{0\mathrm{II}}$ comes as a complete surprise and one can only wonder which general principle is manifesting here itself. We will explore this issue in the next section.

D. Further analysis

We can characterize the solutions $G_{0\mathrm{I}}$, $G_{0\mathrm{II}}$ of the equation (131) found for $n = 4$, $\lambda = 1$, in the preceding section by two properties which may be of general significance. The first one is related to the identity $W = G_0$ [Eqs. (176) and (177)]. One immediately recognizes that for

$$[\mathsf{A}^{(2)}]^2 = -\mathbf{1}_4, \tag{179}$$

$\exp G_0 = \exp G_{0\mathrm{I}} (= \exp G = Z)$ and $\exp G_0 = \exp G_{0\mathrm{II}}$ are *self-reciprocal* Grassmann functions (of course, this is a well-known property of $\exp G_{0\mathrm{I}}$):

$$\int \prod_{l=1}^{4} (\mathrm{d}\chi_l\,\mathrm{d}\overline{\chi_l})\;\; e^{\,G_0[\{\overline{\chi}\},\{\chi\}] + \overline{\eta}\chi + \overline{\chi}\eta} = e^{\,G_0[\{\overline{\eta}\},\{\eta\}]} \tag{180}$$

[$\det \mathsf{A}^{(2)} = 1$, cf. Eq. (179)],[83] i.e., they are eigenfunctions to the Fourier–Laplace transformation[84] to the eigenvalue 1. The term self-reciprocal function is taken from real (complex) analysis where it also denotes eigenfunctions of some integral transformation, in particular, the Fourier transformation.[85–91]

The second property of the solutions $G_{0\mathrm{I}}$, $G_{0\mathrm{II}}$ is related to the identity $G = W$ [apply the fixed point condition $G = G_0$ to the Eqs. (176), (177)]. Taking into account Eqs. (173)–(178), Eqs. (13), (16) tell us that

$$G(G_q)=G(W_q)-\sum_{l=1}^{4}\left(\overline{\eta_l}\Psi_l+\overline{\Psi}_l\eta_l\right), \tag{181}$$

$$\overline{\eta_l}=\frac{\partial G(G_q)}{\partial \Psi_l}=-G'(G_q)(\overline{\Psi}\mathsf{A}^{(2)})_l, \tag{182}$$

$$\eta_l=-\frac{\partial G(G_q)}{\partial \overline{\Psi}_l}=-G'(G_q)(\mathsf{A}^{(2)}\Psi)_l. \tag{183}$$

Here,

$$G'(G_q)=\frac{\partial G(G_q)}{\partial G_q}, \tag{184}$$

where G_q is treated as a formal parameter for the moment. In view of Eqs. (182), (183) it holds

$$W_q=-G_q[G'(G_q)]^2. \tag{185}$$

Taking into account the Eqs. (182), (183), (185), Eq. (181) can be written as

$$G(s)=G(-s[G'(s)]^2)+2sG'(s), \quad s=G_q. \tag{186}$$

Equation (186) is of a very general nature. Its shape does not depend on the value of n. Its derivation depends on the fact only that G, W are functions of G_q, W_q, respectively, and that the identity $G=W$ holds. As we demonstrate in Appendix C, Eq. (186) can also be derived under analogous conditions starting from a (Euclidean space–time) version of Eqs. (1)–(5) for a scalar field on a finite lattice. Consequently, until further notice we disregard the fact that s is a bilinear in the Grassmann algebra generators and simply understand Eq. (186) as an equation for a function $G=G(s)$. We will now analyze Eq. (186) further.

