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Two-loop QCD corrections of the massive fermion propagator [1)]

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Abstract

The off-shell two-loop correction to the massive quark propagator in an arbitrary covariant gauge is calculated and results for the bare and renormalized propagator are presented. The calculations were performed by means of a set of new generalized recurrence relations proposed recently by one of the authors [1]. From the position of the pole of the renormalized propagator we obtain the relationship between the pole mass and the $\overline{\rm MS}$ mass. This relation confirms the known result by Gray et al. [2]. The bare amplitudes are given for an arbitrary gauge group and for arbitrary space-time dimensions.

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[1)] *In memoriam* Leonid Viktorovich Avdeev, 1959-1998

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1 Introduction

In electroweak precision physics there is ongoing interest to improve the status of precise theoretical predictions of the Standard Model. Attempts are being made by several groups to provide more complete two-loop calculations. The major obstacles in doing so are still lacking methods to perform accurate numerical calculations of two-loop integrals. Unlike in the one-loop case integrals cannot be performed analytically in general. Recently substantial progress was made in the development of efficient algorithms which allow us to calculate two–loop diagrams for arbitrary masses in a canonical way [1]. In the present paper we describe a simple application of these techniques: the calculation of the two-loop quark propagator in Quantum Chromodynamics (QCD) with one massive quark and the others massless. Bare expressions are obtained for an arbitrary gauge group and for arbitrary space-time dimensions. The knowledge of the full off-shell propagator is of particular interest for the discussion of the scheme dependence of radiative corrections, since it allows us to relate not only the pole mass M to the $\overline{\rm MS}$ mass m but also to the so called momentum subtraction schemes [3] (MOM) and the Euclidean mass definition [4]. In lattice QCD a MOM subtraction scheme was recently proposed for the non-perturbative operator renormalization with matching conditions to perturbation theory far off-shell [5]. These considerations are of great physical interest in particular in the context of decoupling of heavy particles and the effective QCD schemes where it is required to relate QCD in the $\overline{\rm MS}$ scheme for effective theories with different number of (light) flavors [6].

The two-loop corrections to the electron propagator were first considered in Quantum Electrodynamics (QED) [7]. In QCD so far only the relationship between the pole mass and the $\overline{\rm MS}$ mass was investigated to two-loops [2]. A similar on-shell calculation, within the dimensional reduction scheme, was given in [8]. A detailed discussion of the mass renormalization was presented in [9] for QED, in [10] for the electroweak Standard Model and in [11] for grand unified theories.

In sect. 2, we introduce some notation for the quark propagator. The calculation and the results for the renormalized amplitudes are discussed in sect. 3, while sect. 4 is devoted to the relationship between the pole mass and the $\overline{\rm MS}$ mass. Technical details and bare amplitudes will be given in a few appendices.

2 The quark propagator

The renormalization of the fermion propagator is conveniently discussed by looking at the full inverse bare propagator first

$$S_{\rm F}^{-1}(q) = \not{q} - m_0 - \Sigma(q)\,, \tag{1}$$

where $\Sigma(p)$ is the one-particle irreducible fermion self-energy. We write the inverse bare propagator as

$$S_{\rm F}^{-1}(q) = \not{q} B_{\rm bare} - m_0 A_{\rm bare} \tag{2}$$

in terms of two dimensionless scalar amplitudes $A_{\rm bare}, B_{\rm bare}$, which are functions of the four-momentum square q^2, the bare mass m_0, the bare QCD coupling constant $\alpha_{s0} \equiv g_0^2/(4\pi)$ and the bare gauge parameter ξ_0. Off the mass-shell, when $q^2 \neq M^2$,

the bare amplitudes are *infrared finite* but ultraviolet singular. As usual we start from dimensionally regularized bare amplitudes. Renormalization of the fermion propagator is performed order by order in perturbation theory by writing the bare mass as a sum of the renormalized mass plus a mass counterterm

$$m_0 = m + \delta m = Z_m\, m\,, \qquad Z_m \equiv 1 + \frac{\delta m}{m} \tag{3}$$

and, correspondingly, for the bare strong coupling constant

$$\alpha_{s0} = \alpha_s + \delta\alpha_s = Z_g\, \alpha_s\,, \qquad Z_g \equiv 1 + \frac{\delta\alpha_s}{\alpha_s}\,, \tag{4}$$

while for the bare gauge parameter we have

$$\xi_0 = Z_3 \xi\,. \tag{5}$$

We define ξ such that $\xi = 1$ in the Feynman gauge and Z_3 is the wave-function renormalization factor of the gluon field. Dimensional regularization is used with $d = 4 - 2\varepsilon$ being the dimension of space-time. The Feynman diagrams are divided by $i\pi^{d/2}\exp(-\gamma\varepsilon)\mu^{-2\varepsilon}$ per loop with μ the $\overline{\rm MS}$ renormalization scale, γ the Euler constant. The renormalized fermion propagator is obtained by multiplicative renormalization with a suitable wave-function renormalization factor Z_2

$$S_{\rm F\,ren}(q) = \frac{1}{Z_2} S_{\rm F}(q)\,. \tag{6}$$

We write the inverse propagator in the form

$$S^{-1}_{\rm F\,ren}(q) = \not{q}\, B_{\rm ren} - m\, A_{\rm ren}\,. \tag{7}$$

Explicitly, we obtain the renormalized amplitudes in the $\overline{\rm MS}$ scheme by applying the following $\overline{\rm MS}$ renormalization constants [12]:

$$\begin{aligned}
Z_2 &= 1 + \frac{\alpha_s}{4\pi}\,(-\xi C_F)\,\frac{1}{\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\left(\frac{\xi^2}{2} C_F + (\frac{3\xi}{4} + \frac{\xi^2}{4}) C_A\right)\frac{1}{\varepsilon^2} \right. \\
&\quad \left. + \left(-(\frac{25}{8} + \xi + \frac{\xi^2}{8}) C_A + t n_f + \frac{3}{4} C_F\right)\frac{1}{\varepsilon} \right) C_F\,, \\
Z_3 &= 1 + \frac{\alpha_s}{8\pi}\left((\frac{13}{3} - \xi) C_A - \frac{8}{3} t n_f\right)\frac{1}{\varepsilon}\,, \\
Z_g &= 1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{3} C_A - \frac{4}{3} t n_f\right)\frac{1}{\varepsilon}\,, \\
Z_m &= 1 - \frac{\alpha_s}{4\pi}\,(3 C_F)\,\frac{1}{\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 C_F \left((\frac{11}{2} C_A - 2 t n_f + \frac{9}{2} C_F)\frac{1}{\varepsilon^2} \right. \\
&\quad \left. - (\frac{97}{12} C_A - \frac{5}{3} t n_f + \frac{3}{4} C_F)\frac{1}{\varepsilon} \right)\,.
\end{aligned} \tag{8}$$

