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Exact mass dependent two–loop $\overline{\alpha}_s(Q^2)$ in the background MOM renormalization scheme

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Abstract

A two-loop calculation of the renormalization group β–function in a momentum subtraction scheme with massive quarks is presented using the background field formalism. The results have been obtained by using a set of new generalized recurrence relations proposed recently by one of the authors (O.V.T.). The behavior of the mass dependent effective coupling constant is investigated in detail. Compact analytic results are presented.

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1 Introduction

Particle masses are amongst the most important physical parameters and in many cases their meaning and definition by thresholds (e.g. lepton masses), symmetry breaking parameters (current quark masses, neutrino masses) or scale parameters is quite clear. For particles which exist as free or quasi–free states a definition by the pole mass is most natural and has an unambiguous meaning. The definition of quark masses, in particular for the light quarks, allows for a lot of freedom, mainly because the pole mass is not directly observable due to the confinement property of QCD. Nevertheless, quark masses play a crucial role for the effective behavior of strong interactions at a given scale. The purpose of the present calculation is a precise understanding of the quark mass dependence of QCD, more specifically, of the effective coupling constant $\alpha_s(Q^2) = g_s^2/(4\pi)$, the most important quantity in the description of strong interactions. These considerations are important for a better understanding of the decoupling of heavy particles and of the relationship between QCD with massive quarks and QCD in the $\overline{\rm MS}$ scheme where effective theories with different number of (light) flavors [1–3] must be matched at the different quark thresholds.

When the on-shell renormalization scheme is not adequate we either may use a minimal subtraction scheme (MS or $\overline{\rm MS}$) or some version of a momentum subtraction scheme (MOM), defined by the condition that the radiative corrections of an appropriate set of quantities vanish at a certain (off–shell) momentum configuration. While the $\overline{\rm MS}$ [4] scheme is technically simple and respects the Slavnov-Taylor identities the MOM scheme is more physical since it respects the decoupling theorem [5]. A serious short coming of the standard MOM schemes [6], however, is the fact that they spoil the validity of the canonical form of the Slavnov-Taylor identities. An elegant way out of this difficulty is the use of the so called background field method (BFM) [7]. The latter takes advantage of the freedom to chose a gauge fixing function in a particular way, namely, such that the canonical Slavnov-Taylor identities remain valid also after momentum subtractions. The gauge invariant physical quantities are not affected by the gauge fixing, however, the "background field gauge" selects a particular representative of the gauge variant off–shell amplitudes. The restauration of the Slavnov-Taylor identities in the BFM is achieved solely by changing the vertices with external gluons appropriately. For further details and for the Feynman rules of QCD in the background field (BF) formalism we refer to [8, 9].

In Ref. [9] the renormalization group (RG) β–function of QCD was evaluated at one–loop order in the background field approach using the MOM scheme. In the present article we extend this analysis to a complete mass dependent two–loop calculation. Previously, the two-loop renormalization of the pure Yang-Mills theory in the background field method was first considered in [8] using the $\overline{\rm MS}$ scheme. Fermionic contributions were added later in [10]. Calculations in the background formalism for an arbitrary value of the gauge parameter were presented in [11]. In the standard approach the evaluation of the mass dependent QCD β–function at the two-loop level was performed in [12]. In the latter publication only an approximate expression for the two–loop coefficient was given. Because of the complexity of such calculations the result of [12] was not confirmed by any other group until now. The background field method provides the easiest way to calculate a mass dependent β–function because here only propagator diagrams need to be evaluated. A general method for the evaluation of two-loop

propagator type diagrams with arbitrary masses was recently proposed in [13, 14].
Our paper is organized as follows: in Sec. 2 we describe the calculation of the background field propagator, from which we obtain the RG and the effective running coupling in the BF − MOM scheme in Sec. 3. The relationship between the BF − MOM and the $\overline{\rm MS}$ coupling is presented in Sec. 4 (analytical) and Sec. 5 (numerical). For completeness we include a formula for the bare BF propagator in Appendix A. The BF Feynman rules and the BF propagator diagrams are included in Appendices B and C, respectively.

2 The background field propagator

To regularize divergences we will use the dimensional regularization procedure in $d = 4 - 2\varepsilon$ dimensions. In the background field approach, we only need to calculate the background field renormalization constant Z_A in order to obtain the charge renormalization constant. The complete list of two-loop diagrams as well as the Feynman rules in the background field approach may be found in [8] – [11]. Z_A is determined by renormalizing the background field propagator according to

$$\frac{1}{1+\Pi(Q^2,\mu^2,\{m_i^2\})} = \frac{Z_A}{1+\Pi_0(Q^2,\mu^2,\{m_i^2\})}\ , \tag{1}$$

where $Q^2 = -q^2$ and μ is the subtraction point. Bare quantities carry an subscript $_0$. In the MOM scheme the condition

$$\Pi(Q^2,\mu^2,\{m_i^2\})\,|_{Q^2=\mu^2} = 0 \tag{2}$$

is imposed on the renormalized self–energy function. The renormalized mass m_i in our calculations is defined as a pole of the quark propagator. In the MOM scheme Z_A and therefore also the RG β–function depend on the gauge parameter ξ. The gauge parameter is renormalized by

$$\xi_0 = \xi Z_3\ , \tag{3}$$

where Z_3 is the renormalization constant of the quantum gluon field. To circumvent problems connected with the renormalization of the gauge parameter we have chosen the Landau gauge $\xi = 0$.
We repeated the calculation of all two-loop diagrams in the background formalism keeping non–vanishing quark masses. All calculations have been performed with the help of FORM [15] using the algorithm described in [13]. Our results agree with those presented in [8, 10, 11] for the limit of massless quarks. The sum of all unrenormalized diagrams for an arbitrary value of the gauge parameter is given in the Appendix.
The renormalized self–energy amplitude $\Pi(Q^2)$ has the form:

$$\Pi(Q^2) = \left(\frac{\alpha_s}{4\pi}\right) U_1 + \left(\frac{\alpha_s}{4\pi}\right)^2 U_2 + \cdots \tag{4}$$

where

$$\begin{aligned} U_1(Q^2/\mu^2,\{m_i^2/\mu^2\}) &= \frac{11}{3}C_A \ln\frac{Q^2}{\mu^2} + T_F\sum_{i=1}^{n_F}\left(\Pi_1\left(\frac{Q^2}{m_i^2}\right) - \Pi_1\left(\frac{\mu^2}{m_i^2}\right)\right), \\ U_2(Q^2/\mu^2,\{m_i^2/\mu^2\}) &= \frac{34}{3}C_A^2 \ln\frac{Q^2}{\mu^2} + T_F\sum_{i=1}^{n_F}\left(\Pi_2\left(\frac{Q^2}{m_i^2}\right) - \Pi_2\left(\frac{\mu^2}{m_i^2}\right)\right). \end{aligned} \tag{5}$$

As usual, C_A, C_F and T_F are the group coefficients of the gauge group and n_F is the number of flavors.

