Demons

Demons are auxiliary degrees of freedom that are added to the action. They enable new types of updates of the combined demon-spin system. The combined action is given by

$$S_{comb}(\{\phi\},\{d\}) = S(\{\phi\}) \;\; + S(\{d\})$$

These auxiliary degrees do not change the expectation value of observables:

$$\begin{aligned}
\langle A(\{\phi\})\rangle_{comb} &= \frac{\int \mathrm{D}[d] \int \mathrm{D}[\phi] \exp(-S(\{\phi\}) - S(\{d\}))\, A(\{\phi\})}{\int \mathrm{D}[d] \int \mathrm{D}[\phi] \exp(-S(\{\phi\}) - S(\{d\}))} \\
&= \frac{\int \mathrm{D}[d] \exp(-S(\{d\})) \int \mathrm{D}[\phi] \exp(-S(\{\phi\}))\, A(\{\phi\})}{\int \mathrm{D}[d] \exp(-S(\{d\})) \int \mathrm{D}[\phi] \exp(-S(\{\phi\}))} \\
&= \langle A(\{\phi\})\rangle_{\phi^4}
\end{aligned}$$

Here we shall use

$$S(\{d\}) = \sum_\alpha d_\alpha$$

where $d_\alpha \geq 0$ is a real number.

Probability distribution of the demons

One can use any number of demon degrees of freedom. In the simplest case, there is just one demon degree of freedom. However, often it is useful to have a demon for every site of the lattice. The distribution is given by

$$p(d) = \exp(-d)$$

It can be easily generated by

$$d = -\ln(r)$$

where r is uniformly distributed in $(0, 1]$
Expectation value of the demon:

$$\int_0^\infty \mathrm{d}d \exp(-d) d = 1$$

Elementary local update using the demon

-Proposal as for the Metropolis algorithm:

$$\phi'_x = \phi_x + \eta$$

where η has an even probability distribution.

- Accept this proposal if $\Delta S = S(\phi') - S(\phi) \leq d$
In case the proposal is accepted, also update the demon, following $d^{(i+1)} = d^{(i)} - \Delta S$.

Note that S_{comb} keeps its value.

Further note that if d is updated by $d = -\ln(r)$ before the local update is performed, this procedure is equivalent to the Metropolis algorithm discussed before.

However, here we have the freedom to reuse the updated d.

Generating a proposal without using a random number

Consider

$$S_{Gaussian} = \sum_x \left(-2\kappa \sum_\mu \phi_x \phi_{x+\hat{\mu}} + \phi_x^2 \right)$$

We keep fixed all ϕ_x except for u. The part of $S_{Gaussian}$ that depends on ϕ_u:

$$S_u = -2\kappa\phi_u \sum_\mu (\phi_{u+\hat{\mu}} + \phi_{u-\hat{\mu}}) + \phi_u^2$$

For a fixed value of S_u there are (if any) in general two values of ϕ_u. Going from one to the other:

$$\phi'_u = 2\kappa \sum_\mu (\phi_{u+\hat{\mu}} + \phi_{u-\hat{\mu}}) - \phi_u$$