1 Meanfield Approximation

In order to reduce the problem to one site we replace the neighbours of s_x by an average magnetisation m. As reduced one site Hamiltonian we get

$$H(s_x) = -2dKms_x - hs_x \tag{1}$$

Note that $2d$ is the number of neighbours for the square, simple cubic, ..., lattice, where d is the dimension of the lattice. The expectation value of the spin is given by

$$\begin{aligned}\langle s_x \rangle &= \frac{\sum_{s_x=\pm 1} \exp(-H(s_x)) s_x}{\sum_{s_x=\pm 1} \exp(-H(s_x))} = \frac{\exp(2dKm+h) - \exp(-2dKm-h)}{\exp(2dKm+h) + \exp(-2dKm-h)} \\ &= \tanh(2dKm+h) \end{aligned}\tag{2}$$

Since our original problem is translational invariant, it is natural to set $< s_x >= m$. Hence we have to solve the equation

$$m = \tanh(2dKm + h) \tag{3}$$

In general this can only be done numerically using e.g. the Newton method. Let us discuss the $h = 0$. For small values of m we Taylor-expand the tanh:

$$m = 2dKm - \frac{1}{3}(2dKm)^3 + ... \tag{4}$$

Immediately we see that $m = 0$ is a solution for all values of K. Deviding by m we get

$$2dK - 1 = \frac{1}{3}(2dK)^3 m^2 \tag{5}$$

Hence for $K > K_c = 1/(2d)$ we get the additional solutions

$$m = \pm \sqrt{\frac{3}{(2dK)^3}(2dK-1)} \tag{6}$$

Introducing the reduced temperature $t = \frac{K_c - K}{K_c}$ we see that

$$m = \pm c(-t)^{1/2} \times (1 + b_1 t + b_2 t^2 + ...) \tag{7}$$

and hence the critical exponent of the magnetisation

$$\beta = 1/2 \tag{8}$$

Here we have assumed that the solutions $m \neq 0$ and not $m = 0$ are the physical ones in the low temperature phase, i.e. for $K > K_c$. Below we shall motivate this choice by studying the free energy.

Next compute the magnetic susceptibility in the high temperature phase; i.e. for $K < K_c$. To this end, we expand the tanh to leading order:

$$m = 2dKm + h \tag{9}$$

Hence

$$m = \frac{1}{1 - 2dK} h \tag{10}$$

and therefore

$$\chi = \frac{\partial m}{\partial h} = (1 - 2dK)^{-1} = \left(\frac{K_c - K}{K_c}\right)^{-1} \tag{11}$$

and therefore we get for the critical exponent of the magnetic susceptibility

$$\gamma = 1 \tag{12}$$

The calculation in the low temperature phase is a bit more involved. Here we just mention that is also gives $\gamma = 1$. At the critical point $K_c = 1/(2d)$ we get

$$m = m + h - \frac{1}{3}(m + h)^3 + ... \tag{13}$$

it follows

$$m = h^{1/3} + ... \tag{14}$$

and therefore

$$\delta = 3 \tag{15}$$

Starting point of a more formal derivation of the meanfield approximation is the Bogoliubov inequality:

$$\int \mathrm{d}\xi P(\xi) \exp(A(\xi)) \geq \exp\left(\int \mathrm{d}\xi P(\xi) A(\xi)\right) \tag{16}$$

where $P(\xi)$ is a probability distribution. I.e.

$$P(\xi) \geq 1 \tag{17}$$

for all ξ and

$$\int \mathrm{d}\xi P(\xi) = 1 \tag{18}$$

The Bogoliubov inequality is due to the fact that exp is a convex function.

We write the reduced Hamiltonian as

$$H = H_0 + \Delta H \tag{19}$$

such that the problem associated with H_0 is exactly solvable. The partition function reads

$$\begin{aligned} Z &= \sum_{\{s\}} \exp(-H) = \sum_{\{s\}} \exp(-H_0)\exp(-\Delta H) \\ &= Z_0 \frac{\sum_{\{s\}} \exp(-H_0)\exp(-\Delta H)}{Z_0} = Z_0 \langle \exp(-\Delta H) \rangle_0 \end{aligned} \tag{20}$$

where

$$Z_0 = \sum_{\{s\}} \exp(-H_0) \tag{21}$$

and

$$\langle A \rangle_0 = \frac{\sum_{\{s\}} \exp(-H_0) A}{Z_0} \tag{22}$$

From the Bogoliubov inequality follows that

$$\langle \exp(-\Delta H) \rangle_0 \geq \exp\left(\langle -\Delta H \rangle_0\right) \tag{23}$$

where $P(\{s\}) = \exp(-H_0)/Z_0$. Hence

$$Z \geq Z_0 \exp\left(\langle -\Delta H \rangle_0\right) = \exp\left(-\langle \Delta H \rangle_0\right) \tag{24}$$

and since ln is a monotonically growing function

$$F \leq F_0 + \langle \Delta H \rangle_0 = \tilde{F} \tag{25}$$

The idea is that the splitting of H depends on some parameter (in our case m). The optimal value of this parameter is obtained by minimizing $\tilde{F}$ which gives the best approximation of the free energy F.