The results of our calculations for $\Pi_{1,2}$ read

$$\Pi_1\left(\frac{Q^2}{m^2}\right) = \frac{4}{3z}[1-(1+2z)(1-z)G(z)] \;, \tag{6}$$

$$\begin{aligned}\Pi_2\left(\frac{Q^2}{m^2}\right) &= \frac{(1+2z)}{3z^2}[(C_A+4C_F)\;\sigma(z)-(C_A-2C_F)(1-2z)\;I(z)] \\ &+\frac{2}{9z}\Big\{39+3\tilde{I}_3^{(4)}(z)-[4z^2+134z+57-12(2-5z)zG(z)]\;(1-z)G(z) \\ &\qquad +2[z^2+18z+9-3(3+8z)(1-z)G(z)]\ln(-4z)\Big\}\;C_A \\ &+\frac{2}{3z}\Big\{13-[6(3+2z)+(7+8z-48z^2)G(z)](1-z)G(z)\Big\}\;C_F \;,\end{aligned} \tag{7}$$

where

$$Q^2=-q^2 \;, \qquad z=\frac{q^2}{4m^2} \;, \qquad y=\frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1} \;, \tag{8}$$

denote the kinematic variables and

$$G(z)=\frac{2y\ln y}{y^2-1} \;, \tag{9}$$

$$\begin{aligned}I(z) &= 6[\zeta_3+4\,\mathrm{Li}_3(-y)+2\,\mathrm{Li}_3(y)]-8[2\,\mathrm{Li}_2(-y)+\,\mathrm{Li}_2(y)]\ln y \\ &\qquad -2[2\ln(1+y)+\ln(1-y)]\ln^2 y \;,\end{aligned} \tag{10}$$

$$\tilde{I}_3^{(4)}(z) = 6\zeta_3-6\,\mathrm{Li}_3(y)+6\ln y\,\mathrm{Li}_2(y)+2\ln(1-y)\ln^2 y \;, \tag{11}$$

$$\sigma(z)=\frac{1-y^2}{y}\left\{2\,\mathrm{Li}_2(-y)+\,\mathrm{Li}_2(y)+[\ln(1-y)+2\ln(1+y)-\frac{3}{4}\ln y]\ln y\right\} \tag{12}$$

are our basic integrals. The functions $I(z)$, $\tilde{I}_3^{(4)}(z)$ are master integrals considered in [16, 17].
Setting $C_A=0, C_F=T_F=1, n_F=1$ and taking the limit $\mu^2\to 0$ we reproduce the well known result for the photon propagator [17, 18] in the on-shell scheme.
At large Euclidean momentum $Q^2=-q^2$ we find the asymptotic forms

$$\begin{aligned}\Pi_1\left(\frac{Q^2}{m^2}\right) &\overset{Q^2\to\infty}{\simeq} -\frac{4}{3}\ln\frac{Q^2}{m^2}-8\left(\frac{m^2}{Q^2}\right)+8\left(\frac{m^2}{Q^2}\right)^2\ln\frac{Q^2}{m^2}+\cdots, \\ \Pi_2\left(\frac{Q^2}{m^2}\right) &\overset{Q^2\to\infty}{\simeq} -\frac{4}{3}(5C_A+3C_F)\ln\frac{Q^2}{m^2}+\frac{2}{9}(C_A+36(C_A-2C_F)\zeta_3) \\ &\qquad -6(3C_A-8C_F)\left(\frac{m^2}{Q^2}\right)\ln\frac{Q^2}{m^2}+\cdots.\end{aligned} \tag{13}$$

With these results at hand we are able now to obtain the mass–dependent two–loop β–function.

3 The RG equation and the effective coupling

In the BFM the RG β–function is given by

$$\mu^2 \frac{d}{d\mu^2} \left(\frac{\alpha_s}{4\pi}\right) = \lim_{\varepsilon \to 0} \alpha_s \, \mu \frac{\partial}{\partial \mu} \ln Z_A = -\beta_0 \left(\frac{\alpha_s}{4\pi}\right)^2 - \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^3 - \cdots \tag{14}$$

and hence the coefficients of the β–function may be simply obtained by differentiating (6) and (7). The results read

$$\begin{aligned} \beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_F \sum_{i=1}^{n_F} b_0 \left(\frac{\mu^2}{m_i^2}\right), \\ \beta_1 &= \frac{34}{3} C_A^2 - \quad T_F \sum_{i=1}^{n_F} b_1 \left(\frac{\mu^2}{m_i^2}\right), \end{aligned} \tag{15}$$

where

$$\begin{aligned} b_0 \left(\frac{\mu^2}{m^2}\right) &= 1 + \frac{3}{2x}(1 - G(x)) \, , \\ b_1 \left(\frac{\mu^2}{m^2}\right) &= [16(1-x^2)C_F + (1+8x^2)C_A] \frac{\sigma(x)}{6x^2(1-x)} - \frac{2}{3x^2}(C_A - 2C_F) I(x) \\ &+ \frac{2}{3x} \tilde{I}_3^{(4)} C_A + [(1 + 3x - 10x^2 + 12x^3)C_A - 3(3 - 3x - 4x^2 + 8x^3)C_F] \frac{4}{3x} G^2(x) \\ &- [(147 - 4x - 100x^2 + 8x^3)C_A + 168(1-x)C_F + 6(9+4x)\ln(-4x)C_A] \frac{1}{9x} G(x) \\ &+ [(99 + 62x)C_A + 12(11 + 3x)C_F + 2(27 + 24x - 2x^2)\ln(-4x)C_A] \frac{1}{9x} \, , \end{aligned} \tag{16}$$

with $x = -\mu^2/(4m^2)$. This is our main result.

In Ref. [12] the β–function for QCD ($C_A = 3$, $C_F = 4/3$, $T_F = 1/2$) was evaluated in the standard approach with a renormalized coupling constant defined via the gluon-ghost-ghost vertex in the Landau gauge taken at the symmetric Euclidean point. The authors presented only an approximate result for the function

$$B_1(r) = \frac{34 C_A^2 - 3\beta_1}{4 T_F \, (5 C_A + 3 C_F)} \tag{17}$$

which corresponds to our function $b_1(r)$ and which they parametrized as

$$B_1(r) = \frac{(-0.45577 + 0.26995 r) r}{1 + 2.1742 r + 0.26995 r^2} \tag{18}$$

with $r = \mu^2/m^2$. As asserted in [12] the parametrization (18) has the maximum deviation from the true value in the entire range $0 \le r \le \infty$ smaller than 0.005. We find that the difference between our expression and (18) in the same region is less than 0.015, which is also very small. This is somewhat surprising, since we are comparing couplings in different schemes.

For a mass–dependent renormalization schemes the RG equations

$$\mu\frac{d}{d\mu}g_s(\mu) = \beta[g_s(\mu), m_j(\mu)/\mu] \ , \quad \mu\frac{d}{d\mu}m_i(\mu) = -\gamma_m[g_s(\mu), m_j(\mu)/\mu]\, m_i(\mu) \tag{19}$$

in general can be solved only by numerical integration. However, an approximate solution for the mass dependent effective QCD coupling was proposed in [19, 20]. Indeed, at the two-loop level the expression

$$\overline{\alpha_s}(Q^2) = \frac{\alpha_s}{1+\alpha_s/(4\pi)U_1 + \alpha_s/(4\pi)(U_2/U_1)\ln(1+\alpha_s/(4\pi)U_1)} \ , \tag{20}$$

with $U_{1,2}$ given in (5), correctly sums up all leading as well as "next-to-leading" terms $\alpha_s U_2(\alpha_s U_1)^n$ though it is not an exact solution of the two-loop differential RG equation. We will compare (20) with the result of the numerical integration of the RG equation below.

