1 The Ising model in 1 and 2 dimensions

The reduced Hamiltonian of the one-dimensional Ising model is given by

$$H = -K \sum_x s_x s_{x+1} - h \sum_x s_x \tag{1}$$

where $x \in \{1, 2, ..., L\}$. We take periodic boundary conditions, which means that 1 and L are nearest neighbours and hence $L + 1$ is identified with 1. The partition function can be written in the form

$$Z = \sum_{\{s\}} \exp(-H) = \sum_{\{s\}} \prod_x T_{s_x, s_{x+1}} \tag{2}$$

where

$$T_{s_x, s_{x+1}} = \exp\left(K s_x s_{x+1} + \frac{h}{2}(s_x + s_{x+1})\right) \tag{3}$$

We can interprete T as a 2×2 matrix where s_x and s_{x+1} are the indices of the matrix. In order to make this more clear we might introduce indices $i, j \in \{1, 2\}$, which are functions of s_x and s_{x+1}

$$T_{i(s_x), j(s_{x+1})} \tag{4}$$

where $i(-1) = 1$, $i(1) = 2$, $j(-1) = 1$ and $j(1) = 2$. In matrix form we get

$$T = \begin{pmatrix} \exp(K-h) & \exp(-K) \\ \exp(-K) & \exp(K+h) \end{pmatrix} \tag{5}$$

Now we rewrite eq. (2); we first perform the sum over s_1; the prime in the first sum indicates that s_1 is taken out.

$$Z = \sum_{\{s\}'} \sum_{s_1=\pm 1} \prod_x T_{s_x, s_{x+1}} = \sum_{\{s\}'} \prod_{x \neq L,1} T_{s_x, s_{x+1}} \sum_{s_1=\pm 1} T_{s_L, s_1} T_{s_1, s_2} = \sum_{\{s\}'} \prod_{x \neq L,1} T_{s_x, s_{x+1}} (T^2)_{s_L, s_2} \tag{6}$$

In the next step we take perform the sum over s_2:

$$Z = \sum_{\{s\}''} \prod_{x \neq L,1,2} T_{s_x, s_{x+1}} \sum_{s_2=\pm 1} (T^2)_{s_L, s_2} T_{s_2, s_3} = \sum_{\{s\}''} \prod_{x \neq L,1,2} T_{s_x, s_{x+1}} (T^3)_{s_L, s_3} \tag{7}$$

This procedure can be further iterated by summing over s_3, s_4, ... up to s_{L-1}. This way we arrive at

$$Z = \sum_{s_L=\pm 1} (T^L)_{s_L, s_L} = \mathrm{Tr} T^L = \lambda_1^L + \lambda_2^L \tag{8}$$

where λ_1, λ_2 are the eigenvalues of T. These can be obtained by solving the characteristic equation (or secular equation)

$$\det(T - \lambda_{1,2} I) = 0 \tag{9}$$

which amounts to solve a quadratic equation in the case of a 2×2 matrix. One gets

$$\lambda_{+,-} = \exp(K)\left[\cosh(h) \pm \sqrt{\sinh^2(h) + \exp(-4K)}\right] \tag{10}$$

The reduced free energy density is given by

$$\begin{aligned} f = -\frac{1}{L}\ln Z &= -\frac{1}{L}\ln(\lambda_+^L + \lambda_-^L) = -\frac{1}{L}\ln(\lambda_+^L(1 + (\lambda_-/\lambda_+)^L)) \\ &= -\ln(\lambda_+) - \frac{1}{L}\ln(1 + (\lambda_-/\lambda_+)^L) \end{aligned} \tag{11}$$

In the thermodynamic limit $L \to \infty$ we get

$$f = -\ln(\lambda_+) \tag{12}$$

The magnetisation is given by

$$m(h) = -\frac{\partial f}{\partial h} = \frac{1}{\lambda_+}\frac{\partial \lambda_+}{\partial h} = \frac{1}{\lambda_+}\exp(K)\left[\sinh(h) + \frac{\sinh(h)\cosh(h)}{\sqrt{\sinh^2(h) + \exp(-4K)}}\right] \tag{13}$$

Since $\sinh(h)$ is an odd and $\cosh(h)$ is an even function, $m(h)$ is an odd function. Furthermore $m(h)$ is a continuous function. Therefore $\lim_{h\to 0} m(h) = 0$. This means that for any finite K, there is no spontaneous magnetisation.