Application to the Ising model:

$$\begin{aligned} H &= -K \sum_{<xy>} ((s_x - m) + m)((s_y - m) + m) - h \sum_x s_x \\ &= dVKm^2 - Km \sum_{<xy>} (s_x + s_y) - h \sum_x s_x - K \sum_{<xy>} (s_x - m)(s_y - m) \\ &= dVKm^2 - (2dKm + h) \sum_x s_x - K \sum_{<xy>} (s_x - m)(s_y - m) \end{aligned} \tag{26}$$

where we chose

$$H_0 = dVKm^2 - (2dKm + h) \sum_x s_x \tag{27}$$

and

$$\Delta H = -K \sum_{<xy>} (s_x - m)(s_y - m) \tag{28}$$

Note that V is the number of sites of the lattice.

Now let us compute the free energy density $\tilde{f} = \frac{1}{V} \tilde{F}$:

$$f_0 = dKm^2 - \ln(\exp(2dKm + h) + \exp(-2dKm - h)) \tag{29}$$

and

$$\frac{1}{V} \langle \Delta H \rangle_0 = -dK(\langle s_x \rangle_0 - m)^2 \tag{30}$$

where

$$\langle s_x \rangle_0 = \tanh(2dKm + h) \tag{31}$$

Now let us consider the case of a vanishing external field $h = 0$.

$$\frac{\partial \tilde{f}}{\partial m} = \frac{\partial f_0}{\partial m} + \frac{1}{V} \frac{\partial \langle \Delta H \rangle_0}{\partial m} \tag{32}$$

We get

$$\frac{\partial f_0}{\partial m} = 2dKm - \frac{2dK(\exp(2dKm) - \exp(2dKm))}{\exp(2dKm) + \exp(-2dKm)} = 2dK[m - \tanh(2dKm)] \tag{33}$$

and

$$\frac{1}{V} \frac{\partial \langle \Delta H \rangle_0}{\partial m} = -2dK(\tanh(2dKm) - m)(2dK(1 - \tanh^2(2dKm)) - 1) \tag{34}$$

We immediately see that both terms vanish for

$$m = \tanh(2dKm) \tag{35}$$

which confirms our choice eq. (3). In order to simplify the analysis let us Taylor-expand $\tilde{f}$:

$$f_0 = dKm^2 - \ln(2) - \frac{1}{2}(2dKm)^2 + \frac{1}{12}(2dKm)^4 + ... \tag{36}$$

and

$$\frac{1}{V}\langle \Delta H \rangle_0 = -dK\left(2dKm - \frac{1}{3}(2dKm)^3 + ... - m\right)^2 \tag{37}$$

Taking the first and second derivative we get

$$\frac{\partial f_0}{\partial m} = 2dKm - (2dK)^2 m + \frac{1}{3}(2dK)^4 m^3 + ... \tag{38}$$

$$\frac{\partial^2 f_0}{\partial m^2} = 2dK - (2dK)^2 + (2dK)^4 m^2 + ... \tag{39}$$

and

$$\frac{1}{V}\frac{\partial \langle \Delta H \rangle_0}{\partial m} = 2dK\left(2dKm - \frac{1}{3}(2dKm)^3 + ... - m\right)\left(2dK - (2dK)^3 m^2 + ... - 1\right) \tag{40}$$

and

$$\begin{aligned} \frac{1}{V}\frac{\partial^2 \langle \Delta H \rangle_0}{\partial m^2} &= 2dK\left(2dK - 1 - (2dK)^3 m^2\right)^2 \\ &- 4dK\left(2dKm - \frac{1}{3}(2dKm)^3 + ... - m\right)(2dK)^3 m \end{aligned} \tag{41}$$

Plugging in the solution $m = 0$ into the second derivative of $\tilde{f}$ we get

$$\frac{\partial^2 \tilde{f}}{\partial m^2} = 2dK - (2dK)^2 + 2dK(2dK - 1)^2 = 2dK(1 - 2dK)(2 - 2dK) \tag{42}$$

which is positive for $K < K_c = 1/(2d)$; I.e. $m = 0$ is indeed a minimum of $\tilde{f}$. For $K > K_c = 1/(2d)$ it becomes negative which shows that it is a maximum of $\tilde{f}$ and is therefore not physically realized. The sign turns again at $k = 1/d$ which is an artifact of the Taylor-expansion. Plugging in

$$m^2 = \frac{3}{(2dK)^3}(2dK - 1) \tag{43}$$

into the second derivative of $\tilde{f}$ we see that it takes postive values; indicating a minimum of the reduced free energy.

Finally we discuss the behaviour of the specific heat. In the high temperature phase $K < K_c$ the reduced free energy $\tilde{f}$ is constant. Therefore the energy and the specific heat vanish.

In the low temperature phase, plugging eq. (43) into eqs. (36,37) we get a behaviour of the type

$$C = c_0 + c_1 t + ... \tag{44}$$

where $c_0 \neq 0$. It later it will turn out that this jump in the specific heat is an artifact of the meanfield approximation. Since there is no power law behaviour we conclude that the critical exponent of the specific heat vanishes

$$\alpha = 0 \tag{45}$$