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Algebraic reduction of one-loop Feynman graph amplitudes

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Abstract

An algorithm for the reduction of one-loop n-point tensor integrals to basic integrals is proposed. We transform tensor integrals to scalar integrals with shifted dimension [1, 2] and reduce these by recurrence relations to integrals in generic dimension [3]. Also the integration-by-parts method [4] is used to reduce indices (powers of scalar propagators) of the scalar diagrams. The obtained recurrence relations for one-loop integrals are explicitly evaluated for 5- and 6-point functions. In the latter case the corresponding Gram determinant vanishes identically for $d = 4$, which greatly simplifies the application of the recurrence relations.

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1 Introduction

Radiative corrections contain not only most essential information about quantum properties of a quantum field theory but moreover, their knowledge is indispensable for the interpretation of precision experiments. Higher order calculations in general have been performed for processes with at most four external legs. With increasing energy the observation of multi particle events is becoming more and more important and at least one-loop diagrams must be calculated for these cases as well. One of the important examples is the W–pair production reaction $e^+e^- \to W^+W^- \to 4$ fermions (γ), being experimentally investigated at LEP2, which allows us to accurately determine mass, width and couplings of the W-boson. At future high–energy and high–luminosity e^+e^- linear colliders radiative corrections will become even more important for a detailed understanding of Standard Model processes. Thus their computation will be crucial for a precise investigation of particle properties and the expected discovery of new physics. At present, a full $O(\alpha)$ calculation for a four–fermion production process is not available, mainly because, such a calculation is complicated for various reasons [5, 6]. One of the problems is addressed in the present article and concerns the calculation of one-loop diagrams with special emphasis on processes with five and six external legs.

The evaluation of one-loop Feynman graph amplitudes has a long history [7] - [19] and at the present time many different methods and approaches do exist. However, the most frequently used Passarino-Veltman [11] approach is rather difficult to use in calculating diagrams with five, six or more external legs. The tensor structure of such diagrams is rather complicated. To obtain the coefficients in the tensor decomposition of multi-leg integrals one needs to solve algebraic equations of high order which often are not tractable even with the help of modern computer algebra systems. Additional complications occur in the case when some kinematic Gram determinants are zero.

In the present paper we propose an approach which simplifies the evaluation of one-loop diagrams in an essential way. It allows one to evaluate multi-leg integrals efficiently without solving systems of algebraic equations. Integrals with five and six external legs are worked out explicitly. Those with seven and more external legs need special investigation because there are different types of Gram determinants vanishing and their consideration is postponed for this reason. The above cases with more than four external legs allow particular simplifications: for n=5 due to the property (51) and the property of the recurrence relation (57) and for $n > 5$ due to properties of Gram determinants. Our scheme allows also to evaluate diagrams with $n \leq 4$, but here we cannot use the above mentioned properties. A simplification is achieved in this case since solving of linear equations is avoided and the algorithm is implemented in a FORM [20] program for these cases.

2 Recurrence relations for n-point one-loop integrals

First we consider scalar one-loop integrals depending on $n-1$ independent external momenta:

$$I_n^{(d)} = \int \frac{d^d q}{\pi^{d/2}} \prod_{j=1}^{n} \frac{1}{c_j^{\nu_j}}, \tag{1}$$

where

$$c_j = (q - p_j)^2 - m_j^2 + i\epsilon \quad \text{for} \quad j < n \quad \text{and} \quad c_n = q^2 - m_n^2 + i\epsilon. \tag{2}$$

The corresponding diagram and the convention for the momenta are given in Fig.1.

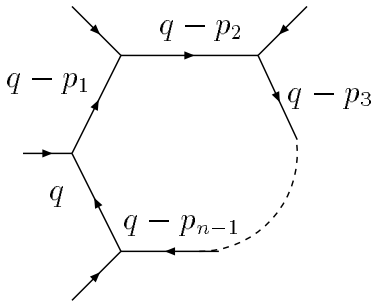

Fig. 1: One-loop diagram with n external legs.

Tensor integrals

$$I^{(d)}_{n,r} = \int \frac{d^d q}{\pi^{d/2}} \prod_{j=1}^{n} \frac{q_{\mu_1} \dots q_{\mu_r}}{c_j^{\nu_j}}, \tag{3}$$

can be written as a combination of scalar integrals with shifted space-time dimension multiplied by tensor structures in terms of external momenta and the metric tensor. This was shown in Ref.[1] for the one-loop case and in Ref.[3] for an arbitrary case. We shall use the relation proposed in [3]

$$I^{(d)}_{n,r} = T_{\mu_1 \dots \mu_r}(\{p_s\}, \{\partial_j\}, \mathbf{d}^+) I^{(d)}_n, \tag{4}$$

where T is a tensor operator, $\partial_j = \frac{\partial}{\partial m_j^2}$, and $\mathbf{d}^+$ is the operator shifting the value of the space-time dimension of the integral by two units: $\mathbf{d}^+ I^{(d)} = I^{(d+2)}$. On the right-hand side of (4) it is assumed that the invariants c_i have arbitrary masses and only after differentiation with respect to m_i^2 these are set to their concrete values.

To derive an explicit expression for the tensor operator $T_{\mu_1 \dots \mu_r}(\{p_s\}, \{\partial_j\}, \mathbf{d}^+)$ we introduce an auxiliary vector a and write the tensor structure of the integrand as

$$q_{\mu_1} \dots q_{\mu_r} = \frac{1}{i^r} \frac{\partial}{\partial a_{\mu_1}} \dots \frac{\partial}{\partial a_{\mu_r}} \exp\left[iaq\right] \Big|_{a_i = 0}. \tag{5}$$

Next we transform the integral

$$I^{(d)}_n(a) = \int \frac{d^d q}{\pi^{d/2}} \prod_{j=1}^{n} \frac{1}{c_j^{\nu_j}} \exp\left[i(aq)\right]; \quad I^{(d)}_n \equiv I^{(d)}_n(0). \tag{6}$$

into the α-parametric representation by means of

$$\frac{1}{(k^2 - m^2 + i\epsilon)^\nu} = \frac{i^{-\nu}}{\Gamma(\nu)} \int_0^\infty d\alpha \; \alpha^{\nu - 1} \exp\left[i\alpha(k^2 - m^2 + i\epsilon)\right], \tag{7}$$

and perform the d-dimensional Gaussian integration

$$\int d^dk \exp\left[i(Ak^2 + 2(pk))\right] = i\left(\frac{\pi}{iA}\right)^{\frac{d}{2}} \exp\left[-\frac{ip^2}{A}\right]. \tag{8}$$

