1 Meanfield approximation: Two-point correlation function

Here we intend to compute the two-point function

$$G(r) = \langle s_0 s_r \rangle - \langle s_0 \rangle \langle s_r \rangle \tag{1}$$

for vanishing external field $h = 0$. For simplicity we shall restrict ourselves to the high temperature phase, where the magnetisation vanishes and therefore $G(r) = \langle s_0 s_r \rangle$. Here we assume that the linear extension of the lattice is large compared with the distance r and 0 is a point far away from the boundaries.

$$\langle s_0 s_r \rangle = \frac{\sum_{\{s\}} \exp(K \sum_{<xy>} s_x s_y) s_0 s_r}{\sum_{\{s\}} \exp(K \sum_{<xy>} s_x s_y)} \tag{2}$$

Now we treat the spin at the site 0 in a special way. Let us consider the denominator first:

$$\begin{aligned} \sum_{\{s\}} \exp(K \sum_{<xy>} s_x s_y) s_0 s_r &= \sum_{\{s\}} (\delta_{s_0,1} + \delta_{s_0,-1}) \exp(K \sum_{<xy>} s_x s_y) s_0 s_r \\ = \sum_{\{s\}} \delta_{s_0,1} \exp(K \sum_{<xy>} s_x s_y) s_r &- \sum_{\{s\}} \delta_{s_0,-1} \exp(K \sum_{<xy>} s_x s_y) s_r \end{aligned} \tag{3}$$

Notice that both terms are exactly equal. This is due to the fact that for $s_x \to -s_x$ for all sites x the value of $\exp(K \sum_{<xy>} s_x s_y) s_0 s_r$ does not change, since only pairs of spins appear in the expression. The same holds for the Boltzmann factor in the nominator. Hence we can write the two-point function as

$$\langle s_0 s_r \rangle = \frac{\sum_{\{s\}} \delta_{s_0,1} \exp(K \sum_{<xy>} s_x s_y) s_r}{\sum_{\{s\}} \delta_{s_0,1} \exp(K \sum_{<xy>} s_x s_y)} \tag{4}$$

In the meanfield approximation we again replace the spins by the average magnetisation $m(x)$ at the site x. Note that now the translational invariance is broken by the requirement $m(0) = 1$. We arrive at

$$m(x) = \tanh(K \sum_{y.nn.x} m(y)) \tag{5}$$

with the boundary conditions $m(0) = 1$ and $\lim_{|x| \to \infty} m(x) = 0$. $y.nn.x$ means that y is a nearest neighbour of x. We can not analytically solve the system of eqs. (5). In order to understand the behaviour of the solution for large $|x|$ we expand the $tanh$ around 0 and get

$$m(x) = K \sum_{y.nn.x} m(y) \tag{6}$$

Here we get a solution by going to momentum space. Transforming the solution back to real space one arrives at

$$G(r) \propto \frac{1}{r^{(d-1)/2}} \exp(-r/\xi(K)) \tag{7}$$

where

$$\xi(K) \simeq (K_c - K)^{-1/2} \tag{8}$$

hence $\nu_{MF} = 1/2$. For $K \to K_c$ the pre-factor in eq. (7) gives the behaviour of the two-point function. We read off

$$\eta_{MF} = 0 \tag{9}$$

The behaviour given by eq. (7) is called Ornstein-Zernicke form of the correlation function.

A more detailed discussion can be found in Cardys book sections 2.3 and 2.4.