For $SU(N_c)$ we have $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, $t = 1/2$. Since the latter coefficient does not depend on the number of colors N_c its value will be inserted in most of the formulae presented below.

3 Renormalized Amplitudes

There are six diagrams contributing to $\Sigma(q)$ at the two-loop level (see Fig. 1).

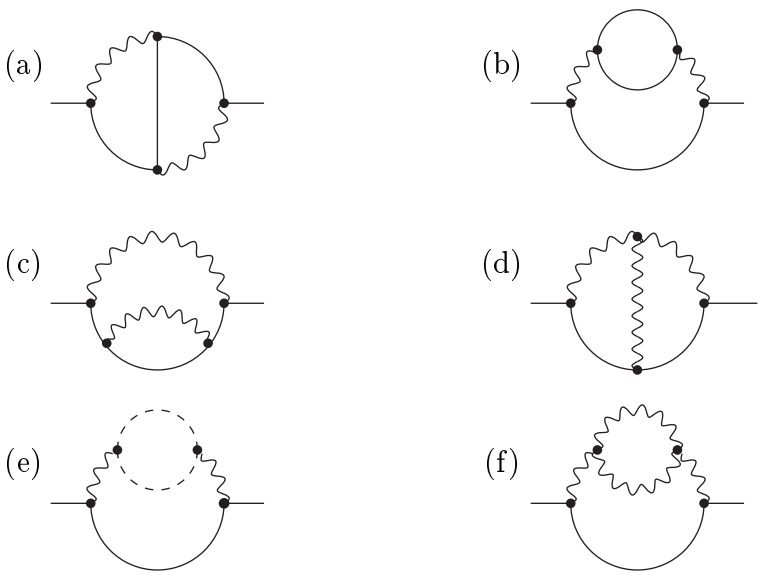

Fig. 1. Fermion self-energy diagrams to two-loops. Solid lines denote quarks, wavy lines gluons and dashed lines Fadeev–Popov ghosts.

In our calculation we consider the linear covariant gauge with arbitrary gauge parameter ξ. UV and IR singularities are dealt with by dimensional regularization and cancel in the observable quantities. A major problem in performing higher loop calculations with arbitrary masses is to find an appropriate basis of integrals which allows us to present results in a compact form. In most cases integrals can be reduced to a set of master integrals by using partial fraction decomposition, by differentiation and by integration by parts. The remaining problem of integrals with irreducible numerators was solved recently in [13]. Such integrals may be expressed in terms of scalar integrals with dimensions shifted by multiples of 2 which then are reduced again to integrals of dimension $d = 4 - 2\varepsilon$ (generic). The resulting complete set of relations between integrals, a set of generalized recurrence relations, was considered in [1] and has been implemented in a computer program written in FORM [14]. All diagrams finally could be expressed in terms of 5 two-loop integrals I_3, $J_{111}(m^2,m^2,m^2)$, $J_{112}(m^2,m^2,m^2)$, $J_{111}(0,0,m^2)$, $J_{112}(0,0,m^2)$ and 3 products of the one-loop integral structures $G_{11}(0,m^2)$ and $G_{01}(0,m^2)$, where

$$
\begin{aligned}
I_3(q^2/m^2) &= -q^2 \iint \frac{d^dk_1\; d^dk_2}{\pi^d\; k_1^2(k_2^2-m^2)((k_1-q)^2-m^2)(k_2-q)^2((k_1-k_2)^2-m^2)}\;, \\
J_{\alpha\beta\gamma}(m_1^2,m_2^2,m_3^2) &= \iint \frac{d^dk_1\; d^dk_2}{\pi^d\; (k_1^2-m_1^2)^\alpha((k_2-q)^2-m_2^2)^\beta((k_1-k_2)^2-m_3^2)^\gamma}\;, \\
G_{\alpha\beta}(m_1^2,m_2^2) &= \int \frac{d^dk_1}{\pi^{d/2}} \frac{1}{(k_1^2-m_1^2)^\alpha((k_1-q)^2-m_2^2)^\beta}\;. \qquad (9)
\end{aligned}
$$

The two-loop integrals J_{111}, J_{112} and the one-loop integral G_{11} with zero masses can be expressed in terms of hypergeometric functions [15] as follows

$$J_{111}(0,0,m^2) = \quad i^2(m^2)^{d-3}\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2}-1)\Gamma(3-d)\,{}_2F_1\left[\begin{matrix}2-\frac{d}{2}, 3-d\,; \\ \frac{d}{2}\,;\end{matrix}\ \frac{q^2}{m^2}\right],$$

$$J_{112}(0,0,m^2) = -i^2(m^2)^{d-4}\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2}-1)\Gamma(4-d)\,{}_2F_1\left[\begin{matrix}2-\frac{d}{2}, 4-d\,; \\ \frac{d}{2}\,;\end{matrix}\ \frac{q^2}{m^2}\right],$$

$$G_{11}(0,m^2) = -i(m^2)^{d/2-2}\Gamma(1-\frac{d}{2})\,{}_2F_1\left[\begin{matrix}1, 2-\frac{d}{2}\,; \\ \frac{d}{2}\,;\end{matrix}\ \frac{q^2}{m^2}\right]. \qquad (10)$$