Observables in the transfer matrix formalism:

$$\langle s_L \rangle = \frac{1}{Z}\sum_{\{s\}} \exp(K\sum_x s_x s_{x+1} + h\sum_x s_x) s_L = \frac{\sum_{s_L=\pm 1}(T^L)_{s_L,s_L} s_L}{\mathrm{Tr} T^L} = \frac{\mathrm{Tr} T^L S}{\mathrm{Tr} T^L} \tag{14}$$

where the matrix S contains the values of s_L on its diagonal:

$$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

Expressed in terms of the eigenvalues and eigenvectors of the transfer matrix we arrive at

$$\langle s_L \rangle = \frac{\sum_{i=\pm} \lambda_i^L \langle i|S|i\rangle}{\sum_{i=\pm} \lambda_i^L} \tag{16}$$

in the thermodynamic limit $L \to \infty$

$$\langle s_L \rangle = \langle +|S|+\rangle \tag{17}$$

remains. In a similar fashion we can compute the two-point correlation function:

$$G(r) = \langle s_y s_{y+r} \rangle = \frac{\mathrm{Tr} T^y S T^r S T^{L-(y+r)}}{\mathrm{Tr} T^L} = \frac{\mathrm{Tr} S T^r S T^{L-r}}{\mathrm{Tr} T^L} \tag{18}$$

Again expressing things in terms of eigenvalues and eigenvectors of the transfer matrix we arrive at

$$G(r) = \frac{\sum_i \sum_j \langle i|S|j\rangle \langle j|S|i\rangle \lambda_i^{L-r} \lambda_j^r}{\lambda_+^L + \lambda_-^L} \tag{19}$$

In the limit $L \to \infty$ we get

$$G(r) = \langle +|S|+\rangle^2 + \langle +|S|-\rangle^2 \left(\frac{\lambda_-}{\lambda_+}\right)^r \tag{20}$$

For $h = 0$ the magnetisation vanishes and therefore $\langle +|S|+\rangle^2 = 0$. Hence for $h = 0$ we get

$$G(r) = \langle +|S|-\rangle^2 \left(\frac{\lambda_-}{\lambda_+}\right)^r = c \exp(-r/\xi) \tag{21}$$

where

$$\xi = -1/\ln(\lambda_-/\lambda_+) \tag{22}$$

For $h \neq 0$ we have to consider

$$G(r) = \langle s_y s_{y+r} \rangle - \langle s_y \rangle \langle s_{y+r} \rangle \tag{23}$$

to get the exponential decay.

1.1 The Ising model on the square lattice

The Ising model on the square lattice was solved by Lars Onsager (1944) for vanishing external field $h = 0$. This solution is based on the diagonalization of the transfer matrix. Let us consider a lattice of the size $L_1 \times L_2$. The sites of the lattice are denoted by $x = (x_1, x_2)$ where $x_i \in \{1, 2, ..., L\}$. A matrix element of the transfer matrix is then given by

$$
\begin{aligned}
& T_{(s_{(x_1,1)}, s_{(x_1,2)}, \ldots, s_{(x_1,L_2)}),(s_{(x_1+1,1)}, s_{(x_1+1,2)}, \ldots, s_{(x_1+1,L_2)})} = \\
& \exp\Bigg(K \sum_{x_2=1}^{L_2} s_{(x_1,x_2)} s_{(x_1+1,x_2)} + \\
& \frac{1}{2}[K \sum_{x_2=1}^{L_2} (s_{(x_1,x_2)} s_{(x_1,x_2+1)} + s_{(x_1+1,x_2)} s_{(x_1+1,x_2+1)}) + h \sum_{x_2=1}^{L_2} (s_{(x_1,x_2)} + s_{(x_1+1,x_2)})] \Bigg)
\end{aligned}
\tag{24}
$$

Note that an index of the transfer matrix is made up by the configuration of the spins in one column. The number of these configurations is 2^{L_2}. Hence the transfer matrix is a $2^{L_2} \times 2^{L_2}$. For $h \neq 0$ one can compute the eigenvalues and eigenvectors numerically. The few largest eigenvalues and the corresponding eigenvectors can be computed for up to $L_2 \approx 30$.

Here we shall only report briefly the results of the exact solution. There is a continuous transition at

$$K_c = \frac{1}{2} \ln\left(1 + \sqrt{2}\right) \quad . \tag{25}$$

(See e.g. Bruria Kaufman, Phys. Rev. 76 (1949) 1232.)

The magnetisation for $K > K_c$ behaves as

$$m = \left(1 - \sinh^{-4}(2K)\right)^{1/8} \tag{26}$$

(C.N. Yang, Phys. Rev. 85 (1952) 808) Note that

$$\sinh^{-4}(2K_c) = \sinh^{-4}\left(\ln\left(1 + \sqrt{2}\right)\right) = \left(\frac{1 + \sqrt{2} - \frac{1}{1+\sqrt{2}}}{2}\right)^{-4} = 1 \tag{27}$$

We can Taylor expand

$$\sinh^{-4}(2K) = 1 + a_1(K - K_c) + a_2(K - K_c)^2 + ... \tag{28}$$

where $a_1 < 0$ we can write the magnetisation as

$$\begin{aligned} m &= \Big(a_1(K-K_c) + a_2(K-K_c)^2 + ...\Big)^{1/8} \\ &= b(K-K_c)^{1/8} \times (1 + c_1(K-K_c) + c_2(K-K_c)^2 + ... \end{aligned} \tag{29}$$

Hence the critical exponent of the magnetisation is

$$\beta = 1/8 \tag{30}$$

which is quite different from the meanfield (MF) result $\beta_{MF} = 1/2$.

From the result for the correlation length ξ one gets

$$\nu = 1 \tag{31}$$

The specific heat has a logarithmic divergence (which is weaker than any power law divergence). Therefore

$$\alpha = 0 \tag{32}$$