Equation (186) appears to be somewhat involved but it can be simplified the following way. We can differentiate both sides of Eq. (186) with respect to s. The resulting equation can be transformed to read

$$\{2s\, G''(s)+G'(s)\}\{1-G'(s)G'(-s[G'(s)]^2)\}=0\,. \tag{187}$$

Equation (187) is being obeyed if either one of the two following equations of very different mathematical nature is respected:

$$2sG''(s)+G'(s)=0, \tag{188}$$

$$G'(s)G'(-s[G'(s)]^2)=1. \tag{189}$$

The solution of the linear differential equation (188) reads

$$G'(s)\sim e^{-\sqrt{s}} \tag{190}$$

entailing

$$G(s)\sim(1+\sqrt{s})e^{-\sqrt{s}}. \tag{191}$$

As $G(s)$ depends on $\sqrt{s}$ this solution is of no relevance in the context of Grassmann algebras. To see this note that the function $G(s)$ contains odd powers of $\sqrt{s}$ in its (Taylor) expansion (in terms of $t=\sqrt{s}$) around $s=0$. If s is being interpreted as a bilinear form in the generators of the

Grassmann algebra these terms have no interpretation within the Grassmann algebra framework.[92] Consequently, in the following we can concentrate our attention onto the nonlinear functional equation (189).

To gain further insight it turns out to be convenient now to define the following functions [the definition in Eq. (193) could equally well read $d(t) = -i\ b(t)$]:

$$b(t) = t\ G'(t^2) = \frac{1}{2}\frac{\partial}{\partial t}G(t^2), \tag{192}$$

$$d(t) = ib(t). \tag{193}$$

Then, having multiplied both sides by $-\sqrt{s}$ Eq. (189) can be written as $(t=\sqrt{s})$

$$d^2(t) = d(d(t)) = -t. \tag{194}$$

This is an iterative functional equation: the function $d(t)$ is the (second) iterative root of $-\mathbf{1}$ (for a review of iterative functional equations see Ref. 93, in particular Chap. 11, p. 421, Ref. 94, in particular Chap. XV, p. 288, also see Ref. 95, Chap. 2, p. 36). The functional equation (194) has been studied by Massera and Petracca[96] who have pointed out its relation to the equivalent functional equation

$$h(h(x)) = \frac{1}{x}. \tag{195}$$

[Define the involution $q(x) = (1-x)/(1+x)$. If $h(x)$ is a solution of Eq. (195) the function $q \circ h \circ q$ is a solution of Eq. (194).] This functional equation characterizes functions h for which $h^{-1} = 1/h$ (note in this context Refs. 97–101, in particular Ref. 101, p. 712). Equation (194) has also been studied for real functions in Ref. 102, Chap. II, Sec. 5, p. 54, and in Refs. 103–106. Of course, in view of Eq. (193) in general we are concerned with complex solutions of Eq. (194).

If the function $G'(s)$ has a definite symmetry under $s \to -s$ Eq. (194) can be simplified to some extent [getting rid of the imaginary unit i present in Eq. (193)]. If $G'(s)$ is an even function [i.e., up to some constant $G(s)$ is odd] Eq. (194) can be written as

$$b^2(t) = b(b(t)) = t. \tag{196}$$

This iterative functional equation is a special case of the functional equation $b^k(t) = t$ which is being called the *Babbage equation* (it has been studied first by Charles Babbage.[107–110] See Ref. 94, Chap. XV, Sec. 1, p. 288, Ref. 93, Sec. 11.6, p. 450, for more information and references, recent references not referred to in Refs. 94, 93 are Refs. 111, 112). Solutions $b(t)$ of Eq. (196) (i.e., solutions of the Babbage equation for $k=2$) are called *involutory functions* [*(second) iterative roots of unity/identity, periodic functions/maps*]. If, for example, the function $G(s)$ stands in correspondence to a Gaussian integral [cf. Eq. (133)], $G(s) = s$ and, consequently,

$$b(t) = t. \tag{197}$$

This is the most elementary involutory function one can think of. Note, that the set of solutions of Eq. (196) is very large as this set is equivalent to the set of even function (see, e.g., Refs. 113, 114, Ref. 93, p. 451). If $G'(s)$ is an odd function [i.e., $G(s)$ is even] Eq. (194) can be written as

$$b^2(t) = b(b(t)) = -t. \tag{198}$$

However, this case is not very interesting as real functions solving Eq. (198) are necessarily discontinuous (Ref. 94, Chap. XV, §4, p. 299, Refs. 103, 97–101, 104, Ref. 93, Subsec. 11.2B, p. 425).

