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Delta Potentials on Planes in QED

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Abstract

QED is considered in the presence of delta shaped external gauge potentials with support on one or two planes. Using the propagators determined in these special field configurations the parameter dependence of the vacuum energy (similar to the Casimir effect) is calculated. Thereby, it turns out that also in the case of massive fields nonrelativistic field theory is unable to approximate the results of relativistic field theory. Surprisingly, a parallel calculation using the zeta function method leads to a wrong result if one does not investigate the nonleading terms carefully. First loop calculations exhibit an unexpected renormalization behaviour which may be typical for certain singular background fields.

Introduction

δ-functions are broadly used idealized elements of theoretical physics. With its help it is possible to formulate models which in many cases can be solved explicitly. In quantum mechanics quite a number of such investigations exists [1] whereas in quantum field theory investigations of this kind are just at the beginning. Here, we consider the case of δ-functions with support on (parallel) planes so that they effectively depend on one coordinate only:

$$e\mathbf{A} = 0 \quad , \quad eA_0 = \sum_{i=1}^{n} a_i \delta(x_3 - d_i) \quad . \tag{1}$$

With such a procedure we in fact introduce a more general type of boundary conditions in field theory which generalizes the mostly used Dirichlet boundary condition. In physical terms, such a δ-function may be seen as a model of a penetrable boundary. From another point of view, it can be considered as a generalized potential pot which contains at most one bound state for each degree of freedom.

Quantized Fields in External Delta Potentials

The most simple case seems to be the charged scalar field, described by the Klein-Gordon equation $[(\partial_\mu - ieA_\mu)^2 + m^2]\phi(x) = 0$. Inserting (1) for the potential A_μ there a difficulty connected with the product of two δ-functions appears. One possible solution consists in the choice of

$$[\Box + m^2 - 2\sum_{i=1}^{n} b_i \delta(x_3 - d_i)]\phi(x) = 0 \tag{2}$$

as new field equation. The drawback of this equation is that due to the simple coupling $2b^i\delta(x_3 - d_i)\phi(x)$ (resulting from the term $(eA_0)^2$) the charge sensitivity has been lost. Nevertheless, we will study this equation because it is much simpler than the Dirac equation which will be considered later. The δ-potential leads to the additional boundary condition for the scalar field at the position of the δ-function

$$\partial_3\phi|_{x_3=d_i+\epsilon} - \partial_3\phi|_{x_3=d_i-\epsilon} = -2b_i\phi|_{x_3=d_i}. \tag{3}$$

The positive (negative) energy solutions of this field equation consist of one bound state, symmetric, and antisymmetric scattering states. The quantum field can be constructed with the help of a mode decomposition containing creation a^+, b^+ and destruction operators a^-, b^-.

For later calculations we need the Feynman propagator. It can be written as follows:

$$^sD^c(x,y) = D^c(x-y) + \bar{D}(x,y) \tag{4}$$

$$D^c(x-y) = \frac{i}{2}\int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\Gamma}\; e^{i\tilde{p}(\tilde{x}-\tilde{y}) + i\Gamma|x_3 - y_3|} \tag{5}$$

$$\bar{D}(x,y) = -\frac{b}{2}\int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\Gamma - ib}\frac{1}{\Gamma}\; e^{i\tilde{p}(\tilde{x}-\tilde{y}) + i\Gamma(|x_3| + |y_3|)} \tag{6}$$

(the unusual notations are $\tilde{p} = (p_0, p_1, p_2)$, $\tilde{x} = (x_0, x_1, x_2)$ and $\Gamma = \sqrt{\tilde{p}^2 - m^2 + i\epsilon}$), where $D^c(x-y)$ is the standard propagator of free field theory and $\bar{D}(x,y)$ an additional term containing the correction due to the δ-function potential. This unusual representation is quite appropriate for all further calculations. The second part of the propagator explicitly contains the bound state $\Gamma = ib$ for $b > 0$ as pole in the physical sheet $Im\,\Gamma > 0$. For $b \to -\infty$ the propagator satisfies the Dirichlet boundary condition.