In appendix A their expansions up to order $O(\varepsilon^2)$ are given. G_{01} is elementary:

$$G_{01}(0,m^2) = -i(m^2)^{d/2-1}\Gamma(1-\frac{d}{2}).$$

Finally, J_{111} and J_{112} with equal masses may be written in the form

$$J_{111}(m^2,m^2,m^2) = m^{2-4\varepsilon}\Gamma^2(1+\varepsilon)\left[-\frac{3}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{z}{4}-\frac{9}{2}\right)+J_3(z)\right],$$
$$J_{112}(m^2,m^2,m^2) = m^{-4\varepsilon}\Gamma^2(1+\varepsilon)\left[-\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}+3-\frac{z}{6}+\frac{1}{3}J_3(z)-\frac{1}{3}J_3'(z)\right], \quad (11)$$

where $z = q^2/m^2$ and $J_3'(z) = zd/dzJ_3(z)$. The numerical evaluation of I_3, J_3 and J_3' will be discussed in appendix B. Let us mention that results from individual diagrams are rather lengthy such that we must refrain from presenting individual contributions.

The amplitudes in (7) are calculated from the diagrams of Fig. 1 where diagram (b) was taken with $n_f - 1$ massless fermions and one massive one in the loop. We find

$$\begin{aligned}
A_{\rm ren} &= 1+\frac{\alpha_s}{4\pi}C_F\left\{4+2\xi+(\xi+3)(\frac{\bar z}{z}\ln\bar z+L_\mu)\right\}\\
&+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{C_F^2\left[\frac{3}{z}-\frac{260}{\bar z}-103-4z-8\xi-\frac{1+z}{z}\xi^2-(1-6\xi+\xi^2)\Big(\zeta_2+\mathrm{Li}_2(z)\Big)\right.\right.\\
&-\frac{2}{z^2}\left(1-31z+18z^2-\xi z(9-5z)-\xi^2(1-z^2)\right)\ln\bar z+2(\xi+2)(\xi+3)L_\mu\\
&+\frac{2(1+z)}{z}I_3+\frac{8(z+3)}{3\bar z}J_3-\frac{8(z-9)}{3\bar z}J_3'\\
&-\frac{\bar z}{z^3}\left(1-5z+8z^2+6\xi z^2-\xi^2(1+z)\right)\ln^2\bar z\\
&\left.-\frac{(\xi+3)}{z}(3z-9-\xi+\xi z)\ln\bar z L_\mu+\frac{1}{2}(\xi+3)^2L_\mu^2\right]\\
&+C_FC_A\left[\frac{1}{24\bar z}\left(235-1291z+16z^2+15\bar z\xi(16+3\xi)\right)+(7-\frac{1}{2}\xi+\frac{1}{2}\xi^2)\Big(\zeta_2+\mathrm{Li}_2(z)\Big)\right.\\
&-\frac{(1+z)}{z}I_3-\frac{4(z+3)}{3\bar z}J_3+\frac{4(z-9)}{3\bar z}J_3'+\frac{\bar z}{z}(\xi+5)\Big(\zeta_2\ln\bar z+\mathrm{Li}_2(z)\ln\bar z+3S_{1,2}(z)\Big)\\
&+\frac{\bar z}{12z}(301+60\xi+15\xi^2)\ln\bar z-\frac{\bar z}{2z^2}(1+14z-\bar z\xi+z\xi^2)\ln^2\bar z\\
&\left.+5\frac{89+12\xi+3\xi^2}{12}L_\mu+\frac{1}{2z}(\xi^2+3\xi+22)(\bar z\ln\bar z+\frac{z}{2}L_\mu)L_\mu\right]
\end{aligned}$$

$$
\begin{aligned}
&+ C_F t \left[4(1-n_f) \left(\zeta_2 + \mathrm{Li}_2(z) + \frac{8\bar{z}}{3z} \ln \bar{z} - \frac{\bar{z}}{z} \ln^2 \bar{z} \right) \right. \\
&- \frac{52}{3} n_f - \frac{32}{3} n_f L_\mu - 4 n_f \frac{\bar{z}}{z} L_\mu \ln \bar{z} \\
&\left. - 2 n_f L_\mu^2 + \frac{1}{3\bar{z}^2}(314 - 95z + 15z^2 + 2z^3) - \frac{4(z^2-9)}{\bar{z}^2} J_3 - \frac{4(z-9)}{3\bar{z}} J_3' \right] \Bigg\} \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
B_{\rm ren} = 1 &+ \frac{\alpha_s}{4\pi} C_F \xi \left\{ \frac{1+z}{z} + \frac{1-z^2}{z^2} \ln \bar{z} + L_\mu \right\} \\
&+ \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F^2 \left[\frac{1}{z}(4\bar{z} + 6z\xi - z\xi^2)\left(\zeta_2 + \mathrm{Li}_2(z)\right) \right. \right. \\
&+ \frac{1}{24z}\Big(8z^2 + 183z - 1140 + 48\xi(8+\xi) \Big) \\
&- \frac{1}{2z^2}\Big(-2(z+3)\bar{z}\xi^2 + 4(3z-11)\xi + 11z^2 - 28z + 17 \Big) \ln \bar{z} \\
&- \frac{4}{z^2} I_3 - \frac{2(z+6)}{3z} J_3 + \frac{2(z-9)}{3z} J_3' \\
&- \frac{\bar{z}}{z^3}\Big(3 + 3z - z^2 + 3\xi(z^2+z-2) - \xi^2 \Big) \ln^2 \bar{z} \\
&+ \frac{1}{2z}\Big(2\xi^2(1+z) + 24\xi - 3z \Big) L_\mu + \frac{1}{z^2}(12 + \xi - z^2\xi)\xi L_\mu \ln \bar{z} + \frac{1}{2}\xi^2 L_\mu^2 \Big] \\
&+ C_F C_A \left[\frac{1}{2z}(4 + 2z - \xi z + z\xi^2)\Big(\zeta_2 + \mathrm{Li}_2(z) \Big) - \frac{1}{24z}\Big(4z^2 - 147z - 714 \right. \\
&\left. - 12(7+13z)\xi - 9(2+3z)\xi^2 \Big) + \frac{\bar{z}}{z^2}(z + 4 + \xi z)\Big(\zeta_2 \ln \bar{z} + \mathrm{Li}_2(z) \ln \bar{z} + 3 S_{1,2}(z) \Big) \right. \\
&+ \frac{2}{z^2} I_3 + \frac{z+6}{3z} J_3 - \frac{(z-9)}{3z} J_3' + \frac{\bar{z}}{4z^2}\Big(11z + 35 + 14\xi(1+z) + 3\xi^2(1+z) \Big) \ln \bar{z} \\
&- \frac{\bar{z}}{2z^2}(1+\xi)\Big(1 + \xi(1+z) \Big) \ln^2 \bar{z} + \frac{1}{4z}\Big(\xi^2(2+3z) + 2\xi(3+7z) + 25z \Big) L_\mu \\
&\left. + \frac{(1-z^2)}{2z^2}\xi(3+\xi) L_\mu \ln \bar{z} + \frac{1}{4}\xi(3+\xi) L_\mu^2 \right] \\
&+ C_F t \left[\frac{236}{5\bar{z}^2} + \frac{11}{15\bar{z}} - \frac{9z^2 - 120z - 56}{30z} + \frac{8(13-z)}{15\bar{z}^2} J_3 - \frac{4(z-2)(z-9)}{15z\bar{z}} J_3' \right. \\
&\left. \left. - \frac{1}{2z}(7z+4) n_f - 2 n_f L_\mu - 2\frac{\bar{z}}{z^2}\Big((1+z) n_f + \frac{1}{15}(z^2 - 17z - 14) \Big) \ln \bar{z} \right] \right\}, \qquad (13)
\end{aligned}
$$