The above consideration can be applied to the solutions of the Grassmann integral equation found in Sec. III C. Equation (173) is of course being described by Eq. (197) $[b_{\rm I}(t)=t]$. From Eq. (174) we recognize that the function $G_{\rm II}(s)$ does not have a definite symmetry under $s\rightarrow -s$. We find

$$b_{\rm II}(t)=t\ (1-t^2+\tfrac{3}{2}\,t^4-\tfrac{3}{2}\,t^6). \tag{199}$$

and one can check that the corresponding function $d_{\rm II}(t)=ib_{\rm II}(t)$ indeed fulfills Eq. (194) at order t^7. [Going through the above arguments one can convince oneself that this is the appropriate order in t one has to take into account for the Grassmann algebra $\mathcal{G}_8$. Order t^7 corresponds to order s^3 in Eq. (189).]

IV. DISCUSSION AND CONCLUSIONS

While most of the explicit expressions obtained in the present paper for the Grassmann algebras $\mathcal{G}_{2n}$, $n=2,3,4$, have been obtained here for the first time, some of them can be compared to results derived earlier by other authors. The point is that partition functions for specific (finite-dimensional) fermionic systems have been obtained by a number of authors and some of these results can be used for direct comparison with the present results. For example, our expressions (40), (60), (84), can be seen to agree with Eq. (8), p. 694, of Ref. 115. Furthermore, our Eq. (40) is in line with Eq. (13), p. 1298, of Ref. 116, the same applies to our Eq. (60) and its counterpart, Eq. (14), p. 1298, Ref. 116. Also Eq. (16), p. 1298, Ref. 116 (for $n=3$, $l=3$ and $n=4$, $l=2,3,4$) gives the same results as our Eqs. (60), (84). And finally, our Eq. (84) agrees with Eq. (10), p. 1083, of Ref. 117 (for $N=4$).

Our consideration of the action map f in the present paper has been motivated by the formalism of (lattice) quantum field theory. However, the consideration of certain modifications of the map f might also be of some interest from various points of view. Let us consider a special set of modifications which can be described by replacing the Eqs. (16) by the equations

$$\overline{\eta}_l=\frac{\partial\tilde{G}[\{\overline{\Psi}\},\{\Psi\}]}{\partial\Psi_l},\quad \eta_l=-\frac{\partial\tilde{G}[\{\overline{\Psi}\},\{\Psi\}]}{\partial\overline{\Psi}_l} \tag{200}$$

(G is replaced by $\tilde{G}$). For example, if one is just interested in the fixed point condition for the action map f [i.e., in the Eq. (131) for $\lambda=1$] it might make sense to consider instead of the action map f a different map $\tilde{f}$ [described by the Eqs. (16), (200), respectively] having the same set of fixed points but which is algebraically or numerically easier to study. One such modification consists in choosing $\tilde{G}=G_0$ [cf. Ref. 73, p. 291, Eq. (2.9)]. The implicit representation of the map f given in Eqs. (15), (16) would then turn into an explicit representation of the map $\tilde{f}$ which might be easier to handle in some respect. As an aside in this context, we mention that for this map $\tilde{f}$ the equations (43), (64), (99) (replace $\mathsf{A}^{(2)\prime}$ on the rhs by $\mathsf{A}^{(2)}$) exhibit a *formal* similarity to the main equation for the Schulz iteration scheme for the calculation of the inverse of a matrix [see Eq. (7), p. 58, in Ref. 118].[119–123] The similarity, however, is only formal as in general the matrix $\mathsf{P}^{(2n-2)\star}/\mathsf{P}^{(2n)\star}$ is not invariant under the map $\tilde{f}$ [for the simplest case, $n=2$, for example, one can convince oneself of this fact starting from Eqs. (43), (44) where one has to omit in this case the primes on the rhs].