Generalizations

If we want to discuss Casimir-like configurations with two planes represented by δ-functions then we have to repeat the same construction like above for the field equation with $i = 1, 2$ and $b_1 = b_2$. Here, the field modes are much more complicated. Again, they contain bound states, symmetric, and antisymmetric scattering states. Without going into detail [2],[3] we quote the result for the propagator only

$$\bar{D}(x,y) = -\frac{b}{2}\int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{e^{i\tilde{p}(\tilde{x}-\tilde{y})}}{\Gamma}\frac{1}{(\Gamma - ib)^2 + b^2 e^{2i\Gamma d}}\,.$$

$$\cdot \left\{(\Gamma - ib)e^{i\Gamma(|x_3-d_1|+|y_3-d_1|)} + ibe^{i\Gamma(|x_3-d_1|+|y_3-d_2|+d)} + (d_1 \leftrightarrow d_2)\right\} \quad . \tag{7}$$

Note that also here the discrete eigenstates (bound states) appear as zeros of the denominator. In addition, there zeros of the denominator exist which do not lie on the real axis in the p_0-plane (so that they do not belong to the spectrum) and which could be interpreted as resonance states.

Let us now turn to the more interesting case of the Dirac equation which looks for the special potential as follows

$$[i\gamma^\mu \partial_\mu - m + \gamma^0 a^i \delta(x_3 - d_i)]\psi(x) = 0 \quad . \tag{8}$$

The substitution of the δ-function by a boundary condition for the Dirac spinor is also nontrivial. We have to take into account that the field itself cannot be continuous at the position of the δ-function. So, the following boundary condition can be derived [2],[3]:

$$\psi|_{x_3=d_i+\epsilon} = R\psi|_{x_3=d_i-\epsilon} \quad , \quad R = \exp\left(i\gamma^0\gamma^3\Theta\right) \quad , \quad \sin\Theta = \frac{a}{(1+a^2/4)} \tag{9}$$

Again, the energy eigenstates are found. In opposition to the approximated Klein-Gordon equation the charge sensitivity is preserved. As it should be, there are either bound states for the particle or for the antiparticle. For simplicity, we write down the propagator ${}^sS^c(x,y) = S^c(x-y) + \bar{S}(x,y)$ corresponding to one δ-function only where

$$\begin{aligned}\bar{S}(x,y) &= \frac{a}{4}\int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{e^{i\tilde{p}(\tilde{x}-\tilde{y}) + i\Gamma(|x_3|+|y_3|)}}{\Gamma^2}(\tilde{p}\tilde{\gamma} + m - \epsilon(x_3)\gamma^3\Gamma) \\ &\quad \frac{(\gamma^0\Gamma - (ia/2)(\tilde{p}\tilde{\gamma} - m))}{\lambda_-\Gamma - iap^0}(\tilde{p}\tilde{\gamma} + m + \epsilon(y_3)\gamma^3\Gamma) \quad .\end{aligned} \tag{10}$$

The energy eigenfunctions as well as the propagators for one or two delta functions can be found in [2],[3].

Vacuum energy

As the simplest quantity of physical interest we calculate the vacuum energy per unit area corresponding to two δ-potentials separated by the distance d. This is a slight generalization of the classical Casimir problem where the plates are now idealized by δ-functions. We illustrate the procedure for the scalar field. The vacuum energy per unit area is given by

$$E_{vac} = \int_{-\infty}^{+\infty} dx_3 < 0|T_{00}|0 >, \quad T_{00} = P(\partial_x, \partial_y)\bar{\phi}(x)\phi(y)|_{x\to y} \tag{11}$$

where $T_{\mu\nu}$ is the energy momentum tensor for the scalar field written here in a symbolic notation containing a point splitting procedure (useful for the regularization process) and the differentiation operator P. So, by formally taking the vacuum expectation value we arrive at an expression containing the Feynman propagator as an essential element.