4 BF – MOM coupling in terms of the $\overline{\rm MS}$ coupling

Let us define the auxiliary functions

$$\begin{aligned} z_{1i} &= -\Pi_1(r_i) - \frac{20}{9} - \frac{4}{3}\, l_i \ , \\ z_{2i} &= -\Pi_2(r_i) - \left(\frac{52}{3} + \frac{20}{3}l_i\right) C_A - \left(\frac{55}{3} + 4l_i\right) C_F \ , \end{aligned} \tag{21}$$

where

$$l_i = \ln r_i \ , \qquad r_i = \frac{\mu^2}{m_i^2} \ .$$

For later use we note that for light fermions, utilizing the expansion (13), we obtain

$$\begin{aligned} z_{1i} &= -\frac{20}{9} + O(m_i^2/\mu^2) \ , \\ z_{2i} &= -\left(\frac{158}{9} + 8\,\zeta_3\right) C_A - \left(\frac{55}{3} - 16\,\zeta_3\right) C_F + O(m_i^2/\mu^2) \ . \end{aligned} \tag{22}$$

The relationship between the renormalized coupling constants may then be written in the form

$$\bar{h} \ = \ \frac{\alpha_{s\,\rm MOM}}{4\pi} = H(h,\mu^2) = h + k_1(\mu^2)h^2 + (k_2(\mu^2) + k_1^2(\mu^2))h^3 + \cdots \tag{23}$$

where

$$\begin{aligned} k_1(\mu^2) &= \frac{205}{36}C_A + T_F\sum_{i=1}^{n_F} z_{1i} \ , \\ k_2(\mu^2) &= \left(\frac{2687}{72} - \frac{57}{8}\zeta_3\right) C_A^2 + T_F\sum_{i=1}^{n_F} z_{2i} \ , \end{aligned} \tag{24}$$

and

$$h \equiv h_{\overline{\rm MS}} \;\; = \;\; \frac{\alpha_{s\,\overline{\rm MS}}}{4\pi} \; .$$

Differentiating the relation with respect to μ^2 we obtain:

$$\begin{aligned} \mu^2 \frac{d\bar{h}}{d\mu^2} \;\; &= \;\; \beta_{\rm MOM}(\bar{h}) = \frac{\partial H(h,\mu^2)}{\partial h}\beta(h) + \mu^2 \frac{\partial H(h,\mu^2)}{\partial \mu^2} \\ &= \;\; -\beta_{0\,\rm MOM}\,\bar{h}^2 - \beta_{1\,\rm MOM}\,\bar{h}^3 - \cdots \; , \end{aligned} \tag{25}$$

where $\beta(h)$ is the β function in the $\overline{\rm MS}$ scheme:

$$\beta(h) = -\beta_0 h^2 - \beta_1 h^3 - \cdots \; , \tag{26}$$

with

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F n_F \; , \\ \beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F n_F - 4 C_F T_F n_F \; . \end{aligned} \tag{27}$$

From the above equation we obtain

$$\begin{aligned} \beta_{0\,\rm MOM} &= \beta_0 - \mu^2 \frac{\partial}{\partial \mu^2} k_1(\mu^2), \\ \beta_{1\,\rm MOM} &= \beta_1 - \mu^2 \frac{\partial}{\partial \mu^2} k_2(\mu^2). \end{aligned} \tag{28}$$

As it should be, in the massless limit the β–functions agree.

5 BF − MOM versus $\overline{\rm MS}$ coupling: numerical aspects

For numerical studies we use the following pole quark masses [23]

$$m_u \sim m_d \sim m_s \sim 0 \; ; \;\; m_c = 1.55{\rm GeV} \; ; \;\; m_b = 4.70{\rm GeV} \; ; \;\; m_t = 173.80{\rm GeV} \; .$$

For the strong interaction coupling we take $\alpha_{s\,\overline{\rm MS}}^{(5)} = 0.12 \pm 0.003$ at scale M_Z=91.19 GeV [24]. In Fig. 1 we show that Shirkov's formula (20) provides an excellent approximation to the exact solution of the two–loop RG equation. At sufficiently large scales the mass effects in the β–function are small and we expect no large numerical differences between different schemes. This is illustrated in Fig. 2, where the evolution of the running couplings is shown for a common start value of $\alpha_s = 0.12$ at the scale $M_Z = 91.19$ GeV. Only the space–like $E = \sqrt{-q^2}$ is considered. We see that the mass effects are of comparable size as the 3–loop contribution [21, 22] in the $\overline{\rm MS}$ scheme (see Tab. 1 given below). The $\overline{\rm MS}$ results were obtained by adopting the Bernreuther–Wetzel (BW) [2] matching scheme between the effective theories with different flavors. We checked that utilizing Marciano (M) matching [3], instead, leads to answers somewhat closer to our MOM results. Only the latter one exhibit the correct physical mass behavior.

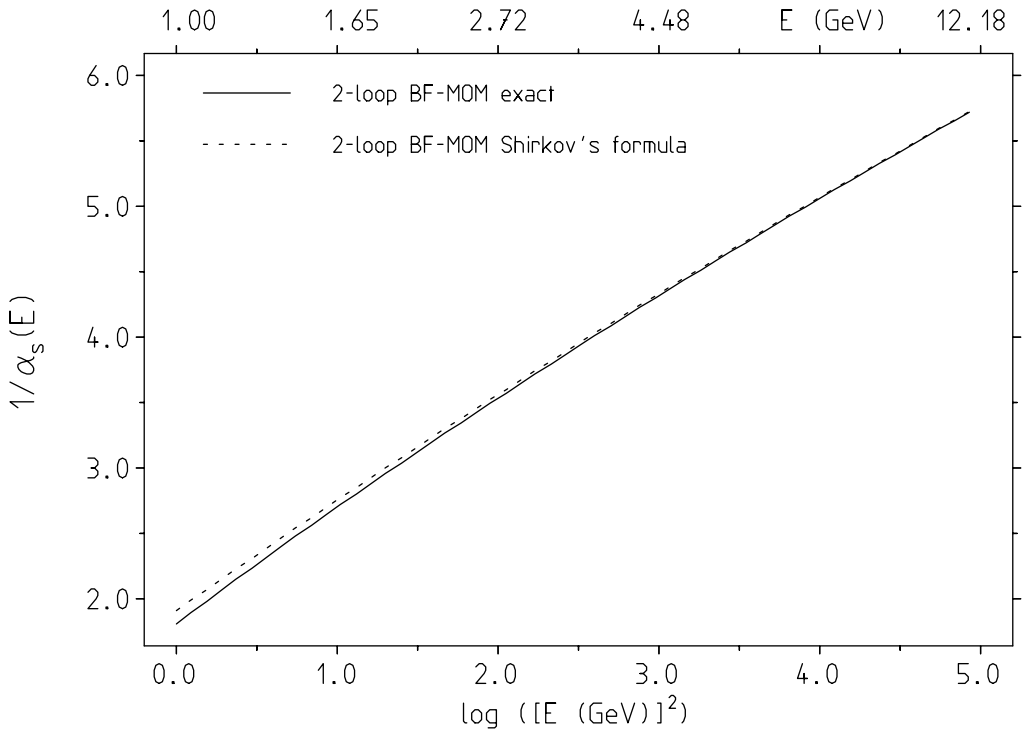

Figure 1: Evolution of α_s in the BF $-$ MOM scheme normalized to $\alpha_s = 0.12$ at the scale $M_Z = 91.19$ GeV. The dotted line represents the approximation by Shirkov's formula.