The final result is:

$$I_n^{(d)}(a) = i\left(\frac{1}{i}\right)^{d/2} \prod_{j=1}^{n} \frac{i^{-\nu_j}}{\Gamma(\nu_j)} \int_0^\infty \cdots \int_0^\infty \frac{d\alpha_j \alpha_j^{\nu_j - 1}}{[D(\alpha)]^{\frac{d}{2}}} \exp\left[i\left(\frac{Q(\alpha, \{p_s\}, a)}{D(\alpha)} - \sum_{l=1}^{n} \alpha_l (m_l^2 - i\epsilon)\right)\right], \tag{9}$$

where

$$D(\alpha) = \sum_{j=1}^{n} \alpha_j, \tag{10}$$

$$Q(\alpha, \{p_s\}, a) = Q(\alpha, \{p_s\}) + \sum_{k=1}^{n-1} (ap_k)\alpha_k - \frac{1}{4}a^2 \tag{11}$$

and $Q(\alpha, \{p_s\})$ is the usual a-independent Q-form of the graph. From the representation (9) together with (5) it is straightforward to work out that

$$T_{\mu_1 \ldots \mu_r}(\{p_s\}, \{\partial_j\}, \mathbf{d}^+) = \frac{1}{i^r} \prod_{j=1}^{r} \frac{\partial}{\partial a_{\mu_j}} \exp\left[i\left(\sum_{k=1}^{n-1} (ap_k)\alpha_k - \frac{1}{4}a^2\right)\rho\right] \Bigg|_{\substack{a_j = 0 \\ \alpha_j = i\partial_j \\ \rho = i\mathbf{d}^+}}. \tag{12}$$

This representation is particularly well suited and effective for a computer implementation of the tensor integrals (4).

2.1 Integrals with non zero Gram determinants

The purpose of this Sect. is to develop an algorithm for reducing the above mentioned scalar integrals to standard integrals in generic dimension $d = 4 - 2\varepsilon$.

Recurrence relations which reduce the index of the j-th line without changing the space-time dimension are obtained by the integration-by-parts method [4]:

$$2\Delta_n \nu_j \mathbf{j}^+ I_n^{(d)} = \sum_{k=1}^{n} (1 + \delta_{jk}) \left(\frac{\partial \Delta_n}{\partial Y_{jk}}\right) \left[d - \sum_{i=1}^{n} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1)\right] I_n^{(d)}, \tag{13}$$

where δ_{ij} is the Kronecker delta symbol, the operators $\mathbf{j}^\pm$ etc. shift the indices $\nu_j \to \nu_j \pm 1$ and

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}.$$

Taking derivatives of Δ_n one should consider all Y_{ij} as independent variables and for $j > k$ assume $\partial/\partial Y_{jk} = \partial/\partial Y_{kj}$. After taking derivatives one should set

$$Y_{ij} = -(p_i - p_j)^2 + m_i^2 + m_j^2, \tag{14}$$

where p_i, p_j are external momenta flowing through $i-, j$-th lines, respectively, and m_j is the mass of the j-th line ($p_n = 0$). At this stage the external momenta are not yet restricted to dimension 4. We will specify later when this property is used.

A recurrence relation for reducing dimension and index of the j-th line is obtained from [3]:

$$G_{n-1}\nu_j \mathbf{j}^+ I_n^{(d+2)} = \left[(\partial_j \Delta_n) + \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n)\mathbf{k}^-\right] I_n^{(d)}, \tag{15}$$

where $\partial_j \equiv \partial/\partial m_j^2$ and

$$G_{n-1} = -2^n \begin{vmatrix} p_1p_1 & p_1p_2 & \dots & p_1p_{n-1} \\ p_1p_2 & p_2p_2 & \dots & p_2p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1p_{n-1} & p_2p_{n-1} & \dots & p_{n-1}p_{n-1} \end{vmatrix}. \tag{16}$$

We may also reduce the space-time dimension of $I_n^{(d)}$ by means of:

$$(d - \sum_{i=1}^{n} \nu_i + 1) G_{n-1} I_n^{(d+2)} = \left[2\Delta_n + \sum_{k=1}^{n} (\partial_k \Delta_n)\mathbf{k}^-\right] I_n^{(d)}. \tag{17}$$

Equations (13), (15) and (17) are our starting point. Some simplifications of these equations can still be obtained. In particular we give in the following a compact representation for the mass-derivatives of Δ_n. First of all we mention the following useful relation between G_{n-1} and Δ_n:

$$\sum_{j=1}^{n} \partial_j \Delta_n = -G_{n-1}. \tag{18}$$

The basic object for the purpose of expressing the derivatives in (13), (15) and (17) turns out to be the "modified Cayley determinant" of the diagram with internal lines $1 \dots n$ [10], namely

$$()_n \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}, \tag{19}$$

labeling elements $0 \dots n$. "Signed minors"

$$\begin{pmatrix} j_1 j_2 \dots \\ k_1 k_2 \dots \end{pmatrix}_n \tag{20}$$

will be labeled by the rows $j_1, j_2, \ldots$ and columns $k_1, k_2, \ldots$ excluded from $()_n$ [10, 22]. E.g. we have

$$\Delta_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \tag{21}$$

Further relations are

$$\partial \Delta_n / \partial Y_{jk} = (2 - \delta_{jk}) \begin{pmatrix} 0j \\ 0k \end{pmatrix}_n \tag{22}$$

$$\partial_j \Delta_n = -2 \begin{pmatrix} j \\ 0 \end{pmatrix}_n \tag{23}$$

$$\partial_j \partial_k \Delta_n = 2 \begin{pmatrix} j \\ k \end{pmatrix}_n \tag{24}$$

and for $p_n = 0$ we have

$$G_{n-1} = 2 ()_n. \tag{25}$$

One obvious advantage of the above relations is that they can be easily used for numerical evaluation. Recursion (13) then reads