$$E_{vac} = -i \int_{-\infty}^{+\infty} dx_3 \quad \partial_{x_o}\partial_{y_o} \left[D^c(x-y) + \bar{D}(x,y)\right] |_{x\to y} \tag{12}$$

The aim of our calculation is to determine the distance dependent part of the vacuum energy, therefore all other distance independent contributions will be omitted. Obviously, this concerns the contribution from the free field propagator D^c as well as further parameter independent contributions. After some algebra we obtain an expression which for large distances leads to

$$E_{vac} = \begin{cases} \dfrac{-b^2}{8(m-b)^2}\left(\dfrac{m}{\pi d}\right)^{3/2} e^{-2md} & , \quad m \neq 0, \quad b < m \\ -\dfrac{\pi^2}{720}\dfrac{1}{d^3} & , \quad m = 0 \quad . \end{cases} \tag{13}$$

The spinor case which corresponds to the field equation

$$[i\gamma^\mu\partial_\mu - m + \gamma^0 a(\delta(x_3-d_1) + \delta(x_3-d_2))]\psi(x) = 0 \tag{14}$$

can be treated in the same manner but the algebra is much more involved. The result is

$$E_{vac} = \begin{cases} \dfrac{1}{4}\dfrac{a^2}{\lambda_-}\left(\dfrac{m}{\pi d}\right)^{3/2} e^{-2md} & , \quad \lambda_- = 1 - \dfrac{a^2}{4} \quad , \quad m \neq 0 \\ \dfrac{1}{6\pi^2}\dfrac{a^2}{d^3} & , \qquad m = 0 \quad . \end{cases} \tag{15}$$

The conclusions following from these calculations are: for large distances (which is the physically interesting limit in any case) the contributions of massive fields to the Casimir effect (electrons contained in metallic plates etc.) are exponentially suppressed. For massless scalar fields, the well-known Casimir result is recovered. In the spinor case, opposite to the scalar theory the resulting Casimir force is repulsive. One further interesting point concerns a corresponding nonrelativistic calculation. Usually one believes that the essential impact of metallic plates is to change the low energy spectrum of the fluctuations of the electromagnetic field, therefore the Casimir effect is considered as an infrared effect. If this would be true for the case of massive fields in the presence of delta functions then a nonrelativistic calculation should be possible. An explicit nonrelativistic calculation [3] shows that this is not the case, the distance dependent part of the vacuum energy (at least for $a < 0$) vanishes. This means that the deformation of the energy spectrum caused by a nonrelativistic approximation is so serious that it leads to a wrong approximation for the Casimir energy.

Zeta Function Method

The ζ-function method is a very powerful method for calculating effective actions and vacuum energies. The mathematical background is as follows: Let K be a self-adjoint operator with a discrete spectrum $K\phi_n = \lambda_n \phi_n$ and nonzero eigenvalues λ_n corresponding to the normalized eigenfunctions $\int dx\ \bar{\phi}_n(x)\phi_m(x) = \delta_{nm}$. Then, we define the ζ-function of the operator K as

$$\zeta_K(s) = Tr[K]^{-s} = \int dxdy\ \delta(x-y) \sum_{n=1}^{\infty} \lambda_n^{-s} \bar{\phi}_n(x)\phi_n(y) \tag{16}$$

$$= \sum_{n=1}^{\infty} \lambda_n^{-s} < \infty \quad \text{for} \quad Re\ s > s_o \quad . \tag{17}$$