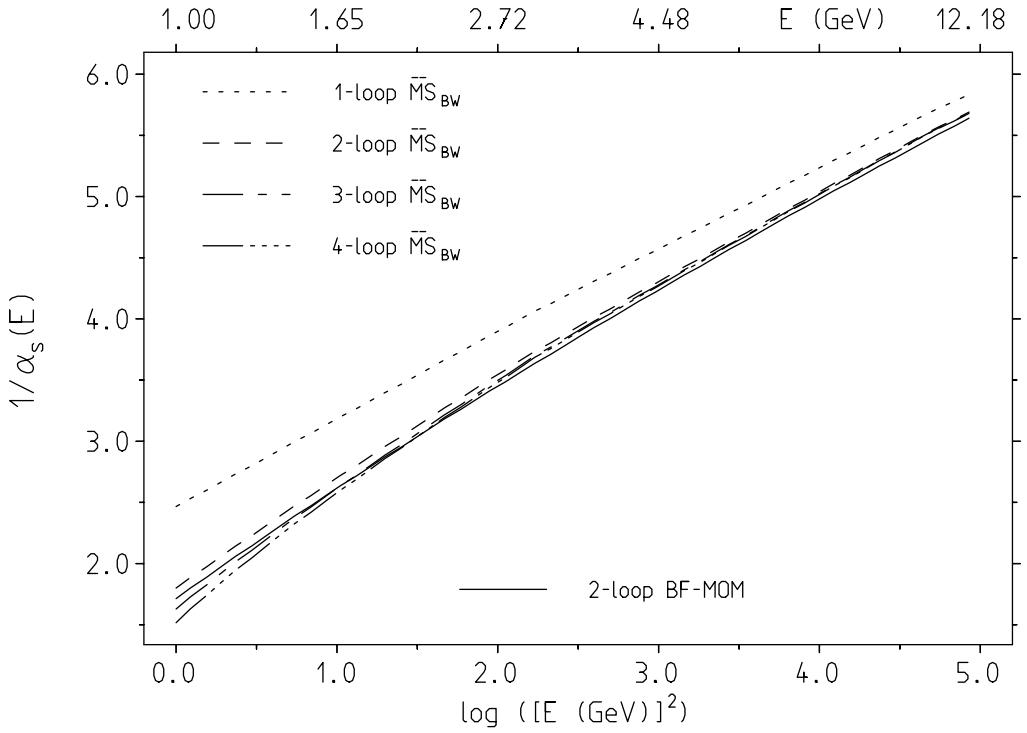

Figure 2: Comparison of the α_s evolution in the space–like region normalized to a common value $\alpha_s = 0.12$ at scale $M_Z = 91.19$ GeV. The dotted, dashed, dash–dot and the dash–dot–dot–dot curves show, respectively, the one–loop, two–loop, three–loop and the four–loop $\overline{\rm MS}$ evolution for BW–matching. The full line represents the exact BF $-$ MOM running coupling.

Although BW–matching seems to be better justified from a field theoretical point of view, it leads to "threshold jumps" which of course are not physical in the space–like region. In contrast M–matching assumes continuity of α_s across the matching scale ("thresholds").

While there are no really large numerical differences in the β–functions, i.e., the derivatives of α_s with respect to μ, down to moderately low scales, there are large μ– and hence mass–independent terms in the relationship between the coupling constants (23), as follows from (24) and (22), and as we can see in Fig. 3. A large constant shift in

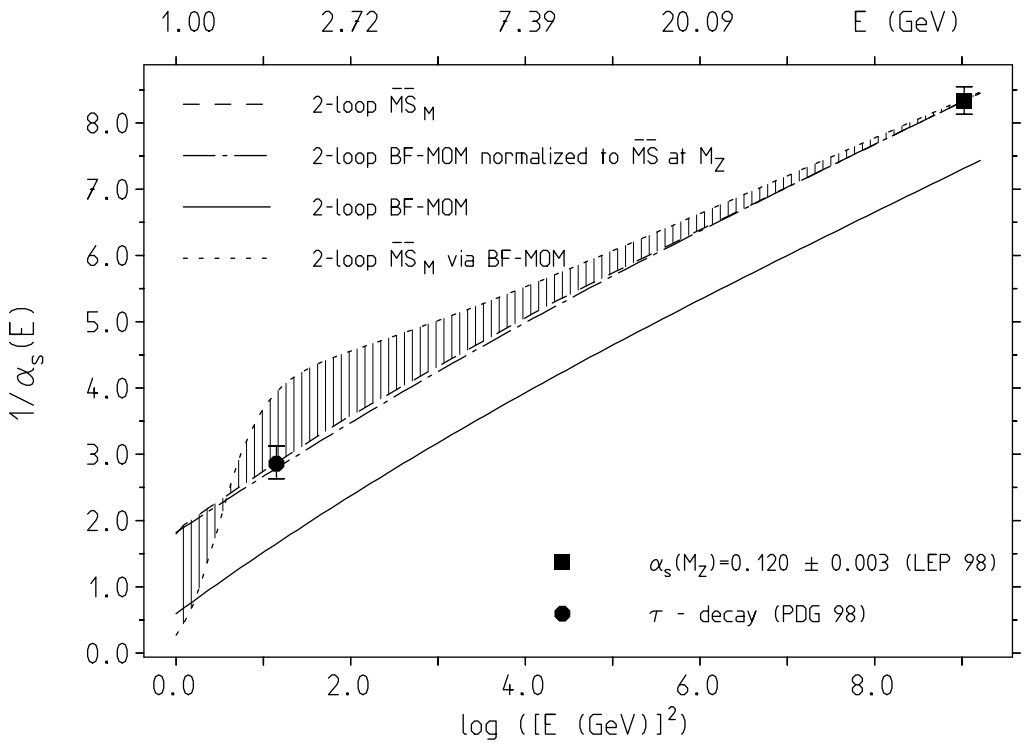

Figure 3: Comparison of α_s in the BF – MOM and the $\overline{\rm MS}$ schemes with $\alpha_{s\,\overline{\rm MS}}^{(5)} = 0.12$ at scale M_Z=91.19 GeV and $\alpha_{s\,\rm BF-MOM}$ at this scale calculated using (23). The dotted line is the $\overline{\rm MS}$ coupling calculated from $\alpha_{s\,\rm BF-MOM}(E)$ by inverting (23).

α_s of about plus 14% at M_Z is obtained when we go from the $\overline{\rm MS}$ to the MOM scheme. In principle, this does not affect the prediction of physical observables. However, the scheme dependence which is due to truncation errors of the perturbation expansion is different for different renormalization schemes. The shaded area of Fig. 3 reflects the theoretical uncertainty at the two–loop level which shows up in the comparison of the two schemes. Below about 1.15(2.92) GeV the one–loop correction $k_1(\mu^2)\,h$ in (23) exceeds the leading trivial term by 100(50)% and the perturbation expansion cannot be applied any longer (see Fig. 3).