where $\bar{z} = 1 - z$ and $L_\mu = \ln(\mu^2/m^2)$. The polylogarithms $\mathrm{Li}_n(z)$ and $S_{n,p}(z)$ are defined by the integrals

$$
S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!\, p!} \int_0^1 \frac{\ln^{n-1}(x)\, \ln^p(1-xz)}{x}\, dx \;\; ; \; \mathrm{Li}_n(z) = S_{n-1,1}(z)
$$

and $\zeta_n = \zeta(n) = \sum_{i=1}^{\infty} 1/n^i$ is the Riemann zeta function, with values $\zeta_2 = \pi^2/6$, $\zeta_3 = 1.202057...$, $\zeta_4 = \pi^4/90$ etc..

We observe that in order to evaluate the on-shell values for $A_{\rm ren}$ and $B_{\rm ren}$ ($\bar{z} \to 0$) we need to know the on-shell value of $I_3(z)$ and the expansions of $J_3(z)$ up to order

$\bar{z}^2$ and of $J_3'(z)$ up to order $\bar{z}$. The on-shell value of $I_3(z)$ (at $z = 1$) is known [16] and is given for completeness in appendix B. The expansions of the integrals J_{111} and J_{112}, and the corresponding expansions of $J_3(z)$ and $J_3'(z)$, can be obtained by the second-order differential equation

$$2(q^2+m^2)(q^2+9m^2)m^2 J_3'' - \Big((d-4)q^4 + 10(3d-10)q^2m^2 + 9(5d-16)m^4\Big) J_3' + 3(3d-8)(d-3)(q^2+3m^2)J_3 = \frac{48q^2m^{2d-6}}{(d-4)^2},$$

where primes denote differentiation w.r.t. m^2, which was derived in [17]. As a result we find

$$J_3(z) = -\frac{59}{8} + \frac{3}{8}\bar{z} + (\frac{5}{4} - \frac{3}{4}\zeta_2)\bar{z}^2,$$

$$J_3'(z) = -\frac{3}{8} - (\frac{17}{8} - \frac{3}{2}\zeta_2)\bar{z}\,.$$

In appendices D and E we also present the small and large momentum expansions of the renormalized amplitudes $A_{\rm ren}$ and $B_{\rm ren}$. For details concerning the expansions of $I_3(z)$ and $J_3(z)$ we refer to appendix B.

4 Connection between pole mass and $\overline{\rm MS}$ mass

Due to the confinement of QCD, quark masses in general do not have an unambiguous simple physical meaning. This is especially true for the masses of the light quarks which are usually parametrized by $\overline{\rm MS}$ masses, the $\overline{\rm MS}$ renormalized Lagrangian mass parameters. Physical values may be attached to them as current quark masses which cause the observed chiral symmetry breaking (see e.g. [18]). For a very heavy quark, like the top quark, the situation is different. The top quark is so unstable that it decays by weak interactions before strong interactions come into play and form bound states. In this case the complex pole mass gains a physical meaning as it manifests itself in a resonance peak in physical cross sections. In the following we calculate the pole mass at the two-loop level in perturbative QCD. We should mention here that the true pole mass of full QCD is expected to differ by non-perturbative effects from the pole mass obtained in perturbative QCD.

The relationship between the pole mass and the $\overline{\rm MS}$ mass at two-loop order was first obtained in [2]. The two-loop correction turns out to be rather large. This may have physical reasons [19], but there could also be a problem with the procedure of calculation adopted in [2]. In [2] prior to the ε-expansion and the renormalization, the external momentum was taken to be on-shell and recurrence relations were applied directly to the on-shell integrals. Note that the corresponding interchange of integrations with taking the on-shell limit could cause problems due to possible infrared singularities. One expects that such an on-shell algorithm can be applied for infrared-safe quantities. However, since we are lacking a rigorous proof that the recurrence relations for the on-shell integrals yield the correct answer, it is highly desirable to find the connection between the pole mass and the $\overline{\rm MS}$ mass by means of a different method

which avoids possible ambiguities. This is possible by determining the position of the pole of the $\overline{\rm MS}$ renormalized propagator. Thus, setting $\not{q} = M$ in (7), and taking the limit $q^2 \to M^2$ in $A_{\rm ren}$ and $B_{\rm ren}$, we obtain the transcendental equation

$$\lim_{q^2 \to M^2} [m A_{\rm ren}(q^2, m^2) - M B_{\rm ren}(q^2, m^2)] = 0 \tag{14}$$