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \nu_j \mathbf{j}^+ I_n^{(d)} = \sum_{k=1}^{n} \begin{pmatrix} 0j \\ 0k \end{pmatrix}_n \left[d - \sum_{i=1}^{n} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_n^{(d)}. \tag{26}$$

Using [10]

$$\sum_{k=1}^{n} \begin{pmatrix} 0j \\ 0k \end{pmatrix}_n = - \begin{pmatrix} 0 \\ j \end{pmatrix}_n , \tag{27}$$

we write it in the most convenient form for further evaluation as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \nu_j \mathbf{j}^+ I_n^{(d)} = \left\{ \left(1 + \sum_{i=1}^{n} \nu_i - d \right) \begin{pmatrix} 0 \\ j \end{pmatrix}_n - \sum_{k=1}^{n} \begin{pmatrix} 0j \\ 0k \end{pmatrix}_n (\nu_k - 1) \right\} I_n^{(d)} - \sum_{\substack{i,k \\ i \neq k}}^{n} \begin{pmatrix} 0j \\ 0k \end{pmatrix}_n \nu_i \mathbf{k}^- \mathbf{i}^+ I_n^{(d)} . \tag{28}$$

Here the 'deviations' $(\nu_k - 1)$ of the indices from 1 are explicitly separated so that all indices $\nu_k = 1$ do not contribute in the second sum in curly brackets on the r.h.s. Finally the double sum can be completely reduced to a single sum by means of [1, 3]

$$\sum_{j=1}^{n} \nu_j \mathbf{j}^+ I_n^{(d+2)} = -I_n^{(d)} . \tag{29}$$

This relation reduces simultaneously indices and dimension, which is what one wants in general. It is not possible, however, to introduce it directly into (28) since in (28) we explicitly have to separate the term $i = k$. Therefore, further details of how to apply recurrence relation (13) can be given only in the case of explicit examples (see Sect. 3). In our notation recurrence relation (15) now reads

$$()_n \, \nu_j \mathbf{j}^+ I_n^{(d+2)} = \left[- \begin{pmatrix} j \\ 0 \end{pmatrix}_n + \sum_{k=1}^{n} \begin{pmatrix} j \\ k \end{pmatrix}_n \mathbf{k}^- \right] I_n^{(d)}, \tag{30}$$

and recurrence relation (17)

$$(d - \sum_{i=1}^{n} \nu_i + 1) \, ()_n \, I_n^{(d+2)} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n - \sum_{k=1}^{n} \begin{pmatrix} 0 \\ k \end{pmatrix}_n \mathbf{k}^- \right] I_n^{(d)}. \tag{31}$$

Relations (28) (including (29)), (30) and (31) now replace (13),(15) and (17). For $\nu_i = 1$ (30) and (31) correspond to eqs. (20) and (18) of the first paper of [15]. In Sect. 3 we demonstrate how to reduce tensor integrals to scalar ones, to which these recurrence relations are then applied.

Furthermore the following general observation is needed in what follows. Applying the recursion relations, contractions of the n-th line occur. In this case we encounter integrals with $p_n \neq 0$. In order to apply our recursion relations for this case, we must properly define G_{n-1}. We can either shift all momenta by p_n or we can use from the very beginning the definition of G_{n-1} in terms of (19) with $p_n \neq 0$. In this manner the recurrence relations remain unchanged.

If our recurrence relations are to be implemented in terms of a computer-algebra program, the following steps are recommended. At first, by using (4) we reduce tensor integrals to scalar ones with shifted dimension (see also below). In a second step we apply (15,30) and then (17,31). Both relations produce the same factor G_{n-1} in the denominator and therefore one expects some similarity of the obtained expressions. Finally, relations (13,28) are applied, bringing all integrals to a set of master integrals in the generic dimension with powers 1 of the scalar propagators.

The properties mentioned in the introduction leading to simplifications for $n \geq 5$, which will be explained in more details below, are difficult to be implemented in computer algebra systems. Therefore we prefer specific representations, e.g. to avoid Gram determinants [3]. In particular, to obtain results as compact as possible, we will do a careful analysis of the recurrence relations and different properties of Gram determinants in Sect. 3.

2.2 Integrals with zero kinematic determinants

When one or both of the determinants G_{n-1} and Δ_n are equal to zero the reduction procedure must be modified. In such cases it is possible to express integrals with n lines as a combination of integrals with $n-1$ lines.

First let us consider $G_{n-1} = 0$. This is the case for n-point functions with $n \geq 6$ for external 4-dimensional vectors since then the order of $()_n$ is $n+1$ but its rank is 6. In this case the recurrence relations (30) and (31) cannot be directly used. Prior to the reduction of d one can remove one of the lines of the diagram by using a relation which follows from (15):

$$I_n^{(d)} = -\sum_{k=1}^{n} \frac{(\partial_j \partial_k \Delta_n)}{(\partial_j \Delta_n)}\, \mathbf{k}^- I_n^{(d)}. \tag{32}$$

Here we keep the original form in terms of derivatives w.r.t. the masses since in some computer algebra systems these derivatives may be easier to calculate than determinants.

By repeated application of this relation one of the lines will be contracted. The procedure must be repeated until one obtains integrals with non-vanishing determinant G. After that the reduction of the space-time dimension can be done for integrals with the smaller number

[3] Strictly speaking, here we only mean G_{n-1}.

of lines. For $G_{n-1} = 0$ yet another relation can be obtained from (17):

$$I_n^{(d)} = -\sum_{k=1}^{n} \frac{(\partial_k \Delta_n)}{2\Delta_n}\ \mathbf{k}^- I_n^{(d)}. \tag{33}$$

In fact both decompositions are equivalent since for $G_{n-1} = 0$, cancelling the common factor in Δ_n and $\partial_k \Delta_n$, the following relation holds

$$\frac{\partial_k \Delta_n}{2\Delta_n} = \frac{\partial_k \partial_i \Delta_n}{\partial_i \Delta_n}. \tag{34}$$

We will see in the next Sect. that indeed further important consequences follow from $G_{n-1} = 0$.