The occurrence of the disturbing large numerical constants in the relationship between the renormalized couplings belonging to different renormalization schemes is not a

peculiar feature in the relation between the MOM and $\overline{\rm MS}$ schemes. Similar worrying lare terms, long time ago, were the reason for replacing the original MS by the $\overline{\rm MS}$ scheme [4], which are related by a simple rescaling of the scale parameter μ. Other, more sophisticated, examples of eliminating leading terms by rescaling were proposed in Ref. [25]. Also, for the comparison of non-perturbative calculations of running couplings in lattice QCD with perturbative results, the adequate choice of a relative scale factor turns out to be crucial [26]. The rescaling usually leads to dramatically improved agreement. A condition for the rescaling to make sense is that the β–functions of the two schemes under consideration do not differ too much numerically. For our two schemes this condition is fairly well satisfied (see Fig. 2). In fact at higher energies the β–functions become identical. Such a rescaling procedure thus looks natural if we tune the running couplings to agree with good accuracy at high energies. This can be achieved as follows: While (23) reads ($\tilde{k}_2 = k_2 + k_1^2$)

$$\bar{h}(\mu^2) = h(\mu^2) + k_1 h^2(\mu^2) + \tilde{k}_2 h^3(\mu^2) + O(h^4) \tag{29}$$

we may absorb the disturbing large term k_1 into a rescaling of μ by a factor x_0 such that [26]

$$\bar{h}((x_0\mu)^2) = h(\mu^2) + 0 + O(h^3)\ . \tag{30}$$

Expanding the RG solution (20) we have ($\tilde{U}_2 = U_2 - U_1^2$)

$$\begin{aligned} \bar{h}((x_0\mu)^2) &= \bar{h}(\mu^2) - U_1(x_0^2, \{m_i^2/\mu^2\})\bar{h}^2(\mu^2) - \tilde{U}_2(x_0^2, \{m_i^2/\mu^2\})\bar{h}^3(\mu^2) + O(\bar{h}^4) \\ &= h(\mu^2) + \Big(k_1 - U_1(x_0^2, \{m_i^2/\mu^2\})\Big)\ h^2(\mu^2) \\ &\quad + \Big(\tilde{k}_2 - 2U_1(x_0^2, \{m_i^2/\mu^2\})k_1 - \tilde{U}_2(x_0^2, \{m_i^2/\mu^2\})\Big)\ h^3(\mu^2) + O(h^4) \end{aligned} \tag{31}$$

and the rescaling factor x_0 is determined by the equation

$$k_1 = U_1(x_0^2, \{m_i^2/\mu^2\})\ . \tag{32}$$

In our mass dependent scheme we require this to be true only at very large scales $\mu^2 \gg m_f^2$ for all flavors f including the top quark. This convention is simple and most importantly, it does not conflict with the manifest decoupling property of the MOM scheme. As a consequence we obtain a running coupling which depends very little on the scheme at large energies, a property which looks most natural in an asymptotically free theory like QCD. For the BF − MOM scheme the rescaling factor x_0 is determined by

$$\ln(x_0^2) = \left(\frac{205}{36}C_A - \frac{20}{9}T_F n_F\right) / \left(\frac{11}{3}C_A - \frac{4}{3}T_F n_F\right) = 125/84 \tag{33}$$

for QCD with $n_F = 6$ flavors. Numerically we find $x_0 \simeq 2.0144$.

In order to check whether the above rescaling makes sense, we must inspect the change of the 2–loop coefficient in the rescaled relationship (32) between MOM and $\overline{\rm MS}$. Indeed, the rescaling changes the coefficients from $k_1 \simeq 10.42$, $\tilde{k}_2 \simeq 126.35$ to $k_{1\,\rm eff} = 0$, $\tilde{k}_{2\,\rm eff} \simeq -32.46$ and thus we get a substantial improvement for the next to leading coefficient too, as it should be. We note that the rescaling improved MOM

perturbation expansion at low energies does not any longer deviate substantially from the $\overline{\rm MS}$ results. Of course, only the appropriate higher order calculations of observables in the BF − MOM scheme could reveal the true convergence properties of the perturbation series in this scheme.

In the MOM scheme the energy scale comes in by a momentum subtraction and the location of the thresholds of course cannot depend on the rescaling "reparametrization". This means that actually the scale must be changed in the $\overline{\rm MS}$ scheme, where the scale parameter μ enters in a purely formal way and "thresholds" are put in by hand for switching between the effective theories of different numbers of flavors. Since, conventionally, μ in the $\overline{\rm MS}$ scheme has already been identified with the c.m. energy, for example, in the LEP determination of $\alpha_s(M_Z)$ which we use as an input, we have to apply the rescaling to the MOM calculation. As the thresholds must stay at their "physical" location, i.e., $4m^2/q^2$ must remain invariant, we have to perform the scaling simultaneously to the energy and the masses.

The result from utilizing this rescaling procedure is displayed in Fig. 4. The large deviations seen in Fig. 3 have disappeared now. The sizes of effects are still illustrated by what we observe in Fig. 2 except that the initial values at M_Z differ. In Fig. 4 we have recalculated the input values of $\alpha_s^{(5)}(M_Z)$ as a function of the perturbative order, assuming the observable $R(s)$ to have a given experimental value. $R(s)$ is the ratio of hadronic to leptonic e^+e^-–annihilation cross sections at sufficiently large s, from which a precise determinations of $\alpha_s(s)$ is possible. At our reference scale M_Z we may use perturbative QCD in the massless approximation [27]

$$R(s) \;=\; 3\sum_f Q_f^2\,\left(1 + a + c_1 a^2 + c_2 a^3 + \cdots\right) \tag{34}$$

where Q_f denotes the charge of the quark, $a = 4h = \alpha_s(s)/\pi$, and

$$\begin{aligned} c_1 &= 1.9857 - 0.1153\, n_F \\ c_2 &= -6.6368 - 1.2002\, n_F - 0.0052\, n_F^2 - 1.2395\,(\textstyle\sum Q_f)^2/(3\sum Q_f^2) \end{aligned}$$

in the $\overline{MS}$ scheme, with $n_F = 5$ active flavors.

Some concluding remarks: We have investigated a MOM renormalization scheme in the background field gauge at the two–loop order in QCD and shown that a substantial scheme dependence is observed relative to the $\overline{\rm MS}$ scheme, unless we apply a suitable rescaling. These findings are in accord with earlier investigations at the one–loop [6,28] and two–loop [12] level. Mass effects in any case are non-negligible at a level of precision where also higher order corrections are relevant. The calculation in full QCD includes the exact mass effects and is smooth and analytic at all scales and in particular across thresholds. It thus avoids problems with the $\overline{\rm MS}$ scheme addressed in a recent article by Brodsky et al. [29] which were cured by an analytic extension of the $\overline{\rm MS}$ renormalization scheme.

We note that the use of the BF − MOM scheme, particularly when using the compact form obtained for Shirkov's approximation, is much easier in practice because decou-

pling is manifest at any threshold and there are no matching conditions to be imposed.

