1 The Gaussian integral

In the case of a single real variable we get

$$z(a) = \int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{a}{2}\phi^2\right) \tag{1}$$

where a and ϕ are real numbers. Using $\tilde{\phi} = a^{1/2}\phi$ and $\mathrm{d}\tilde{\phi} = a^{1/2}\mathrm{d}\phi$ we arrive at

$$z(a) = a^{-1/2} z(1) \tag{2}$$

where

$$z(1) = \sqrt{2\pi} \tag{3}$$

Prove:

$$z(1)^2 = \int_{\phi=-\infty}^{\infty} \mathrm{d}\phi \int_{\psi=-\infty}^{\infty} \mathrm{d}\psi \exp\left(-\frac{1}{2}[\phi^2 + \psi^2]\right) \tag{4}$$

Now we write:

$$\phi = r\cos(\alpha) \qquad \psi = r\sin(\alpha) \tag{5}$$

with

$$\mathrm{d}\phi\mathrm{d}\psi = \mathrm{d}r\; r\mathrm{d}\alpha \tag{6}$$

It follows

$$z(1)^2 = \int_{r=0}^{\infty} \mathrm{d}r\; r \int_0^{2\pi} \exp\left(-\frac{1}{2}r^2\right) = 2\pi \int_0^{\infty} \mathrm{d}r\; r \exp\left(-\frac{1}{2}r^2\right) \tag{7}$$

$$= 2\pi \left[-\exp\left(-\frac{1}{2}r^2\right)\right]_0^{\infty} = 2\pi \tag{8}$$

I.e. $z(1) = \sqrt{2\pi}$.

Next let us study

$$\langle\phi^2\rangle_a := \frac{1}{z(a)} \int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{a}{2}\phi^2\right)\phi^2 \tag{9}$$

To this end we introduce the generating function

$$z(a,j) = \int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{a}{2}\phi^2 + j\phi\right) \tag{10}$$

Taking the second derivative with respect to j we get

$$\left.\frac{\partial^2 z(a,j)}{\partial j^2}\right|_{j=0} = \int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{a}{2}\phi^2\right)\phi^2 = z(a,0)\langle\phi^2\rangle_a \tag{11}$$

on the other hand, by completing the square we get

$$\begin{aligned} z(a,j) &= \int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\left[\sqrt{\frac{a}{2}}\phi - j\frac{1}{\sqrt{2a}}\right]^2\right)\exp\left(\frac{j^2}{2a}\right) \\ &= \int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{a}{2}\left[\phi - j\frac{1}{2a}\right]^2\right)\exp\left(\frac{j^2}{2a}\right) \\ &= z(a,0)\exp\left(\frac{j^2}{2a}\right) \end{aligned} \tag{12}$$

where we have substituted $\tilde{\phi} = \phi - j\frac{1}{2a}$. Taking again the second derivative with respect to j we arrive at

$$\left.\frac{\partial^2 z(a,j)}{\partial j^2}\right|_{j=0} = z(a,0)a^{-1} \tag{13}$$

Hence

$$\langle\phi^2\rangle_a = a^{-1} \tag{14}$$

In a statistics context a^{-1} is dentoted by σ^2 and called *variance*.

Next we generalize the problem to many real variables ϕ_i, where $i = 1, 2, ..., N$. Let us consider the multi-dimensional integral

$$Z(A) = \int \mathrm{D}[\phi] \exp\left[-\frac{1}{2}(\phi, A\phi)\right] \tag{15}$$

where we have introduced the notation

$$\int \mathrm{D}[\phi] := \int_{-\infty}^{\infty} \mathrm{d}\phi_1 ... \int_{-\infty}^{\infty} \mathrm{d}\phi_N \tag{16}$$

and

$$(\phi, A\psi) = \sum_{k,l} A_{kl}\phi_k\psi_l \tag{17}$$

where A is a symmetric positive definite matrix. The problem can be solved by a similarity transformation that diagonalises the matrix A:

$$(\phi, A\psi) = (S^{-1}\phi, S^{-1}ASS^{-1}\psi) \tag{18}$$

where S is an orthogonal matrix

$$S^{-1} = S^T\ . \tag{19}$$

Therefore the determinant of the Jacobi matrix is equal to one. One gets

$$\begin{aligned} Z(A) &= \int \mathrm{D}[\tilde{\phi}] \prod_{i=1}^{N} \exp\left(-\frac{1}{2}\lambda_i \tilde{\phi}^2\right) &(20)\\ &= (2\pi)^{N/2} \left(\prod_{i=1}^{N} \lambda_i\right)^{-1/2} = (2\pi)^{N/2} \det(A)^{-1/2} &(21)\end{aligned}$$

where λ_i are the eigenvalues of A and $\tilde{\phi} = S^{-1}\phi$. Again let us introduce sources

$$Z(A, j) = \int \mathrm{D}[\phi] \exp\left(-\frac{1}{2}(\phi, A\phi) + (j, \phi)\right) \tag{22}$$

where now j is a vector with N real components. Taking partial derivatives with respect to the components of j we get

$$\langle \phi_k \phi_l \rangle_A = \left.\frac{\partial^2 Z(A, j)}{\partial j_k \partial j_l}\right|_{j=0} \tag{23}$$

where

$$\langle \phi_k \phi_l \rangle_A = \frac{1}{Z(A)} \int \mathrm{D}[\phi] \exp\left(-\frac{1}{2}(\phi, A\phi)\right) \phi_k \phi_l \tag{24}$$

On the other hand, by completing the square

$$Z(A, j) = Z(A, 0) \exp\left(-\frac{1}{2}(j, A^{-1}j)\right) \tag{25}$$

It follows

$$\langle \phi_k \phi_l \rangle_A = (A^{-1})_{kl} \tag{26}$$

In the same fashion one can compute $2n$-point functions:

$$\langle \phi_{l_1} \phi_{l_2} ... \phi_{l_{2n}} \rangle := \frac{1}{Z(A)} \int \mathrm{D}[\phi] \exp\left(-\frac{1}{2}(\phi, A\phi)\right) \phi_{l_1} \phi_{l_2} ... \phi_{l_{2n}} \tag{27}$$

It turns out that these $2n$-point functions can be written in terms of 2-point functions, using the so called Wick rule. In the case of 4 points it looks like

$$\langle \phi_{l_1}\phi_{l_2}\phi_{l_3}\phi_{l_4}\rangle = \langle \phi_{l_1}\phi_{l_2}\rangle\langle \phi_{l_3}\phi_{l_4}\rangle + \langle \phi_{l_1}\phi_{l_3}\rangle\langle \phi_{l_2}\phi_{l_4}\rangle + \langle \phi_{l_1}\phi_{l_4}\rangle\langle \phi_{l_2}\phi_{l_3}\rangle \tag{28}$$

I.e. it is summed over all possible decompositions into 2-point functions. If some of the points are identical, do as if the were different and identify them in the result e.g.

$$\langle \phi_l^4\rangle = 3\langle \phi_l^2\rangle^2 \tag{29}$$