Zeros of Δ_n may occur in special domains of the phase space for n=6. In this case the reduction procedure will be as follows. First, applying the relation

$$(d - \sum_{i=1}^{n} \nu_i - 1) G_{n-1} I_n^{(d)} = \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d-2)}, \tag{35}$$

which follows from (17), the integral with n propagators must be reduced to a sum of integrals with $n-1$ propagators but in lower space-time dimension. As in the case with $G_{n-1} = 0$ the procedure of contracting lines can be repeated until we obtain integrals with non-vanishing Δ_n. After that integrals with space-time dimension $d < 4$ can be expressed in terms of integrals in the generic dimension d by using (29) and indices can be reduced to 1 by using relations (13) and (15).

Explicit calculation [4] shows that $\Delta_n = 0$ for n-point functions with $n \geq 7$, again for external 4-dimensional vectors - and in particular for arbitrary masses. Then, due to (23) also $\binom{0}{j}_n = 0$ for $n \geq 7$. These properties eliminate quite many terms in relations (28), (30) and (31) [5]. This is the reason why the investigation of $n \geq 7$ is postponed at present. Nevertheless, we write the recurrence relations in what follows for arbitrary n. In particular for external vectors not of dimension $d = 4$ the Gram determinants do not vanish as described and for these cases anyway our relations hold in the given form for arbitrary n.

3 Explicit recursions

In this section we give details of how to calculate in particular 5- and 6- point functions since these are of greatest actuality for present day experiments. Our goal is to present the most compact formulae for direct applications. Throughout we use the notation

$$\int^d \equiv \int \frac{d^d q}{\pi^{d/2}} \tag{36}$$

[4] This has been performed for $n = 7$ and 8 in terms of a FORM program, using component representations of fourvectors.

[5] See also the discussion after (60).

and for the scalar integrals we introduce the explicit notation

$$I^{[d+]^l,stu\ldots}_{p,ijk\ldots} = \int^{[d+]^l} \prod_{r=1}^{n} c_r^{-(1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\ldots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\ldots)}, \tag{37}$$

where $[d+]^l = 4+2l-2\varepsilon$. Observe that (number of entries $s,t,u,\ldots \neq i,j,k,\ldots)+p = n$ and equal upper and lower indices 'cancel'. The index p specifies the 'actual' number of external legs. We consider now Feynman diagrams in the 'generic' dimension $d = 4-2\varepsilon$. For scalar n-point integrals in the generic dimension we use the notation I_n.

For the reduction of the tensor integrals to scalar ones with shifted dimension, we wrote a FORM [20] program which applies (4) to integrals of rank 1, 2 and 3. After inspection the result reads for arbitrary n and powers of the scalar propagators equal 1 (the latter is the most frequent case in the electroweak Standard Model in the Feynman gauge; otherwise, as mentioned above, we can reduce higher indices by recurrence relations):

$$\begin{aligned}
I^{\mu}_{n} &= \int^{d} q^{\mu} \prod_{r=1}^{n} c_r^{-1} \\
&= \sum_{i=1}^{n-1} p_i^{\mu} I^{[d+]}_{n,i} \qquad (38) \\
I^{\mu\nu}_{n} &= \int^{d} q^{\mu} q^{\nu} \prod_{r=1}^{n} c_r^{-1} \\
&= \sum_{i,j=1}^{n-1} p_i^{\mu} p_j^{\nu} \cdot n_{ij} I^{[d+]^2}_{n,ij} - \frac{1}{2} g^{\mu\nu} I^{[d+]}_{n} \qquad (39) \\
I^{\mu\nu\lambda}_{n} &= \int^{d} q^{\mu} q^{\nu} q^{\lambda} \prod_{r=1}^{n} c_r^{-1} \\
&= -\sum_{i,j,k=1}^{n-1} p_i^{\mu} p_j^{\nu} p_k^{\lambda} \cdot n_{ijk}! I^{[d+]^3}_{n,ijk} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n-1} (g^{\mu\nu} p_i^{\lambda} + g^{\mu\lambda} p_i^{\nu} + g^{\nu\lambda} p_i^{\mu}) I^{[d+]^2}_{n,i}, \qquad (40)
\end{aligned}$$

where $n_{ij} = 1+\delta_{ij}$ and $n_{ijk} = 1+\delta_{ij}+\delta_{ik}+\delta_{jk}-\delta_{ij}\delta_{ik}\delta_{jk}$ is the number of equal indices among i,j,k which can be written in this symmetric manner.

We first consider now $n = 5$, using recursion relation (30) we have:

$$\nu_{ijk}\ I^{[d+]^3}_{5,ijk} = -\frac{\binom{0}{k}_5}{\binom{}{}_5} I^{[d+]^2}_{5,ij} + \sum_{s=1}^{5} \frac{\binom{s}{k}}{\binom{}{}_5} I^{[d+]^2,s}_{p,ij}\ , \qquad \nu_{ijk} = 1+\delta_{ik}+\delta_{jk} \tag{41}$$

where in the second term on the r.h.s. we have introduced the index p. In general $p = 4$, but for $s = i,j$ we have $p = 5$ (recall that equal upper and lower indices cancel). Let us now

consider the first term on the r.h.s. of (41):

$$\nu_{ij}\ I_{5,ij}^{[d+]^2} = -\frac{\binom{0}{j}_5}{\binom{}{}_5} I_{5,i}^{[d+]} + \sum_{s=1}^{5} \frac{\binom{s}{j}_5}{\binom{}{}_5} I_{p,i}^{[d+],s}\ , \qquad \nu_{ij} = 1 + \delta_{ij} \tag{42}$$

$$I_{5,i}^{[d+]} = -\frac{\binom{0}{i}_5}{\binom{}{}_5} I_5 + \sum_{s=1}^{5} \frac{\binom{s}{i}}{\binom{}{}_5} I_4^s \tag{43}$$

$$I_{p,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{t=1}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st} \qquad (s \neq i,\ p = 4), \tag{44}$$

where we indicated two more steps of the reduction. Like in (41) we have again introduced the index p in the second term on the r.h.s. of (42): for $s = i,\ p = 5$. At this point it is also worth pointing out that

$$n_{ij} = \nu_{ij}, \qquad n_{ijk}! = \nu_{ij}\nu_{ijk}. \tag{45}$$

Now consider the second term on the r.h.s. of (41):

$$\nu_{ij}\ I_{p,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \sum_{t=1}^{5} \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{q,i}^{[d+],st} \qquad (s \neq i, j,\ p = 4). \tag{46}$$