As already mentioned the investigation performed in the present study within the framework of Grassmann algebras has been inspired by a problem in quantum field theory which in its simplest version (within zero-dimensional field theory) is a problem in real/complex analysis. The standard analysis analogue of the Grassmann integral equation studied in Chap. 3 (for $\lambda=1$) reads [cf. Eq. (8)]

$$e^{g(y)}=C\int_{-\infty}^{+\infty}\mathrm{d}x\, e^{g(x+y)-g'(y)x}. \tag{201}$$

This is a nonlinear integro-differential equation for the real function $g(x)$. Clearly, the elementary function $g(x)=-ax^2/2$, $0<a\in\mathbf{R}$ $[C=\sqrt{a/(2\pi)}]$ solves this equation. However, the interesting question is if this equation has any other (nonelementary) solution which stands in correspondence to a non-Gaussian integral. No qualitative information seems to be available in the mathematical literature in this respect. As pointed out in Ref. 28, Sec. 4, p. 859 (p. 475 of the English transl.), Eq. (201) is a very complicated equation. The main difficulty in explicitly finding any nonelementary solution to it (if it exists at all—we just assume this for the time being) consists in the fact that it is very difficult if not impossible to calculate for an arbitrary function $\exp g(x)$ its Fourier (or Laplace) transform explicitly. The question now arises if the analysis in Sec. III D of the solutions of the Grassmann integral equation found for $n=4$, $\lambda=1$, might help in overcoming this problem. We do not have any final answer on this but in our view it makes sense to say: perhaps. The solutions of the Grassmann integral equation found for $n=4$, $\lambda=1$, are characterized by two properties which are not related to the anticommuting character of Grassmann variables. The solutions were related, first, to eigenfunctions of the Fourier–Laplace transformation to the eigenvalue 1 (i.e., to self-reciprocal functions) and, second, to some iterative functional equation. Now, it seems to be reasonable to assume that also (some) solutions of Eq. (201) might be characterized by these two properties. The two sets of functions obeying one of these two principles are very large and one might think that the intersection of these two sets contains also other functions than just the functions given by $g(x)=-ax^2/2$. The task of solving Eq. (201) then is equivalent to studying eigenfunctions of the Fourier transformation to the eigenvalue 1, i.e., self-reciprocal functions $\exp g(x)$.[124–126] They obey the equation

$$e^{g(y)}=\int_{-\infty}^{+\infty}\frac{dx}{\sqrt{2\pi}}e^{iyx}e^{g(x)}. \tag{202}$$

The consideration of eigenfunctions of the Fourier transformation solves the above mentioned problem of finding their Fourier transforms at once.[127] There is a vast mathematical literature on self-reciprocal functions (in particular for the Fourier transformation) but in our context it makes sense to concentrate on a certain subclass of self-reciprocal functions. Klauder (Ref. 128, p. 375, Ref. 59, Subsec. 10.4, p. 246) has pointed out the relevance of infinitely divisible characteristic functions in a quantum field theoretic context. This entails in our context that the self-reciprocal functions $\exp g(x)$ should be self-reciprocal probability densities (positive definite ones, in addition: without zeros—this follows from infinite divisibility). The subject of self-reciprocal (positive definite) probability densities has been studied for some time in probability theory (Refs. 129–135, Ref. 136, Subsec. 7.5, p. 122, Refs. 137, 138, Ref. 139, Chap. 6, p. 148, Ref. 140; see Refs. 137, 139 for some further references). Of most relevance to the present problem is the work by Teugels[130] who describes explicit methods to construct solutions of Eq. (202) (also note Ref. 140 in this respect). From the solutions $\exp g(x)$ of (202) (which are even functions) we define the function $G=G(-x^2/2)=g(x)$.[141,142] The function $d(t)$ [Eq. (193)] associated with it then has to obey the functional equation (194) in order to ensure that the function $g(x)$ solves Eq. (201). In the case under discussion, the equations (192)–(194) can be reformulated the following way. Define the functions

$$\tilde{b}(x)=-\frac{\partial g(x)}{\partial x}=-\frac{\partial}{\partial x}G\left(-\frac{x^2}{2}\right)=xG'\left(-\frac{x^2}{2}\right), \tag{203}$$

$$\tilde{d}(x)=i\tilde{b}(x). \tag{204}$$

Then, from Eq. (194) one can derive the following iterative functional equation which has to be fulfilled:

$$\tilde{d}^2(x) = \tilde{d}(\tilde{d}(x)) = -x. \tag{205}$$

As in Sec. III D, one can now assume a certain behavior of the function $g(x)$. Assuming again that the function $G'(s)$ is an even function [i.e., up to some constant $G(s)$ is odd] Eq. (205) can be written as