for the pole mass M. Its iterative solution yields the desired relation

$$M = m \left(1 + c_1 \left(\frac{\alpha_s}{4\pi} \right) + c_2 \left(\frac{\alpha_s}{4\pi} \right)^2 + \ldots \right), \tag{15}$$

with

$$c_1 = C_F (4 + 3L), \tag{16}$$

$$\begin{aligned} c_2 = {} & C_F C_A \left(\frac{1111}{24} - 8\zeta_2 - 4 I_3(1) + \frac{185}{6} L + \frac{11}{2} L^2 \right) \\ & - C_F t n_f \left(\frac{71}{6} + 8\zeta_2 + \frac{26}{3} L + 2L^2 \right) \\ & + C_F^2 \left(\frac{121}{8} + 30\zeta_2 + 8 I_3(1) + \frac{27}{2} L + \frac{9}{2} L^2 \right) - 12 C_F t \left(1 - 2\zeta_2 \right). \end{aligned} \tag{17}$$

where $I_3(1)$ is given in appendix B and $L = \ln(\mu^2/M^2)$. Setting $\mu^2 = M^2$ we arrive at the relation already known from [2].

Note that as long as we consider QCD corrections only, the location of the pole of the quark propagator M is real. Only after switching on the weak interaction the quarks become unstable and the pole moves off the real axis.

Our result confirms the validity of the on-shell approach, i.e. the possibility of setting the external momentum squared on-shell and applying the on-shell recurrence relations from the very beginning. This check thus enhances our confidence that it is possible to derive the same relation on the three-loop level by starting from the on-shell recurrence relations, which for this task evidently is much simpler than its "indirect" evaluation from the renormalized propagator.

After completion of this paper we learned about related work [20] where the 2-loop relation between the quark mass in the $\overline{\rm MS}$ and MOM schemes has been obtained in the NNLO approximation. The result [20] corresponds to the limit $q^2/m^2 \to \infty$ of the complete 2-loop massive quark propagator which has been presented here.

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Appendix A

Here the two-loop integrals $J_{111}(0,0,m^2)$, $J_{112}(0,0,m^2)$ and G_{11} of (10) are expanded in ε. This is the form which is needed to obtain the renormalized amplitudes A and B.

$$
\begin{aligned}
e^{2\gamma\varepsilon}m^{4\varepsilon-2}J_{111}(0,0,m^2) &= -\frac{1}{2\varepsilon^2}+\frac{z-6}{4\varepsilon}-3+\frac{13}{8}z-\frac{3}{2}\zeta_2+\frac{(1-z^2)}{2z}\ln\bar{z}-\ \mathrm{Li}_2(z)\\
&+\varepsilon\left(-\frac{15}{4}+\frac{115}{16}z+\frac{3(z-6)}{4}\zeta_2+\frac{4}{3}\zeta_3+\frac{1-z^2}{z}\left(\frac{13}{4}-\ln\bar{z}\right)\ln\bar{z}\right.\\
&\left.+\frac{2z^2-6z-1}{2z}\,\mathrm{Li}_2(z)-\ \mathrm{Li}_3(z)-4S_{1,2}(z)\right)+\varepsilon^2\left(\frac{21}{8}+\frac{865}{32}z-9\zeta_2+\frac{39}{8}z\zeta_2\right.\\
&+(4-\frac{2}{3}z)\zeta_3-\frac{63}{8}\zeta_4+\frac{1-z^2}{z}\left(\frac{115}{8}+\frac{3}{2}\zeta_2-\frac{13}{2}\ln\bar{z}+\frac{4}{3}\ln^2\bar{z}\right)\ln\bar{z}\\
&-3\,\mathrm{Li}_2^2(z)-\ \mathrm{Li}_4(z)-\frac{2}{z}(z^2+6z-2)S_{1,2}(z)+\frac{1}{2z}(2z^2-6z-1)\,\mathrm{Li}_3(z)\\
&\left.-(6+\frac{13}{4z}-\frac{13}{2}z+3\zeta_2)\,\mathrm{Li}_2(z)+\frac{3(1-z^2)}{z}\ln\bar{z}\,\mathrm{Li}_2(z)-16S_{1,3}(z)+8S_{2,2}(z)\right),
\end{aligned}
\tag{18}
$$

where $S_{n,p}(z)$ is the generalized Nielsen polylogarithm [21].

$$
\begin{aligned}
e^{2\gamma\varepsilon}m^{4\varepsilon-4}J_{112}(0,0,m^2) &= -\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}+\frac{1}{2}-\frac{3}{2}\zeta_2+\frac{\bar{z}}{z}\ln\bar{z}-\ \mathrm{Li}_2(z)\\
&+\varepsilon\left(\frac{11}{2}-\frac{3}{2}\zeta_2+\frac{5\bar{z}}{z}\ln\bar{z}-\frac{2\bar{z}}{z}\ln^2\bar{z}+\frac{4}{3}\zeta_3-\frac{1}{z}\,\mathrm{Li}_2(z)-\ \mathrm{Li}_3(z)\right.\\
&\left.-4S_{1,2}(z)\right)+\varepsilon^2\left(\frac{49}{2}+\frac{3}{2}\zeta_2-\frac{63}{8}\zeta_4+\frac{\bar{z}}{z}(19+3\zeta_2)\ln\bar{z}-10\frac{\bar{z}}{z}\ln^2\bar{z}\right.\\
&+\frac{8\bar{z}}{3z}\ln^3\bar{z}+\frac{4}{3}\zeta_3-\frac{1}{z}\,\mathrm{Li}_3(z)+\ \mathrm{Li}_2(z)(6-\frac{5}{z}-3\zeta_2+6\frac{\bar{z}}{z}\ln\bar{z})\\
&\left.+(\frac{8}{z}-12)S_{1,2}(z)-3\,\mathrm{Li}_2^2(z)-\ \mathrm{Li}_4(z)-16S_{1,3}(z)+8S_{2,2}(z)\right).
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
e^{2\gamma\varepsilon}m^{4\varepsilon-2}G_{11}(0,m^2)G_{01}(0,m^2) &= -\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\left(3+\frac{\bar{z}}{z}\ln\bar{z}\right)-7-\zeta_2+\frac{\bar{z}}{z}\Big(-3\ln\bar{z}+\ln^2\bar{z}+\ \mathrm{Li}_2(z)\Big)\\
&+\varepsilon\left[-15-3\zeta_2+\frac{2}{3}\zeta_3+\frac{\bar{z}}{z}\Big(-7-\zeta_2+3\ln\bar{z}-\frac{2}{3}\ln^2\bar{z}\Big)\ln\bar{z}\right.\\
&\left.+\frac{\bar{z}}{z}\left(\Big(3-2\ln\bar{z}\Big)\,\mathrm{Li}_2(z)+\ \mathrm{Li}_3(z)-2S_{1,2}(z)\right)\right]\\
&+\varepsilon^2\frac{\bar{z}}{z}\left[\frac{z}{\bar{z}}\Big(-31-7\zeta_2+2\zeta_3-\frac{7}{4}\zeta_4\Big)-\Big(15+3\zeta_2-\frac{2}{3}\zeta_3\Big)\ln\bar{z}\right.\\
&+(7+\zeta_2)\ln^2\bar{z}-2\ln^3\bar{z}+\frac{1}{3}\ln^4\bar{z}+\Big(7+\zeta_2-6\ln\bar{z}+2\ln^2\bar{z}\Big)\ \mathrm{Li}_2(z)\\
&\left.+\Big(3-2\ln\bar{z}\Big)\Big(\ \mathrm{Li}_3(z)-2S_{1,2}(z)\Big)+\ \mathrm{Li}_4(z)+4S_{1,3}(z)-2S_{2,2}(z)\right].
\end{aligned}
\tag{20}
$$