Here the index q in the second term on the r.h.s. is $q = 3$ in general, except when $t = i$, in which case $q = 4$ (see (50)). For $q = 3$ we have:

$$I_{3,i}^{[d+],st} = -\frac{\binom{0st}{ist}_5}{\binom{st}{st}_5} I_3^{st} + \sum_{u=1}^{5} \frac{\binom{ust}{ist}_5}{\binom{st}{st}_5} I_2^{stu}\ . \tag{47}$$

For $s = i, j$ in (46) and also in the second part of (40) we have integrals of the type

$$I_{5,i}^{[d+]^2} = -\frac{\binom{0}{i}_5}{\binom{}{}_5} I_5^{[d+]} + \sum_{s=1}^{5} \frac{\binom{s}{i}_5}{\binom{}{}_5} I_4^{[d+],s} \tag{48}$$

and finally, applying recursion (17), we have

$$I_5^{[d+]} = \left[\frac{\binom{0}{0}_5}{\binom{}{}_5} I_5 - \sum_{s=1}^{5} \frac{\binom{0}{s}_5}{\binom{}{}_5} I_4^s \right] \cdot \frac{1}{d-4} \tag{49}$$

$$I_4^{[d+],s} = \left[\frac{\binom{0s}{0s}_5}{\binom{s}{s}_5} I_4^s - \sum_{t=1}^{5} \frac{\binom{0s}{ts}_5}{\binom{s}{s}_5} I_3^{st} \right] \cdot \frac{1}{d-3}. \tag{50}$$

We observe that (49) is unpleasant in the sense that the expression in the square bracket is zero for $d = 4$ and the overall factor is $-\frac{1}{2\varepsilon}$, i.e. we need to expand the 5- and 4- point functions up to order ε in order to get the finite part. In fact, however, we will show that $I_5^{[d+]}$ cancels. To demonstrate this for the 5-point function, we need first of all to express the $g^{\mu\nu}$ tensor in terms of the external vectors. Assuming that the external vectors $p_1, \ldots, p_4$ are 4-dimensional and independent (i.e. no collinearities occur), we can write (see e.g. [23])

$$g^{\mu\nu} = 2 \sum_{i,j=1}^{4} \frac{\binom{i}{j}_5}{\binom{}{}_5} p_i^\mu p_j^\nu \tag{51}$$

Inserting this into (39), we get

$$\sum_{i,j=1}^{4} p_i^\mu p_j^\nu \cdot n_{ij} I_{5,ij}^{[d+]^2} - \sum_{i,j=1}^{4} p_i^\mu p_j^\nu \frac{\binom{i}{j}_5}{()_5} I_5^{[d+]} \ . \tag{52}$$

By inspection of (42) we see that the second term on the r.h.s. for $s = i$ exactly contains the $I_5^{[d+]}$ to cancel the $I_5^{[d+]}$ in (52). In a similar way with the help of (51) we can rewrite the second sum in (40), to get

$$\sum_{i,j,k=1}^{4} p_i^\mu p_j^\nu p_k^\lambda \left[\frac{\binom{j}{k}_5}{()_5} I_{5,i}^{[d+]^2} + \frac{\binom{i}{k}_5}{()_5} I_{5,j}^{[d+]^2} + \frac{\binom{i}{j}_5}{()_5} I_{5,k}^{[d+]^2} \right] . \tag{53}$$

To pick out the $I_5^{[d+]}$ we use (48). Inserting (41) into (40), the $I_5^{[d+]}$ contributions come from the second term in (42) (for $s = i$) and the second term of (41) (for $s = i, j$). Further we need the property (45).

Finally a remark is in order concerning the finiteness of the 5-point function: the only infinities occurring in the above decomposition are coming from the I_2 's in (47). The $\frac{1}{\varepsilon}$ terms in these 2-point functions are independent of masses and momenta, however. Thus according to (see [10])

$$\sum_{j=1}^{n} \binom{i}{j}_n = 0 \quad , \quad i = 1, \cdots n \tag{54}$$

for $n = 3$ these infinite terms cancel. With these considerations the tensor 5-point functions can be completely reduced to scalar 4-, 3- and 2-point functions in generic dimension. We want to point out that use has been made only of relations (30) and (31). We will see below that (43) should be replaced by another relation containing no Gram determinant, which might simplify the numerics in some kinematic domains. At the end the tensor integrals under consideration are finite (after cancellation of the above mentioned terms of order $\frac{1}{\varepsilon}$) and therefore we can put $\varepsilon = 0$, which in particular applies to the scalar 5-point function [7, 10, 15].

For $n = 6$ and $d = 4$ the situation is completely different due to the fact that $()_6 = 0$ as was discussed in Sect. 2.3 . Therefore (15) and (17) both reduce to (32), i.e. instead of those two recursion relations we have only one. Explicitly it reads, introducing the signed minors of the modified Cayley determinant

$$I_{6,i..}^{[d+]^l} = \sum_{r=1}^{6} \frac{\binom{R}{r}_6}{\binom{R}{0}_6} I_{p,i..}^{[d+]^l,r}, \quad R = \text{any value } 0, \ldots, 6. \tag{55}$$

We see that (55) does not allow to reduce the dimension and it can only be used to reduce indices! If $p = 5$ on the r.h.s. of (55), which is the case for $r \neq i..$, then the reduction can be continued as for the 5-point function above. If, however, $p = 6$ on the r.h.s. of (55) in case of $r = i..$, then again (55) has to be applied until all $I_{6,i..}^{[d+]^l}$ are eliminated.

So far only relations (30) and (31) for the Gram determinants G_n have been applied. Now let us investigate (28). It does not contain Gram determinants G_n. A priori it does not allow

to reduce the dimension. In the following, however, we will explicitly show that (29) can be used to perform one sub-summation in (28) and that it is this relation which finally provides a very efficient possibility to reduce also the dimension in the case of the 6-point function. Also some further useful information about the reduction of 5-point functions is obtained in this manner. In the following we investigate explicitly the integrals occurring in (38),(39) and (40).