$$\tilde{b}^2(x) = \tilde{b}(\tilde{b}(x)) = x. \tag{206}$$

However, this case is not very interesting as it does not lead to any non-Gaussian function $\exp g(x)$ [Ref. 143, Theorem 3, p. 117 (Teor. Veroyatn. Prim.), p. 119 (Theor. Prob. Appl.); note that Lukacs uses the term self-reciprocal in this article in a different sense than we do in the present paper]. Assuming that $G'(s)$ is an odd function [i.e., $G(s)$ is even] Eq. (205) can be written as

$$\tilde{b}^2(x) = \tilde{b}(\tilde{b}(x)) = -x. \tag{207}$$

However, this case is also not very interesting as real functions solving Eq. (207) are necessarily discontinuous (Ref. 94, Chap. XV, Sec. 4, p. 299, Refs. 103, 97–101, 104, Ref. 93, Subsec. 11.2B, p. 425). Consequently, Eq. (205) cannot sensibly be simplified by the above considerations. However, the sketched program still faces another challenge. At first glance, it is not obvious how to combine the existent theory of self-reciprocal probability densities with the theory of iterative functional equations in an operationally effective way in order to find nonelementary solutions of Eq. (201) (or its multidimensional generalizations) which correspond to non-Gaussian integrals. This will have to be the subject of further research.

This discussion has brought us to the end of the present study. What are its main results? From a mathematical point of view, the paper introduces a new type of equation which has not been studied before—a Grassmann integral equation. The concrete equation studied has been shown to be equivalent to a coupled system of nonlinear matrix equations which can be solved (Sec. III). From the point of view of standard quantum field theory, the main results of the present article are as follows. For low-dimensional Grassmann algebras the present paper derives explicit expressions for the finite-dimensional analogue of the effective action functional in terms of the data specifying a fairly general ansatz for the corresponding analogue of the so-called "classical" action functional (Sec. II). This is a model study which in some way can be understood as the fermionic (Grassmann) analogue of zero-dimensional field theory and which may turn out to be useful in several respect. Moreover, for an arbitrary Grassmann algebra (related to an arbitrary purely fermionic "lattice quantum field theory"—on a finite lattice) on the basis of the explicit results obtained for low-dimensional Grassmann algebras an exact expression for the four-fermion term of the finite lattice analogue of the effective action functional is derived in a heuristic manner [Sec. II E, Eq. (123)]. From the point of view of the conceptual foundations of quantum field theory, the present study demonstrates on the basis of a finite-dimensional example that the (Grassmann) integral equation proposed in Refs. 28, 73 can have solutions which are equivalent to non-Gaussian integrals (Sec. III). This certainly will be of interest in various respect. To illustrate this point let us repeat in compact form some of the results found for the Grassmann algebra $\mathcal{G}_8$ in Sec. III C (these results are specific for this Grassmann algebra). Define for an arbitrary invertible 4×4 matrix B ($\det \mathsf{B} \neq 0$) the Grassmann bilinears

$$G_q = \sum_{l,m=1}^{4} \mathsf{B}_{lm} \bar{\chi}_l \chi_m = \bar{\chi} \mathsf{B} \chi, \tag{208}$$

$$W_q = -\sum_{l,m=1}^{4} [\mathsf{B}^{-1}]_{lm} \bar{\eta}_l \eta_m = -\bar{\eta} [\mathsf{B}]^{-1} \eta. \tag{209}$$

Then, the following equation applies:

$$\int \prod_{l=1}^{4} (\mathrm{d}\chi_l \, \mathrm{d}\overline{\chi_l}) \, e^{(\overline{\eta}\chi + \overline{\chi}\eta)} \exp[G_q - \tfrac{1}{2} G_q^2 + \tfrac{1}{2} G_q^3 - \tfrac{3}{8} G_q^4] = \det \mathsf{B} \exp[W_q - \tfrac{1}{2} W_q^2 + \tfrac{1}{2} W_q^3 - \tfrac{3}{8} W_q^4]. \tag{210}$$