The last formula can as well be used for the ε-expansion of the one-loop scalar propagator integral with arbitrary masses and momentum (see, for example, [22]).

A surprising fact is that in the final renormalized result of the quark propagator given in sect. 3 various structures like ζ_3, ζ_4, $\mathrm{Li}_2^2(x)$, $\mathrm{Li}_3(x)$, $\mathrm{Li}_4(x)$ and all generalized Nielsen polylogarithms, except $S_{1,2}$, have canceled.

Appendix B

In this appendix we present details about the calculation of the functions $I_3(z)$ and $J_3(z)$ entering the self-energy function given in sect. 3. We can give one-dimensional integral representations which are, however, not the most convenient forms for a numerical evaluation as will be seen. In particular it turns out that the expansions w.r.t. small and large q^2 are very efficient even in the time-like region (on the cut) of the functions. For this reason we give a full account of these expansions, mainly due to [16].

An integral representation for I_3 is given by the following expression [16, 17]

$$q^2 I_3(q^2/m^2) = \int_m^\infty \frac{4w\ dw}{w^2-m^2}\left(\ln\frac{w}{m} - \frac{w^2-m^2}{w^2}\ln\frac{w^2-m^2}{m^2}\right)\ln\frac{w^2+q^2}{w^2}$$
$$+ \int_{2m}^\infty \frac{4\ dw}{(w^2-4m^2)^{1/2}}\left(3\ln\frac{w}{m} - \frac{3w^2-3m^2+q^2}{W_+W_-}\ln\frac{W_++W_-}{W_+-W_-}\right)\ln\frac{w}{2m}, \quad (21)$$

with $W_\pm \equiv ((w\pm m)^2+q^2)^{1/2}$.

We recall that for $q^2 = m^2$ ($z \equiv q^2/m^2 = 1$) the value of I_3 is [16]

$$I_3(1) = \frac{3}{2}\zeta_3 - 6\zeta_2\ln 2. \quad (22)$$

The small q^2 expansion is obtained from [16] in the following manner: with the Ansatz

$$I_3(z) = -\sum_{n=1}^\infty z^n \frac{d_3(n)}{n}, \quad (23)$$

$d_3(n)$ being expansion coefficients and $d_3(1) = -\zeta_2 + \frac{27}{2}S_2$ with S_2 derived from the maximum value of Clausen's integral,

$$S_2 = \frac{4}{9\sqrt{3}}\mathrm{Cl}(\frac{\pi}{3}) = 0.2604341376\ldots \quad . \quad (24)$$

The higher order terms are obtained by solving the following differential equation [16] (the notation is adopted from [16])

$$D_3D_2I_3(z) = D_3D_2I_a(z) + (q^2-9m^2)^2 - 2\pi^2m^2(q^2+3m^2), \quad (25)$$

where

$$D_2 \equiv (q^2-m^2)\frac{d}{dq^2},$$

$$D_3 \equiv 6m^2(q^2+3m^2) + (q^2-9m^2)\left[(q^2-m^2)(q^2-9m^2)\frac{d}{dq^2} - 8m^2\right]q^2\frac{d}{dq^2}$$

and

$$I_a(z) = \sum_{n=1}^{\infty} z^n \left\{ -\frac{\pi^2}{6n} + \sum_{r=1}^{n-1} (\frac{1}{r^2 n} - \frac{2}{r n^2}) \right\} . \tag{26}$$

Inserting (23) into (25), equating equal powers of z and solving iteratively for the coefficients $d_3(n)$ is very conveniently done, e.g., with the help of a REDUCE [23] program, which can also perform the final numerical evaluation. To calculate I_3 on its cut, the method of mapping and Padé approximation is applied as introduced in [24]. Since any number of Taylor coefficients is easily obtained for I_3, it can be calculated with practically any required precision even on the cut. Thus as an example for z=100 with 100 Taylor coefficients we obtain $I_3 = 7.77982600 - 0.77916795i$, where the precision is estimated from the convergence of the Padé approximants. For comparison: the integral representation (21) is not even applicable on the cut and in the space-like region for such large q^2 the two contributions mainly cancel, which seems not the proper way to evaluate I_3 numerically.