For arbitrary n (28) yields for the integral in (38)

$$\binom{0}{0}_n I^{[d+]^l}_{n,i} = [n+1-(d+2l)] \binom{0}{i}_n I^{[d+]^l}_n - \sum_{\substack{r,s \\ r\neq s}}^{n} \binom{0i}{0r}_n I^{[d+]^l,r}_{n-1,s} \tag{56}$$

and using (29) once we obtain :

$$\binom{0}{0}_n I^{[d+]^l}_{n,i} = [n+1-(d+2l)] \binom{0}{i}_n I^{[d+]^l}_n + \sum_{r=1}^{n} \binom{0i}{0r}_n I^{[d+]^{(l-1)},r}_{n-1} \ . \tag{57}$$

Similarly, for the integral in (39) we obtain from (28) for arbitrary n

$$\begin{aligned} \binom{0}{0}_n \nu_{ij} I^{[d+]^l}_{n,ij} &= [n+2-(d+2l)] \binom{0}{j}_n I^{[d+]^l}_{n,i} - \binom{0j}{0i}_n \sum_{s=1}^{n} I^{[d+]^l}_{n,s} \\ &\quad - \sum_{\substack{r=1 \\ r\neq i}}^{n} \binom{0j}{0r}_n \sum_{\substack{s=1 \\ s\neq r}}^{n} \nu_{is} I^{[d+]^l,r}_{n-1,is} \end{aligned} \tag{58}$$

and using (29)

$$\begin{aligned} \binom{0}{0}_n \nu_{ij} I^{[d+]^l}_{n,ij} &= [n+2-(d+2l)] \binom{0}{j}_n I^{[d+]^l}_{n,i} + \binom{0i}{0j}_n I^{[d+]^{(l-1)}}_n \\ &\quad + \sum_{\substack{r=1 \\ r\neq i}}^{n} \binom{0j}{0r}_n I^{[d+]^{(l-1)},r}_{n-1,i}, \end{aligned} \tag{59}$$

It is interesting to note that in (57) for $n = 5$ and $l = 1$ as well as in (59) for $n = 6$ and $l = 2$, the numerical square brackets evaluate to $4 - d = 2\varepsilon$. This means a great simplification for $\varepsilon = 0$ and in particular (57) ought to replace (43) for $n = 5$ (a little algebra shows that indeed for $n = 5$ (57) and (43) are identical including the ε-part). Moreover this is exactly what is needed in (59) for $n = 6$ and $l = 2$, the integral needed in (39) for $n = 6$.

For the integral with three indices occurring in (40) we finally have

$$\begin{aligned} \binom{0}{0}_n \nu_{ijk} I^{[d+]^l}_{n,ijk} &= [n+3-(d+2l)] \binom{0}{k}_n I^{[d+]^l}_{n,ij} \\ &\quad + \left[\binom{0k}{0i}_n I^{[d+]^{(l-1)}}_{n,j} + \binom{0k}{0j}_n I^{[d+]^{(l-1)}}_{n,i} \right] \frac{1}{\nu_{ij}} + \sum_{\substack{r=1 \\ r\neq i,j}}^{n} \binom{0k}{0r}_n I^{[d+]^{(l-1)},r}_{n-1,ij}. \end{aligned} \tag{60}$$

Equations (57),(59) and (60) are valid for any n and they represent in a way an optimal form of the recursion for the considered integrals. For $n \geq 7$ and $d = 4$, as mentioned in Sect. 2.3

already, these relations are still valid but reduce considerably. Since this case is at present of minor physical interest, it will be discussed separately. We only mention that, e.g., using (60) for $i = j = k$ and $n = 7$ we obtain a relation for $I^{[d+]^l}_{7,i}$ in terms of $I^{[d+]^l,r}_{6,ii}$, summed over r, etc.

The integrals with highest dimension in the above relations have coefficients $\binom{0}{i}_n$, $\binom{0}{j}_n$ and $\binom{0}{k}_n$, respectively. In (38), (39) and (40) these are multiplied with p_i^μ, p_j^ν and p_k^λ and summed over i, j, k. Due to

$$\sum_{j=1}^{n-1} p_j^\nu \binom{0}{j}_n = 0 \ , \qquad n \geq 6 \tag{61}$$

all these contributions vanish. To prove (61), we project it on all $p_i, i = 1 \ldots n-1 (p_n = 0)$ in d=4 dimensions, assuming that at least four p_i's are linearly independent. Thus we have to show

$$\sum_{j=1}^{n} p_i p_j \binom{0}{j}_n = 0 \ , \qquad i = 1 \ldots n-1. \tag{62}$$

First of all we write the scalar products in the form [6]

$$p_i p_j = \frac{1}{2} \{ Y_{ij} - Y_{in} - Y_{jn} + Y_{nn} \} \tag{63}$$

so that (62) reads

$$\sum_{j=1}^{n} \{ Y_{ij} - Y_{nj} - (Y_{in} - Y_{nn}) \} \binom{0}{j}_n = 0 \tag{64}$$

Now the term $(Y_{in} - Y_{nn})$ is independent of j and the summation over j can therefore be performed:

$$\sum_{j=1}^{n} \binom{0}{j}_n = ()_n = 0 \ , \qquad n \geq 6. \tag{65}$$

Next we consider the first term in (64)

$$\sum_{j=1}^{n} Y_{ij} \binom{0}{j}_n = \begin{vmatrix} 0 & Y_{i1} & Y_{i2} & Y_{i3} & \ldots & Y_{in} \\ 1 & Y_{11} & Y_{12} & Y_{13} & \ldots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & Y_{23} & \ldots & Y_{2n} \\ \vdots & & & & & \\ 1 & Y_{i1} & Y_{i2} & Y_{i3} & \ldots & Y_{in} \\ \vdots & & & & & \\ 1 & Y_{1n} & Y_{2n} & Y_{3n} & \ldots & Y_{nn} \end{vmatrix} = -\binom{0}{0}_n \tag{66}$$

Subtracting the 0-th line of this determinant from the i-th one, the latter takes the form $(1, 0, \ldots 0)$. Expanding finally the determinant w.r.t. the first column yields the r.h.s. of (66). Thus this determinant is independent of i, which finally proves (64), and hence (61).