This should be compared to the well-known, corresponding result for a Gaussian integral

$$\int \prod_{l=1}^{4} (\mathrm{d}\chi_l \, \mathrm{d}\overline{\chi_l}) \, e^{(\overline{\eta}\chi + \overline{\chi}\eta)} \exp[G_q] = \det \mathsf{B} \exp[W_q]. \tag{211}$$

Moreover, in Sec. III C it has been found that the (Grassmann) function $G_q - \frac{1}{2} G_q^2 + \frac{1}{2} G_q^3 - \frac{3}{8} G_q^4$ is the (first) Legendre transform of the function $W_q - \frac{1}{2} W_q^2 + \frac{1}{2} W_q^3 - \frac{3}{8} W_q^4$ [cf. Eqs. (181)–(183)]. This entails that these functions behave exactly the same way as the functions G_q and W_q. It is clear that any Grassmann algebra $\mathcal{G}_{8k}$, $1 \leqslant k \in \mathbf{N}$, supports equations of the type (210) [simply by multiplying k copies of Eq. (210)]. Given the role that Gaussian integrals and their properties play in quantum field theory, statistical physics and probability theory it will be interesting to explore the implications and applications of the above results in the future.

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APPENDIX A

Here we collect some formulas for compound matrices.[144] Let B, D be $n \times n$ matrices. The *compound matrix* $C_k(\mathsf{B})$, $0 \leqslant k \leqslant n$, is a $\binom{n}{k} \times \binom{n}{k}$ matrix of all order k minors of the matrix B. The indices of the compound matrix entries are given by ordered strings of length k. These strings are composed from the row and column indices of the matrix elements of the matrix B the given minor of the matrix B is composed of. Typically, the entries of a compound matrix are ordered lexicographically with respect to the compound matrix indices. (We also apply this convention.) The *supplementary (or adjugate) compound matrix* $C^{n-k}(\mathsf{B})$ (sometimes also referred to as the *matrix of the* kth *cofactors*) of the matrix B is defined by the equation [cf. Eq. (24)]

$$C^{n-k}(\mathsf{B}) = C_{n-k}(\mathsf{B})^{\star}. \tag{A1}$$

The components of the supplementary compound matrix $C^{n-k}(\mathsf{B})$ can also be defined by means of the following formula [here, $l_1 < l_2 < \cdots < l_k$, $m_1 < m_2 < \cdots < m_k$; Ref. 145, Chap. IV, Sec. 89, p. 75, Ref. 146, Chap. 3, p. 18; also see our Eqs. (31)–(36)]

$$C^{n-k}(\mathsf{B})_{LM} = \frac{\partial}{\partial \mathsf{B}_{l_1 m_1}} \cdots \frac{\partial}{\partial \mathsf{B}_{l_k m_k}} \det \mathsf{B}. \tag{A2}$$

This comparatively little known definition of (matrices of) cofactors (supplementary compound matrices) is essentially due to Jacobi (Ref. 147, Sec. 10, p. 301, p. 273 of the 'Gesammelte Werke,' p. 25 of the German transl.; also see the corresponding comment by Muir in Ref. 148, Part I, Chap. IX, pp. 253–272, in particular pp. 262/263).

For compound matrices holds ($\mathbf{1}_r$ is the $r \times r$ unit matrix, α some constant)

$$C_k(\alpha \mathbf{1}_n) = \alpha^k \mathbf{1}_{\binom{n}{k}}. \tag{A3}$$

Important relations are given by the *Binet–Cauchy formula*

$$C_k(\mathsf{B}) C_k(\mathsf{D}) = C_k(\mathsf{BD}) \tag{A4}$$

from which immediately follows

$$C_k(\mathsf{B}^{-1}) = C_k(\mathsf{B})^{-1}, \tag{A5}$$

the *Laplace expansion*

$$C_k(\mathsf{B})C^{\ n-k}(\mathsf{B}) = C^{\ n-k}(\mathsf{B})C_k(\mathsf{B}) = C_k(\mathsf{B})C_{n-k}(\mathsf{B})^\star = C_{n-k}(\mathsf{B})^\star C_k(\mathsf{B}) = \det \mathsf{B}\ \ \mathbf{1}_{\binom{n}{k}}, \tag{A6}$$