As described in [16] as well, for large q^2 the large momentum expansion can be used: with five terms

$$\begin{aligned}
(I_3)_{\rm asy}(z) &= 6\zeta_3 + (2L_z^2 + 6L_z + 6)/z + (4L_z^2 + \frac{1}{2}L_z - \frac{15}{4})/z^2 + \\
&(\frac{29}{3}L_z^2 - \frac{46}{3}L_z - \frac{257}{18})/z^3 + (32L_z^2 - \frac{1957}{24}L_z - \frac{3613}{96})/z^4 + \\
&(\frac{677}{5}L_z^2 - \frac{30907}{75}L_z - \frac{103577}{1000})/z^5 ,
\end{aligned}$$

and $L_z = \ln z - i\pi$ we obtain the same numerical result as above except for the last decimal in the imaginary part (i.e. Im $(I_3)_{\rm asy}(100) = -0.77916794$).

We also mention that from the small and large momentum expansions of $I_3(z)$ a 3-loop integral D_3 was calculated [25].

The situation is similar for J_3. We found the following integral representation

$$\begin{aligned}
J_3(z) = &-9 + \frac{13}{8}z + \frac{(3+z)(1-z)}{2z}\ln(1-z) \\
&+(1-z)\int_0^1 \frac{dy[-3+y(3-z)-3y^2]}{2zy(1-y)^2}\left(1 - \frac{1-y+y^2-zy}{\sqrt{\Delta}}\right)\ln y,
\end{aligned} \tag{27}$$

with $\Delta = y^4 - 2(1-z)y^3 + (z^2-6z+3)y^2 - 2(1-z)y + 1$. However, this representation is only applicable for $z \leq 1$. Recall that the first threshold starts at z=9 ! Therefore, the small and large momentum expansions will be much more useful again. For $J_3(z)$ we have the special values [26]

$$J_3(1) = -\frac{59}{8} , \quad J_3(9) = \frac{45}{8} - \frac{8\pi}{\sqrt{3}} .$$

Writing

$$J_3(z) = \sum_{n=0}^{\infty} z^n j_3(n), \tag{28}$$

the Taylor coefficients $j_3(n)$ of J_3 are obtained recursively in the following manner

$$j_3(0) = -\frac{21}{2} + \frac{27}{2}S_2,\ j_3(1) = \frac{3}{8} - 3S_2,\ j_3(2) = \frac{11}{108} - \frac{1}{3}S_2,\ j_3(3) = \frac{19}{648} - \frac{1}{9}S_2. \quad (29)$$

Introducing $c_3(j) = -j!(j+1)!j_3(j), j \geq 2$, we have the recursion [17]

$$c_3(j+1) = \frac{1}{9}\left((10j^2-4)c_3(j) - (j-2)(j-1)j(j+1)c_3(j-1)\right). \quad (30)$$

Thus the coefficients are much easier to calculate than in the case of I_3. Furthermore, the precision of the Padé approximants is much higher, i.e. already with 33 coefficients one obtains for $z = 100$ a precision of 10 decimals. The reason for this is of course that the threshold is much higher in this case.

The large q^2 expansion of J_3 is given in [17]. The first six terms (again up to z^{-5}) give the contribution

$$\begin{aligned}(J_3)_{\text{asy}}(z) = &-(\frac{1}{2}L_z - \frac{13}{8})z + \frac{3}{2}L_z^2 - \frac{15}{2} - (3L_z^2 + \frac{9}{2}L_z - \frac{3}{4})/z \\ &- (3L_z^2 - 6L_z - \frac{29}{4})/z^2 - (9L_z^2 - 23L_z - \frac{307}{24})/z^3 - (36L_z^2 - 107L_z - \frac{7927}{240})/z^4 \\ &- (171L_z^2 - \frac{2773}{5}L_z - \frac{14107}{150})/z^5. \end{aligned} \quad (31)$$

For z=100 this yields also a precision of 10 decimals, i.e. the same precision as the small q^2 expansion with 33 coefficients.

What concerns J_3', the Taylor series as well as the large momentum expansion are easily obtained by differentiation from the corresponding representations of J_3. The precision with the same number of coefficients as above is here 9 decimals for the small q^2 expansion and the asymptotic expansion.

Finally we mention that also the Taylor coefficients can be calculated analytically and stored which allows us to write a FORTRAN program in terms of the multiple precision version of D.H. Bailey [27]. Then, calculating these functions from the small momentum expansion, for I_3, J_3 and J_3' only the Padé's have to be performed, which is very little time consuming. For the numerical evaluation of $S_{1,2}$ we refer the reader to [21].

Appendix C

In the following we present the bare amplitudes A_{bare} and B_{bare} in terms of the basic integrals of sect. 3. For these we introduce the following notation:

$$\begin{aligned} I_1 &= G_{11}(0,m^2)G_{11}(0,m^2), \\ I_2 &= \frac{1}{m^2}G_{1,1}(0,m^2)G_{01}(0,m^2), \\ I_3 &\ \text{ see } (9), \\ I_4 &= \frac{1}{m^2}J_{111}(m^2,m^2,m^2), \\ I_5 &= J_{112}(m^2,m^2,m^2), \end{aligned}$$

$$
\begin{aligned}
I_6 &= \frac{1}{m^2} J_{111}(0,0,m^2), \\
I_7 &= J_{112}(0,0,m^2), \\
I_8 &= \frac{1}{m^4} G_{01}(0,m^2) G_{01}(0,m^2)
\end{aligned}
$$