Further we also see that $I_n^{[d+](l-1)}$ in (59) cancels against the $g^{\mu\nu}$ term in (39) ($l = 2$) due to

$$g^{\mu\nu} = \frac{2}{\binom{0}{0}_n} \sum_{i,j=1}^{n-1} \binom{0\ i}{0\ j}_n p_i^\mu p_j^\nu \ , \qquad n \geq 6. \tag{67}$$

[6] This essential decomposition, to our knowledge, was first used in [10] and is the basis of the proof in [15] as well.

As above we show this again by proving that all projections of this tensor on any pair of vectors $p_k, p_l (k, l = 1 \ldots n-1)$ vanish, i.e.

$$\sum_{i,j=1}^{n} (p_i p_k)(p_j p_\ell) \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix}_n = \frac{1}{2} p_k p_\ell \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n . \tag{68}$$

Expressing all scalar products again in terms of the Y_{ik}, this is equivalent to

$$\begin{aligned} &\sum_{i,j=1}^{n} \left[Y_{ik} - Y_{in} - (Y_{kn} - Y_{nn}) \right] \left[Y_{j\ell} - Y_{jn} - (Y_{\ell n} - Y_{nn}) \right] \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix}_n \\ &= \left[Y_{k\ell} - Y_{kn} - (Y_{\ell n} - Y_{nn}) \right] \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \end{aligned} \tag{69}$$

The contribution on the l.h.s. of (69) with Y's independent of i, j is proportional to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix}_n = -\sum_{i=1}^{n} \begin{pmatrix} 0 \\ i \end{pmatrix} = -\begin{pmatrix} \\ \end{pmatrix}_n = 0. \tag{70}$$

The 'linear' terms, according to (27) and (66), read

$$\sum_{i=1}^{n} Y_{ik} \sum_{j=1}^{n} \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \tag{71}$$

and are independent of k, i.e. the contributions of this type cancel. Finally, the 'quadratic' terms are of the form

$$\sum_{i=1}^{n} Y_{ik} \sum_{j=1}^{n} Y_{jl} \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix} = \sum_{i=1}^{n} Y_{ik} \cdot \delta_{il} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n = Y_{k\ell} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n , \tag{72}$$

which proves (69) and thus (67).

We point out that (67) is a representation of the $g_{\mu\nu}$ tensor in close analogy to (51), only that in this case the $n-1$ vectors ($n \geq 6$) are linearly dependent in $d = 4$ dimensions.

Finally we investigate the simplification of (40) due to (60), where we concentrate on the 6-point function, $n = 6$. Let us first consider the $I_{n,i}^{[d+]^2}$ integrals. These are given by (57) for $l = 2$. First of all we again observe that their first part can be dropped due to (61). Their remaining parts for $n = 6$ are only of the $I_5^{[d+]}$ type, which we would like to cancel. This concerns in particular the complete second part in (40). It is easy to see that due to (67) the $g^{\mu\lambda}$ and $g^{\nu\lambda}$ terms cancel against the two terms in the square bracket of (60) by summing in (40) over i, k and j, k , respectively. To show the cancellation of the remaining $I_5^{[d+]}$'s is a bit tricky. At first we observe that the last term in (60) contains the following $I_5^{[d+]}$ term (most easily seen from (42)):

$$\nu_{ij} I_{5,ij}^{[d+]^2,r} = \frac{\binom{ir}{jr}}{\binom{r}{r}} I_5^{[d+],r} + \cdots \quad . \tag{73}$$

Thus the sum of all remaining $I_5^{[d+]}$ terms is given by

$$-\binom{0}{0}_6^{-1} \sum_{i,j,k=1}^{5} p_i^\mu p_j^\nu p_k^\lambda \sum_{r=1}^{6} \frac{\binom{ir}{jr}}{\binom{r}{r}} \binom{0k}{0r} I_5^{[d+],r} + \frac{1}{2} g^{\mu\nu} \binom{0}{0}_6^{-1} \sum_{k=1}^{5} p_k^\lambda \sum_{r=1}^{6} \binom{0k}{0r} I_5^{[d+],r} \qquad (74)$$

Then we observe that for $n = 6$ the $g^{\mu\nu}$ tensor can be written in the following form

$$g^{\mu\nu} = 2 \sum_{i,j=1}^{5} \frac{\binom{ir}{jr}_6}{\binom{r}{r}_6} p_i^\mu p_j^\nu \ , \qquad r = 1 \ldots 6. \qquad (75)$$

For $r \leq 5$ this equation immediately follows from (51), just by scratching one of the 5 momenta and assuming that always four of them are linearly independent. For $r = 6$, (75) can be proven in the same way as (67). Thus, summing over $i, j = 1 \ldots 5$ in (74), the cancellation of all $I_5^{[d+]}$ integrals is also shown for the tensor integrals of rank 3, a result which has been obtained previously in [15]. What remains in (60) is finally again only the last sum with the exclusion of the $I_5^{[d+]}$ contribution. The $I_{5,ij}^{[d+]^2,r}$ integrals in (60) can be calculated, e.g., by means of (42), without the term $s = i$ in the sum on its r.h.s. or from (59) again dropping $I_5^{[d+]}$ on the r.h.s. .

As a final remark we point out that the only Gram determinants occurring are those in the tensor integrals of rank 3, coming from the reduction in (42), where instead of (43) now (57) is to be used. Thus only inverse Gram determinants of the first power appear. Moreover the remaining integrals are all reduced to standard 4- and 5-point functions for $n = 6$.

Conclusion

We have presented an algorithm for the calculation of n-point Feynman diagrams, applicable for any tensorial structure and gave a detailed specification for 5- and 6-point functions. Dependent on the kinematic situation different recurrence relations may be used. A typical example is the equivalence of (43) and (57) for $n = 5$. All recursion relations are valid for arbitrary dimension. In the main part of the paper we have concentrated on the experimentally most relevant cases, i.e. the tensor 5- and 6- point functions, and considered $d = 4$ for the external momenta in order to simplify the results. In this case a regulator mass is needed for infrared divergences. We restricted ourselves to tensors of rank 3 [7]. For the 5-point function, due to the above equivalence, there occur inverse Gram determinants of only second order; for the 6-point function, surprisingly, only of first order. n-point functions with $n \geq 7$ are not considered explicitly since due to the drastic reduction of the recurrence relations for $d = 4$ these cases need a separate investigation.