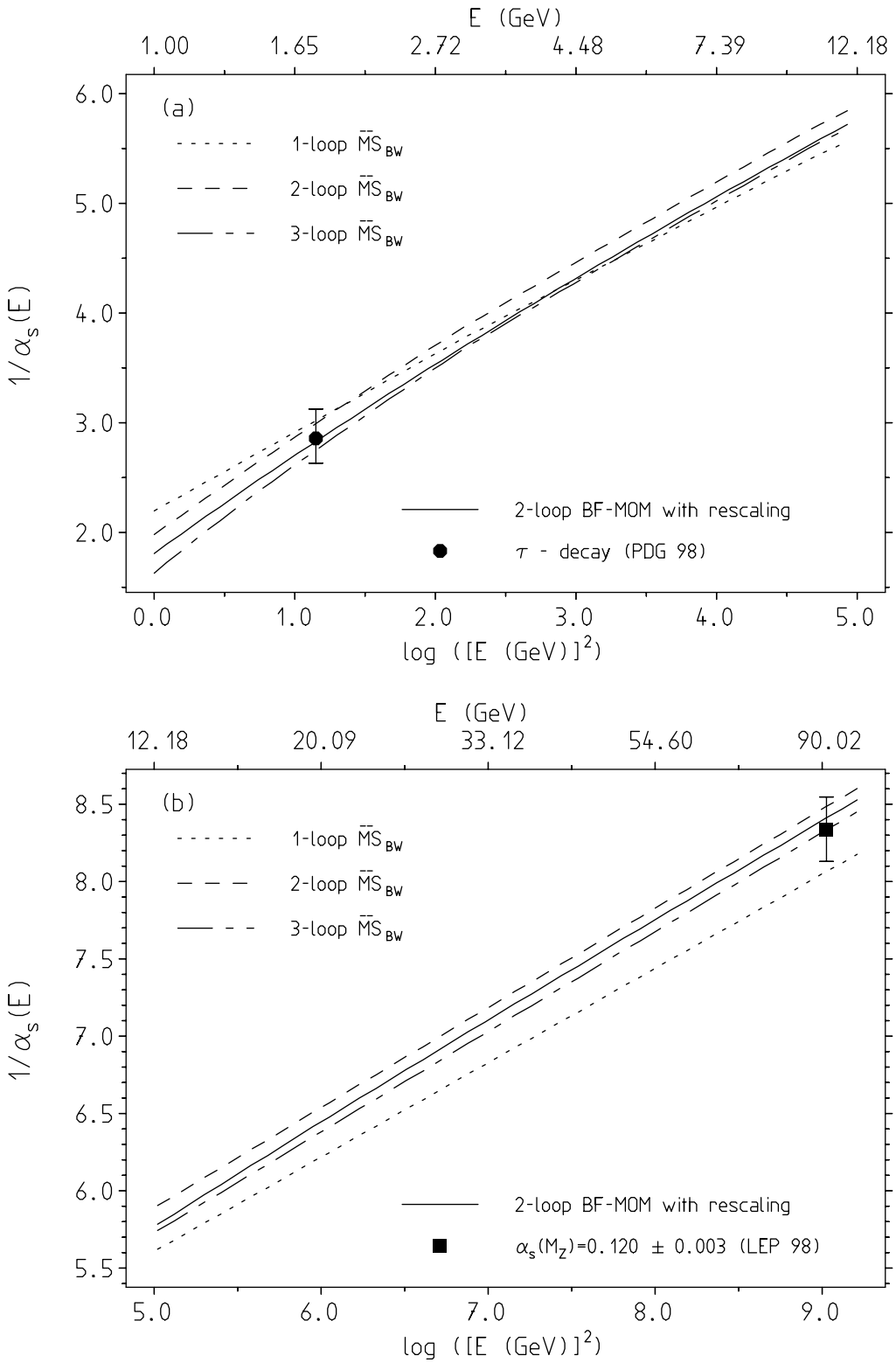

Figure 4: Comparison of $\alpha_s((x_0E)^2)$ in the BF − MOM and $\alpha_s(E^2)$ in the $\overline{\rm MS}$ scheme with input values $\alpha_{s\,\overline{\rm MS}}^{(5)} = 0.120$ at scale M_Z=91.19 GeV and $\alpha_{s\,\rm BF-MOM} = 0.1189$ obtained for the rescaled energy $x_0M_Z \simeq$190.90 GeV ($x_0 \simeq 2.1044$).

We emphasize that the scheme and scale dependence of perturbativ QCD predictions is not a matter of the order of perturbation theory alone but may depend substantially on other details like the kind of matching condition applied in the mass independent $\overline{\rm MS}$ schemes or the threshold and mass effects in MOM schemes. The following table (Tab. 1) may illustrate the kind of uncertainties we expect to encounter. We find that

Table 1: Comparison of predicted α_s values at the masses of the Υ, J/ψ and τ. Here we adopt a common input value $\alpha_s(M_Z) = 0.12$ for the $\overline{\rm MS}$ scheme independent of the perturbative order.

scheme	$\alpha_s(M_Z)$ (input)	$\alpha_s(M_\Upsilon)$	$\alpha_s(M_{J/\psi})$	$\alpha_s(M_\tau)$
$\overline{\rm MS}$ 2-loop (BW)	0.120	0.179	0.260	0.354
$\overline{\rm MS}$ 3-loop (BW)	0.120	0.179	0.262	0.364
$\overline{\rm MS}$ 4-loop (BW)	0.120	0.179	0.263	0.368
$\overline{\rm MS}$ 2-loop (M)	0.120	0.179	0.258	0.348
$\overline{\rm MS}$ 2-loop via BF − MOM	0.120	0.168	0.211	0.254
(BF − MOM 2-loop	0.120	0.180	0.265	0.358)
BF − MOM 2-loop	0.137	0.222	0.372	0.605
BF − MOM 2-loop rescaled	0.121	0.181	0.260	0.345

the 2–loop $\overline{\rm MS}$ value at the τ–mass M_τ is $\alpha_s = 0.254$ when we switch from $\overline{\rm MS}$ to BF − MOM at the Z–mass M_Z use BF − MOM evolution down to M_τ and switch back from BF − MOM to $\overline{\rm MS}$. Standard (direct) $\overline{\rm MS}$ evolution depends on the matching scheme utilized (BW or M) and for BW(M)–matching yields $\alpha_s = 0.354(0.348)$, such that via BF − MOM we get a value which is lower by 0.100(0.094). However, the BF − MOM value obtained with the rescaling is 0.345, not very different from its $\overline{\rm MS}$ value. The Particle Data Group [23] quotes $\alpha_s(M_\tau) = 0.35 \pm 0.03$ for the experimental value obtained from τ–decays (see also [30]).

Experience with many physical applications of the $\overline{\rm MS}$ scheme somehow established this scheme as a preferred one, in the spirit that this prescription is better than others in the sense that it leads to reliable perturbative predictions for many physical observables. In our opinion it remains unclear whether a prefered scheme exist. The problem is the appropriate choice of scale. We advocate here to take more serious the physical mass dependence. In order to get a better understanding of the scheme dependences we need more calculations in different schemes.

Acknowledgments

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Appendix A: Bare BF propagator

$$
\begin{aligned}
(d-1)(d-4)U_{2\,\mathrm{bare}} &= \frac{c_1}{16(d-4)(d-6)}C_A^2 J_{111}(0,0,0) + \frac{q^2 c_2}{64}C_A^2 G_{11}^2(0,0) \\
&+T_F\sum_{i=1}^{n_F}\Big\{32 z_i m_i^4(d-4)\ C_A\ \tilde{I}_3^{(d)}(z_i) + \frac{4f_1(z_i)}{1-z_i}(C_A-2C_F)\ m_i^2 G_{11}^2(m_i^2,m_i^2) \\
&+2(d-4)f_2(z_i)\ C_A\ m_i^2 G_{11}(0,0)G_{11}(m_i^2,m_i^2) \\
&+\frac{4}{1-z_i}\left[\frac{d-2}{d-3}f_3(z_i)C_F + 2f_4(z_i)C_A\right]G_{11}(m_i^2,m_i^2)G_{10}(m_i^2,0) \\
&+\frac{f_5(z_i)}{15}(d-2)(d-4)\ C_A\ G_{11}(0,0)G_{10}(m_i^2,0) \\
&+\left[\frac{4f_6(z_i)}{d-3}C_F + \frac{f_7(z_i)C_A}{15z_i(1-z_i)}\right]4m_i^2 J_{112}(0,m_i^2,m_i^2) \\
&+\left[4(d-1)(d-2)(d-4)C_F + \frac{f_8(z_i)C_A}{15z_i(1-z_i)}\right]\ J_{111}(0,m_i^2,m_i^2) \\
&-\left[\frac{f_9(z_i)C_A}{30z_i(d-5)} + 2\ f_{10}(z_i)C_F\right]\frac{(d-2)}{(1-z_i)(d-3)m_i^2}\ G_{10}^2(m_i^2,0)\Big\}\,.
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
c_1 &= (3d-8)\xi\left[-(d-1)(d^2-9d+22)(d-4)^2\xi^2 - (15d^5-256d^4+1685d^3\right. \\
&\left.-5292d^2+7744d-3960)\xi + 33d^5 - 542d^4 + 3311d^3 - 9378d^2 + 12448d - 6608\right] \\
&+27112d - 8128 - 33312d^2 - 51d^6 + 916d^5 + 20016d^3 - 6169d^4,
\end{aligned}
$$