Jacobi's theorem [a consequence of the Eqs. (A6) and (A5)]

$$C_k(\mathsf{B}^{-1}) = \frac{1}{\det \mathsf{B}} C^{\ n-k}(\mathsf{B}) = \frac{1}{\det \mathsf{B}} C_{n-k}(\mathsf{B})^\star, \tag{A7}$$

and the *Sylvester–Franke theorem*

$$\det C_k(\mathsf{B}) = (\det \mathsf{B})^{\binom{n-1}{k-1}}. \tag{A8}$$

Compound matrices are treated in a number of references. A comprehensive discussion of compound matrices can be found in Ref. 149, Chap. V, pp. 63–87, Ref. 150, Chap. V, pp. 90–110, and, in a modern treatment, in Ref. 151, Chap. 6, pp. 142–155. More algebraically oriented modern treatments can be found in Ref. 75, Part I, Chap. 2, Sec. 2.4, pp. 116–159, Part II, Chap. 4, pp. 1–164 (very thorough), Ref. 152, Chap. 7, Sec. 7.2, pp. 411–420, and Ref. 153, Vol. 3, Chap. 2, Sec. 2.4, pp. 58–68. Concise reviews of the properties of compound matrices are given in Refs. 154, 155. Also note Refs. 156 and 157.

APPENDIX B

Let B be a 3×3 matrix. Then, the following identities apply:

$$\mathrm{adj}\,\mathsf{B} = \mathsf{B}^2 - \mathsf{B}\ \ \mathrm{tr}\mathsf{B} + \tfrac{1}{2}(\mathrm{tr}\,\mathsf{B})^2\mathbf{1}_3 - \tfrac{1}{2}\mathrm{tr}(\mathsf{B}^2)\mathbf{1}_3, \tag{B1}$$

$$\mathrm{tr}(\mathrm{adj}\,\mathsf{B}) = \tfrac{1}{2}(\mathrm{tr}\,\mathsf{B})^2 - \tfrac{1}{2}\mathrm{tr}(\mathsf{B}^2). \tag{B2}$$

Equation (B1) can be derived by means of the Cayley–Hamilton theorem [cf. e.g., Ref. 158, Subsec. 2.4, p. 264, Eq. (2.4.7), Ref. 159, Sec. 7, p. 154, Eq. (29)].

APPENDIX C

In this appendix we want to rederive Eq. (186) starting from a (Euclidean space–time) version of the Eqs. (1)–(5) on a finite lattice with k sites. The equations (3), (5) then read

$$G[\phi] = W[J] - \sum_{l=1}^{k} J_l \phi_l, \tag{C1}$$

$$J_l = -\frac{\partial G}{\partial \phi_l}. \tag{C2}$$

In analogy to the Eqs. (175), (178) we define (B is a symmetric $k\times k$ matrix)

$$G_q = G_q[\phi] = -\frac{1}{2}\sum_{l,m=1}^{k} \mathsf{B}_{lm}\phi_l\phi_m = -\frac{1}{2}\phi\mathsf{B}\phi, \tag{C3}$$

$$W_q = W_q[J] = \frac{1}{2}\sum_{l,m=1}^{k} (\mathsf{B}^{-1})_{lm}J_l J_m = \frac{1}{2}J\mathsf{B}^{-1}J. \tag{C4}$$

Now we assume that G, W depend on ϕ, J only as functions of $G_q[\phi]$, $W_q[J]$, respectively, and, in addition, that the identity $G=W$ holds. Then, in analogy to the Eqs. (181)–(183) the Eqs. (C1), (C2) read

$$G(G_q)=G(W_q)-\sum_{l=1}^{k} J_l\phi_l, \tag{C5}$$

$$J_l=-\frac{\partial G(G_q)}{\partial\phi_l}=-G'(G_q)(\phi\mathsf{B})_l. \tag{C6}$$

Here, again

$$G'(G_q)=\frac{\partial G(G_q)}{\partial G_q}. \tag{C7}$$

In view of Eq. (C6) it holds

$$W_q=-G_q[G'(G_q)]^2. \tag{C8}$$

Taking into account the Eqs. (C6), (C8), Eq. (C5) can be written as

$$G(s)=G(-s[G'(s)]^2)+2sG'(s), \quad s=G_q, \tag{C9}$$

and this equation completely agrees with Eq. (186).