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[7] For more than three integration momenta in the numerator the extension of our procedure is straightforward.

Appendix

In this appendix we present for the 6-point function the 'effective' contributions to the integrals (38) - (40), i.e. leaving out all those contributions which we have shown to cancel. First of all no contributions come from the metric tensor in (39) and (40). The integral in (38) is 'effectively' given by

$$I_{6,i}^{[d+]} := \frac{1}{\binom{0}{0}_6} \sum_{r=1}^{6} \binom{0i}{0r}_6 I_5^r = \frac{1}{\binom{0}{0}_6} \sum_{r=1}^{6} \frac{\binom{0i}{0r}_6}{\binom{0r}{0r}_6} \sum_{s=1}^{6} \binom{0r}{sr}_6 I_4^{rs}, \tag{76}$$

where we have used (57) and expressed the 5-point function I_5^r in generic dimension in terms of 4-point functions. No Gram determinant is involved in this case.

The remaining integral in (39) is 'effectively'

$$n_{ij} I_{6,ij}^{[d+]^2} := \frac{1}{\binom{0}{0}_6} \sum_{\substack{r=1 \\ r\neq i}}^{6} \binom{0j}{0r}_6 I_{5,i}^{[d+],r} := \frac{1}{\binom{0}{0}_6} \sum_{r=1}^{6} \frac{\binom{0j}{0r}_6}{\binom{0r}{0r}_6} \sum_{s=1}^{6} \binom{0ir}{0sr}_6 I_4^{rs}. \tag{77}$$

Here (59) is used and the integral $I_{5,i}^{[d+],r}$ is expressed in terms of (57), where the square bracket of the first contribution evaluates to 2ε and is dropped. The result is very similar to (76), i.e. it is a double sum over 4-point functions and no Gram determinant involved. The latter property is due to (57) , which is used instead of (43).

The integral in (40) is a bit more complicated:

$$\begin{aligned} n_{ijk}! I_{6,ijk}^{[d+]^3} &:= \frac{1}{\binom{0}{0}_6} \sum_{\substack{r=1 \\ r\neq i,j}}^{6} \binom{0k}{0r}_6 \nu_{ij} I_{5,ij}^{[d+]^2,r} \\ &:= \frac{1}{\binom{0}{0}_6} \sum_{\substack{r=1 \\ r\neq i,j}}^{6} \frac{\binom{0k}{0r}_6}{\binom{0r}{0r}_6} \left\{ -\binom{0r}{jr}_6 I_{5,i}^{[d+]^2,r} + \sum_{\substack{s=1 \\ s\neq i}}^{6} \binom{0jr}{0sr}_6 I_{4,i}^{[d+],rs} \right\}. \end{aligned} \tag{78}$$

Here (60) is used together with (59) for the reduction of $I_{5,ij}^{[d+]^2,r}$. In the square bracket of the first part of (59) again $\varepsilon = 0$ is taken. Reducing $I_{5,i}^{[d+]^2,r}$ we see no possibility to avoid the inverse Gram determinant and therefore propose to use (48), i.e. 'effectively' (dropping $I_5^{[d+]}$)

$$I_{5,i}^{[d+]^2,r} := \frac{1}{\binom{r}{r}_6} \sum_{s=1}^{6} \binom{sr}{ir}_6 I_4^{[d+],rs} \tag{79}$$

with ($\varepsilon = 0$)

$$I_4^{[d+],rs} = \frac{1}{\binom{rs}{rs}_6} \left[\binom{0rs}{0rs}_6 I_4^{rs} - \sum_{t=1}^{6} \binom{0rs}{trs}_6 I_3^{rst} \right] \tag{80}$$

from (50). Finally, $I_{4,i}^{[d+],rs}$ should be reduced to the form (see e.g. (44))

$$I_{4,i}^{[d+],rs} = \frac{-1}{\binom{rs}{rs}_6} \left[\binom{irs}{0rs}_6 I_4^{rs} - \sum_{t=1}^{6} \binom{irs}{trs}_6 I_3^{rst} \right] . \tag{81}$$

The factor $\binom{r}{r}_6^{-1}$ in (79) is the only occurrence of the inverse Gram determinant. $\binom{rs}{rs}_6^{-1}$ in (80) and (81) are in principle inverse Gram determinants as well, but since one does not expect the l.h.s. of (80) and (81) to have kinematic singularities for $\binom{rs}{rs}_6 \to 0$, the numerators also vanish in this case, which allows to keep the numerical evaluation under control. The situation is different in (79) : even if the l.h.s. has no kinematic singularity as $\binom{r}{r}_6 \to 0$, on the r.h.s. $I_5^{[d+]}$ has been cancelled and therefore the above argument is not applicable.

It is convenient to add up all contributions in (78) by introducing the quantity

$$\{r\}_{ij}^{st} = \binom{0r}{0r}_6 \left[\binom{tr}{sr}_6 \binom{irs}{jrs}_6 - \binom{rs}{rs}_6 \binom{irs}{jrt}_6 \right] - \binom{rs}{rs}_6 \left[\binom{tr}{0r}_6 \binom{ir0}{jrs}_6 - \binom{sr}{0r}_6 \binom{ir0}{jrt}_6 \right]. \tag{82}$$

With this definition (78) can be written in the form

$$n_{ijk}! I_{6,ijk}^{[d+]^3} := \frac{1}{\binom{0}{0}_6} \sum_{\substack{r=1 \\ r\neq i,j}}^{6} \frac{\binom{0k}{0r}_6}{\binom{0r}{0r}_6} \frac{1}{\binom{r}{r}_6} \sum_{\substack{s=1 \\ s\neq r}}^{6} \frac{1}{\binom{rs}{rs}_6} \left\{ - \{r\}_{ij}^{s0} I_4^{rs} + \sum_{t=1}^{6} \{r\}_{ij}^{st} I_3^{rst} \right\}. \tag{83}$$

As we see, some further cancellations occur. The representations derived here are particularly useful for numerical evaluation.

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