$$
\begin{aligned}
c_2 &= (d-1)(d-4)^2[(d-4)\xi + 6(2d-7)]\xi^3 + 2(11d^4 - 144d^3 + 677d^2 - 1296d + 743)\xi^2 \\
&-(84d^4 - 854d^3 + 2994d^2 - 4304d + 2384)\xi + 49d^4 - 403d^3 + 1106d^2 - 1392d + 664,
\end{aligned}
$$

$$
f_1 = (d-2)[(d^2-7d+16)z - (d-5)(d-4)]z - 2,
$$

$$
f_2 = (d-2)[(d-4)\xi^2 - 7d + 12]z + (d-4)\xi^2 + 2((d-2)z+4)(3d-10)\xi - 7d + 16,
$$

$$
f_3 = 2(d-2)(d^2-5d+8)z^2 + (d-1)(d-3)(d-4)^2 z + d^3 - 6d^2 + 5d + 8,
$$

$$
f_4 = (d-2)(1-2z)[(d-2)z+1],
$$

$$
\begin{aligned}
f_5 &= 4[(d-1)\xi + 3d - 7](d-4)z^2 - 10[(d-1)(3d-8)\xi - 7d^2 + 23d - 28]z \\
&-15(d-4)\xi^2 - 30(3d-10)\xi + 15(7d-16)
\end{aligned}
$$

$$
f_6 = (d-2)[(d^2-5d+8)z - d^2 + 7d - 10],
$$

$$
\begin{aligned}
f_7 &= 2(d-4)^2[(d-1)\xi + 3d - 7]z^4 - \Big[3(7d-20)(d-1)(d-4)\xi - 172d - 17d^3 + 81d^2 \\
&+240\Big]z^3 - [(637d^2 - 2331d - 54d^3 + 2708)\xi - 713d^2 + 2025d + 78d^3 - 1924]z^2 \\
&-[(d-6)(53d^2 - 398d + 729)\xi + 2202 - 81d^3 + 800d^2 - 2447d]z \\
&+(2d-7)(d-7)(9d-41)\xi - 26d^3 + 275d^2 - 892d + 787,
\end{aligned}
$$

$$
\begin{aligned}
f_8 &= 2(d-3)(d-4)(3d-8)[(d-1)\xi+3d-7]z^3 \\
&\quad -[(3d-8)(19d^3-214d^2+715d-712)\xi-3891d^2+7392d-5248+880d^3-69d^4]z^2 \\
&\quad +(d-4)[(5d-17)(7d-39)(3d-8)\xi-165d^3+1424d^2-3819d+3192]z \\
&\quad -(d-7)(9d-41)(2d-7)(3d-8)\xi+(3d-8)(26d^3-275d^2+892d-787),
\end{aligned}
$$

$$
\begin{aligned}
f_9 &= 2(d-5)(d-3)(d-4)^2[(d-1)\xi+3d-7]z^3 \\
&\quad -\Big[(d-1)(d-4)(19d^3-208d^2+711d-754)\xi-6266d+345d^4+2888-23d^5 \\
&\quad +5095d^2-1943d^3\Big)z^2+(d-2)\Big[(d-6)(35d^3-401d^2+1521d-1923)\xi \\
&\quad -(55d^4-817d^3+4375d^2-9799d+7434)\Big]z-[(d-7)(2d-7)(9d-41)\xi \\
&\quad -(26d^3-275d^2+892d-787)\Big](d-2)(d-5),
\end{aligned}
$$

$$
f_{10} = (d-2)[(d^2-5d+8)z+d^3-7d^2+16d-14]. \tag{36}
$$

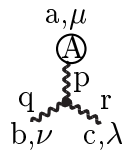
In the above formulae we have used the following notation:

$$
\begin{aligned}
\tilde{I}_3^{(d)} &= \int\int \frac{d^dk_1\ d^dk_2}{\pi^d\ k_1^2(k_2^2-m^2)(k_1-q)^2((k_2-q)^2-m^2)((k_1-k_2)^2-m^2)}\ , \\
J_{\alpha\beta\gamma}(m_1^2,m_2^2,m_3^2) &= \int\int \frac{d^dk_1\ d^dk_2}{\pi^d\ (k_1^2-m_1^2)^\alpha((k_2-q)^2-m_2^2)^\beta((k_1-k_2)^2-m_3^2)^\gamma}\ , \\
G_{\alpha\beta}(m_1^2,m_2^2) &= \int \frac{d^dk_1}{\pi^{d/2}} \frac{1}{(k_1^2-m_1^2)^\alpha((k_1-q)^2-m_2^2)^\beta}\ .
\end{aligned} \tag{37}
$$

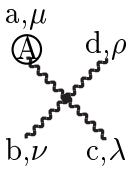
All parameters are the bare one's, $z_i = q^2/(4m_i^2)$ and the coefficient functions $f_n = f_n(z_i)$ are functions of z_i.

Appendix B: The BF Feynman rules.

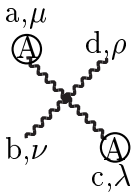
In addition to the conventional QCD Feynman rules we have:



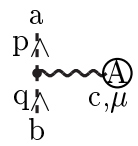
$$gf_{abc}\left[g_{\mu\lambda}\,(p-r-\tfrac{1}{\xi}q)_\nu + g_{\nu\lambda}\,(r-q)_\mu + g_{\mu\nu}\,(q-p+\tfrac{1}{\xi}r)_\lambda\right] \qquad (\xi=\infty \text{ standard triple vertex})$$



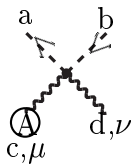
$$-ig^2\left[f_{abx}f_{xcd}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}) + f_{adx}f_{xbc}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}) + f_{acx}f_{xbd}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\nu\lambda})\right] \qquad (=\text{ standard quartic vertex})$$