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a discussion of the Fourier–Laplace transformation in a Grassmann algebra see Ref. 66, Ref. 14, Chap. 1, Sec. 6, Subsec. 2, pp. 62/63 (Subsec. 1.6.2, p. 63 of the English translation), Ref. 67, Appendix A, p. 355, Ref. 68, Appendix A, p. 3709, Ref. 69, Subsec. 10.5.4., p. 339, Ref. 9, Subsec. 2.3, p. 72 (p. 74 of the English translation).

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[74] Only by accident, in preparing the final draft of the present paper we became aware of the fact that the concept proposed in Ref. 73 (preprint version, University of Leipzig Preprint NTZ 16/1993, Physics E-Print arXiv:hep-th/9312137) has also been proposed earlier by Prokhorov (Ref. 28).

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[76] The notation $\mathsf{P}^{(2n)\star}$ is chosen with hindsight. Of course, $\mathsf{P}^{(2n)\star}=\mathsf{P}^{(2n)}$—ignoring the fact that (very formally) these constants live in different spaces, cf. Eq. (24): $\mathsf{P}^{(2n)\star}$ is a constant without indices while $\mathsf{P}^{(2n)}$ is a 1×1 matrix with a length n row and column index.

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[80] Speaking rigorously and for any value of n, this is not possible as due to our choice (29) $A^{(0)}$ cannot be found from $G[\{\bar{\Psi}\},\{\Psi\}]$. The action map f can only be inverted in the subspace of the coefficients $A^{(2k)\prime}$, $A^{(2k)}$, $k>0$. In a way, $A^{(0)}$ is a dummy variable while $A^{(0)\prime}=\ln P$ can be understood as a function of $A^{(2k)\prime}$, $k>0$, via its dependence on $A^{(2k)}$, $k>0$. In this view, the coefficients $A^{(2k)}$, $k>0$, are given in terms of the coefficients $A^{(2k)\prime}$, $k>0$, by means of the inverse action map f^{-1}. On the other hand, $A^{(0)}$ can be given a sensible meaning if one assumes that the action G_0 has been induced by some other action G_{-1} exactly the same way as the action G is being induced by G_0. Then, one can derive an expression for $A^{(0)}$ in terms of the coefficients $A^{(2k)}$, $k>0$, by means of the inverse action map f^{-1}. This way, on the basis of Eq. (54) one finds for $n=2$: $A^{(0)}=2\ln\det\mathsf{A}^{(2)}-\ln\mathsf{P}^{(4)\star}$ [cf. Sec. III A, Eq. (139)].

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[82] Then, $\mathsf{A}^{(2)}$ has a 2×2 block matrix structure and $\mathsf{A}^{(4)}$ has a diagonal matrix structure with $\mathsf{A}^{(4)}=\mathrm{diag}(A^{(4)}_{12,12},0,0,0,0,A^{(4)}_{34,34})$.

[83] We disregard here the generalization to the case $\det\mathsf{A}^{(2)}=-1$.

[84] If we would redefine the Fourier–Laplace transformation in a Grassmann algebra by replacing on the lhs of Eq. (180) $\bar{\eta}$, η by $i\bar{\eta}$, $i\eta$, Eq. (179) would of course read $[\mathsf{A}^{(2)}]^2=\mathbf{1}_4$.

[85] Compare, e.g., Ref. 86, Ref. 88, Chap. IX, p. 245. For an early account of the history of self-reciprocal functions see Ref. 89. Eigenfunctions to the eigenvalue -1 are often called skew-reciprocal. In optics, following a paper by Caola (who seems to not have been aware of the history of the subject in mathematics, Ref. 90) in recent years self-reciprocal functions are often referred to as ''self-Fourier functions'' (in the context of the Fourier transformation). Incidentally, also note the comment in Ref. 91 on the earlier optics literature on the subject.

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