In physics, however, discrete eigenvalues of operators are not the rule. As an example, we study a complex scalar field under the influence of two δ-potentials. First we have to turn to Euclidean field theory. The operator is $K = -(\partial_4^2 + \triangle) + m^2 - 2b(\delta(x_3 - d_1) + \delta(x_3 - d_2))$ where with the help of the generalized boundary conditions the δ-functions determine a self-adjoint operator. If we choose $b < 0$ then this operator possesses a continuous spectrum with no discrete eigenvalue. So, we cannot expect to obtain a physically meaningful result using the ζ-function method. To have discrete eigenvalues we introduce one further boundary condition in x_3-direction namely we are considering a finite interval of lenght L with Dirichlet boundary conditions at its ends. It turns out that this is sufficient for a successful application of the ζ-function method in the present case. In [4] we obtained the following result for the ζ-function:

$$\zeta_K(s) = \frac{V_2 T_E}{2\pi^2} \frac{\Gamma(3/2)\Gamma(s-1/2)}{\Gamma(s)} \left\{ \frac{1}{\pi} \int_0^{\infty} d\kappa (\kappa^2 + m^2)^{(3/2-s)} \left[2L - \frac{b}{(b^2+\kappa^2)} \right] \right.$$
$$\left. + \frac{2b^2}{\pi} \int_m^{\infty} d\kappa (\kappa^2 - m^2)^{(3/2-s)} \frac{\left(d + \frac{1}{\kappa - b}\right) e^{-2d\kappa}}{(\kappa - b)^2 - b^2 e^{-2d\kappa}} \right\} \tag{18}$$

The vacuum energy can be extracted using the formula

$$V_2 T_E\ E_{vac} = Tr \lg K = -\frac{d}{ds} \zeta_K(s)|_{s=0}. \tag{19}$$

where the infinite quantities V_2 (volume of a two-dimensional Euclidean space) and T_E (volume of a one-dimensional Euclidean space, imaginary time) reflecting the symmetries of the problem will drop out for the vacuum energy per unit area by definition. However, the first term in the curly bracket which contains one further infinite contribution ($2L \to \infty$) is unexpected. This is an untypical contribution for ζ-function calculations and it can be omitted by hand because it is parameter independent. This first term would be the result of a naive calculation without imposing additional boundary conditions. The second term describes the dependence of the vacuum energy on the coupling constant and the third term leads to the distance dependent contribution to the vacuum energy already calculated earlier.

INTERACTING QED

Here we discuss quantum electrodynamics at the one loop level containing a δ-function as an external potential ($a_1 = a$) [5]. In perturbation theory the standard Feynman rules are valid with the one exception that we have to use the more complicated spinor propagator ${}^sS^c$. Let us calculate the mass operator for this configuration. Because ${}^sS^c$ is a summed-up propagator we expect that besides the standard divergences of free field theory the self-energy diagram also contains contributions from the triangle diagram (with one external field insertion) which exhibit infinities. According to conventional wisdom that the inclusion of electromagnetic background fields does not change the divergences of QED we would expect no further divergences. This however is not the case here. A direct calculation of the divergent part of the mass operator (using Feynman gauge and a UV cut-off Λ) yields the result that the self-energy part containing the second part $\bar{S}$ of the spinor propagator leads to the expected structure of the divergency however with an unexpected complicated coefficient function of the dimensionless coupling constant a

$$\overline{\Sigma}(x,y)|_{div.} = -i\frac{e^2}{8\pi^2}\gamma^0 f(a)\,\delta(x_3)\,\delta^{(4)}(x-y)\ln\Lambda^2 \qquad (20)$$

$$f(a) = \frac{1}{a}\left[3\left(\frac{\lambda_+}{a}\arctan\frac{a}{\lambda_-}-1\right)+\frac{a^3}{4}\left(\frac{\lambda_+}{a}\arctan\frac{a}{\lambda_-}+1\right)\right], \quad \lambda_\pm = 1\pm\frac{a^2}{4}.$$

Such a function can appear only if each insertion of the δ-function in this diagram produces an additional divergent term. Loosely speaking, the reason is that the δ-function fixes external lines (corresponding to the external field) onto the same point $x_3 = 0$. However, the theory remains renormalizable but one has to use some complicated nonlinear parameter renormalization.

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