Simple sampling
Configurations are generated with a uniform probability (density)

\[ \langle A \rangle \approx \sum_i B(X_i)A(X_i) \sum_i B(X_i) \]

where \( X_i \) are configurations that we have generated. \( A \) is the observable and \( B \) the weight factor; in our context mostly the Boltzmann factor.
Problem: Typically \( B \) only has a significant amplitude for a very small fraction of the configurations.

Importance sampling (achieved by using a Markov chain)
Configurations \( X_i \) are generated with a probability (density) proportional to \( B(X_i) \). We get

\[ \langle A \rangle \approx \frac{1}{N} \sum_{i=1}^{N} A(X_i) \]

(Here we assume that sufficient configurations at the beginning of the chain are discarded)
Generalization: “Reweighting”

We generate configurations with a probability (density) proportional to $B_0(X_i)$. Now we like to get expectation values of observables $A$ with respect to weight factors $B$ which are in some sense close to $B_0$

$$\langle A \rangle_B \approx \frac{\sum_i [B(X_i)/B_0(X_i)] A(X_i)}{\sum_i [B(X_i)/B_0(X_i)]}$$

Actually known for a long time; Most researchers just did not aspect that it could have useful applications. Became popular with

Alan M. Ferrenberg and Robert H. Swendsen

New Monte Carlo technique for studying phase transitions

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Simple example: We generate configurations of the $\phi^4$ model with a Boltzmann factor corresponding to $\kappa = \kappa_0$ and $\lambda = \lambda_0$.

Now we like to compute expectation values of observables at fixed $\lambda$ for various values of $\kappa$ in the neighbourhood of $\kappa_0$.

$$\frac{B([\phi])}{B_0([\phi])} = \exp(-S(\kappa, \lambda_0, [\phi]) + S(\kappa_0, \lambda_0, [\phi]) =$$

$$\exp \left( 2[\kappa - \kappa_0] \sum_x \sum_\mu \phi_x \phi_{x+\mu} \right)$$