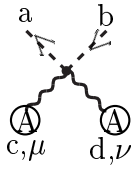
$$-ig^2\left[f_{abx}f_{xcd}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}+\tfrac{1}{\xi}g_{\mu\nu}g_{\lambda\rho}) + f_{adx}f_{xbc}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-\tfrac{1}{\xi}g_{\mu\rho}g_{\nu\lambda}) + f_{acx}f_{xbd}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\nu\lambda})\right] \qquad (\xi=\infty \text{ standard quartic vertex})$$



$$-gf_{abc}\,(p-q)_\mu \qquad (q_\mu=0 \text{ standard gluon–ghost vertex})$$



$$-ig^2 f_{acx}f_{xdb}g_{\mu\nu}$$

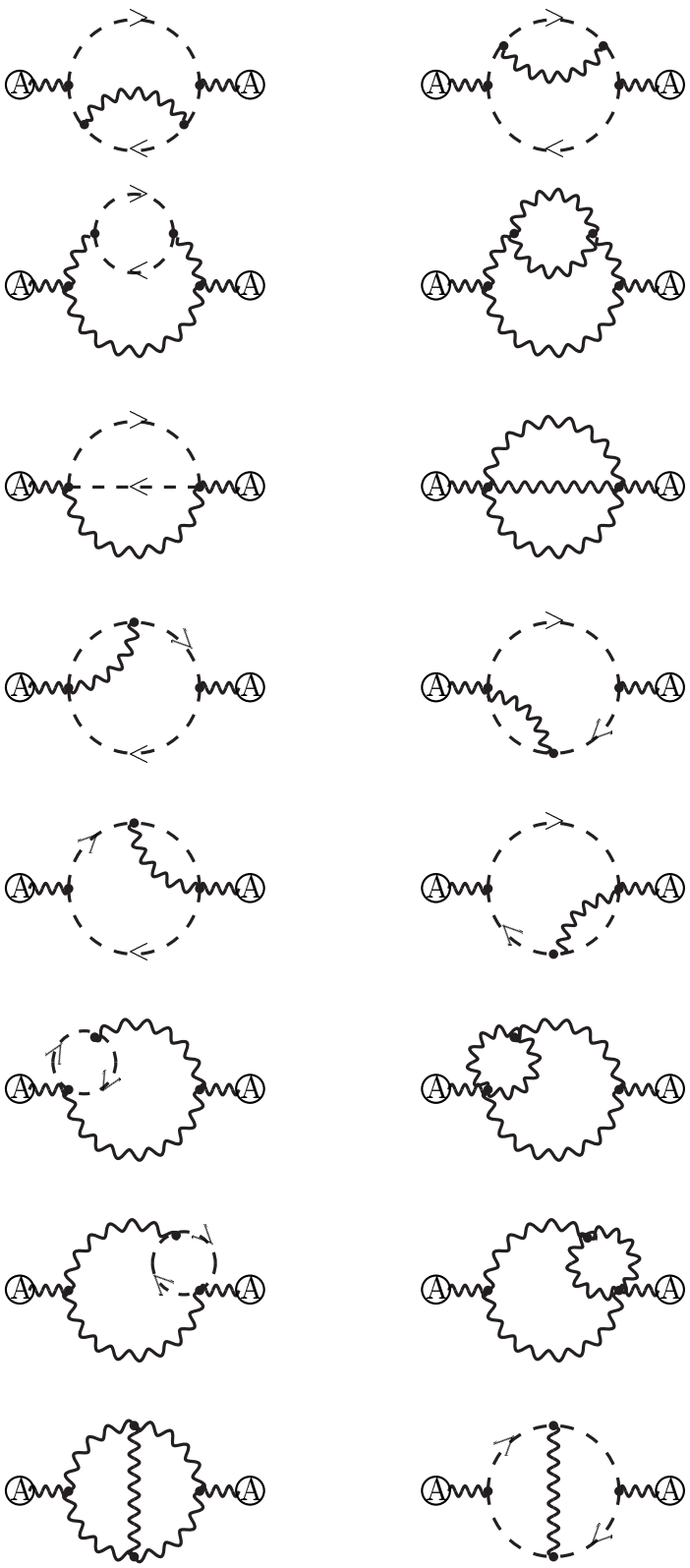


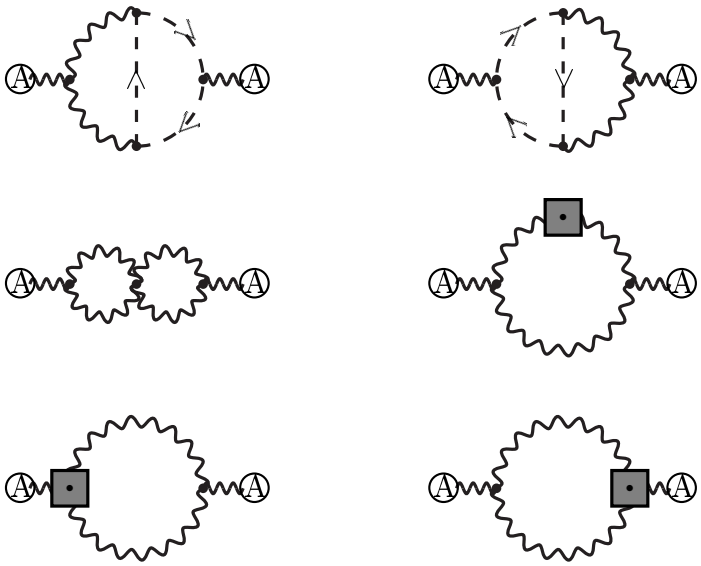
$$-ig^2 g_{\mu\nu}(f_{acx}f_{xdb}+f_{adx}f_{xcb})$$

All momenta are taken to be outgoing.

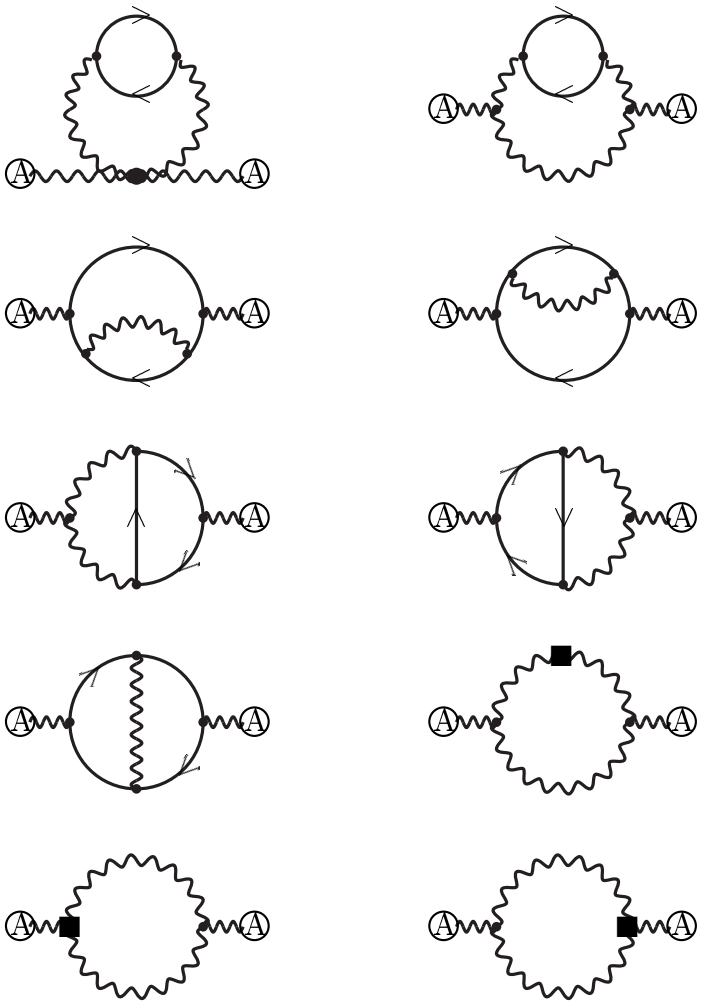
Appendix C: BF propagator diagrams.

a) Pure Yang–Mills contributions to the BF propagator [8]





b) Fermionic contributions to BF propagator [10]



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