The general structure of the bare amplitudes reads

$$
\begin{aligned}
A = & \frac{C_F}{1-z}\left(\frac{C_A a_1}{d-4} + \frac{C_F a_2}{z}\right) I_1 + \frac{C_F}{1-z}\left((C_F - \frac{1}{2}C_A)a_3 + \frac{t a_4 (3d-8)}{3(1-z)^2}\right) I_4 \\
& + \frac{(d-2)C_F}{(1-z)(d-3)}\left(\frac{C_A a_5}{d-4} + \frac{(1-z)(d-3)t}{d-2} a_6 + \frac{C_F a_7}{z}\right) I_2 \\
& + \frac{C_F}{z}\left(C_F - \frac{1}{2}C_A\right) a_8 I_3 + \frac{C_F}{1-z}\left((C_F - \frac{1}{2}C_A)a_9 + \frac{t a_{10}}{3(1-z)^2}\right) I_5 \\
& + \frac{(3d-8)C_F}{(d-4)(1-z)}\left(\frac{C_A a_{11}}{2(d-4)} + \frac{t a_{12}}{3d-8} + C_F a_{13}\right) I_6 \\
& + \frac{C_F}{(d-4)(1-z)}\left(\frac{C_A a_{14}}{(d-4)(d-3)} + t a_{15} + \frac{C_F a_{16}}{d-3}\right) I_7 \\
& + \frac{(d-2)^2 C_F}{(1-z)(d-3)}\left(\frac{C_A a_{17}}{d-4} + \frac{t a_{18}}{3(1-z)^2(d-5)} + \frac{C_F a_{19}}{z}\right) I_8
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
zB = & \frac{C_F}{1-z}\left(\frac{C_A b_1}{d-4} + C_F b_2\right) I_1 + \frac{(d-2)C_F}{(d-3)(1-z)}\left(\frac{C_A b_3}{d-4} + \frac{t(1-z)(d-3)b_4}{15(d-2)}\right. \\
& \left. + C_F b_5\right) I_2 + \frac{C_F}{z}\left(C_F - \frac{1}{2}C_A\right) b_6 I_3 + \frac{C_F}{1-z}\left((C_F - \frac{1}{2}C_A)b_7 + \frac{t b_8}{15(1-z)^2}\right) I_4 \\
& + \frac{C_F}{1-z}\left((C_F - \frac{1}{2}C_A)b_9 + \frac{t b_{10}}{15(1-z)^2}\right) I_5 + \frac{C_F}{1-z}\left(\frac{C_A b_{11}}{2(d-4)^2(d-6)} + \frac{t b_{12}}{d-6} + \frac{C_F b_{13}}{d-4}\right) I_6 \\
& + \frac{C_F}{1-z}\left(\frac{C_A b_{14}}{(d-3)(d-4)^2(d-6)} + \frac{t b_{15}}{d-6} + \frac{C_F b_{16}}{(d-4)(d-3)}\right) I_7 \\
& + \frac{(d-2)C_F}{(d-3)(1-z)}\left(\frac{C_A b_{17}}{d-4} + \frac{t b_{18}}{15(d-5)(1-z)^2} + C_F b_{19}(d-2)\right) I_8,
\end{aligned} \tag{33}
$$

where the coefficients a_i and b_i (i=1,2, ...) are polynomials in the space-time dimension d, the gauge parameter ξ and $z = q^2/m^2$.

$$
\begin{aligned}
a_1 &= (d-4)(d^2-4d+2) - (d-2)(d^2-18d+46)z - 2(d-3)(d-4)\xi - 2(d-3)(d-6)\xi z \\
a_2 &= (d-2)^2 + (-4-4d^2+12d)z + (12-d^2)z^2 + 4(d-1)(d-4)\xi z + 4(d-1)(d-2)\xi z^2 \\
&\quad -(d-2)(d-6)\xi^2 z^2 + (-2d^2-20+12d)\xi^2 z - (d-2)^2\xi^2 \\
a_3 &= 8d(3d-8) \\
a_4 &= -48d + 264 + (-96d+272)z + (16d-24)z^2 \\
a_5 &= -d^3 - 86 + 47d - 3d^2 + (2 + d^3 - 9d^2 + 17d)z + (3d-10)(d-5)\xi + (d-2)(d-5)\xi z \\
a_6 &= 8/3(d-1)(d-2)
\end{aligned}
$$

$$
\begin{aligned}
a_7 &= -2(d-2)(d-3) - 2d(-13d+22+2d^2)z + (-2d+12)z^2 \\
&\quad -2(d-1)(5d-16)\xi z - 2(d-1)(d-2)\xi z^2 + 2(d-3)(d-4)\xi^2 z + 2(d-2)(d-3)\xi^2 \\
a_8 &= 24 + 2d^2 - 12d - (2d^2 + 8 - 12d)z \\
a_9 &= 8zd - 72d \\
a_{10} &= -2160 + 432d + (-2352 + 816d)z + (432 - 240d)z^2 + (-16 + 16d)z^3 \\
a_{11} &= -(d-4)(d^2 - 9d + 16)\xi^2 + (2d^3 - 8 - 18d^2 + 40d)\xi - d^3 - 7d^2 + 60d - 104 \\
a_{12} &= -8(d-2)(3d-8)(n_f - 1) \\
a_{13} &= 2(d-3)(d-6)\xi^2 - 8(d-1)(d-4)\xi - 4 - 14d + 6d^2 \\
a_{14} &= -216d^2 + 628d + 26d^3 - 640 + 2(d-2)(d-4)(d^2 - 3d + 1)z + (332d - 4d^4 + 50d^3 \\
&\quad -128 - 210d^2)\xi - 2(d-5)(d-2)(d-4)\xi z + 2(d-4)(d-3)(d^2 - 9d + 16)\xi^2 \\
a_{15} &= 32(n_f - 1)(d-2) \\
a_{16} &= -144 + 68d^2 - 16d - 16d^3 - 4(d-2)(d-4)(2d-3)z \\
&\quad +4(7d-22)(d-1)(d-4)\xi + 4(d-1)(d-2)(d-4)\xi z - 8(d-6)(d-3)^2\xi^2 \\
a_{17} &= (-d+5)\xi + (d+1)(2d-7) \\
a_{18} &= -8d^2 + 204 - 4d + (80d^2 - 616d + 1112)z - 4(d-1)(2d-9)z^2 \\
a_{19} &= d - 3 + (2d^2 - 10d + 4)z + (4d-4)\xi z + (3-d)\xi^2
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= -2(d-2)(d-3)\xi z^2 - 2(d-3)(d-4)\xi z + (d-2)(d^2 - 10d + 22)z^2 \\
&\quad +(56 - 34d + 8d^2 - d^3)z \\
b_2 &= -3(d-2)^2 + (36 - 12d)z - (d-2)(d-6)z^2 + 4(d-1)(d-2)\xi + 4(d-1)(d-4)\xi z \\
&\quad -(d-2)^2\xi^2 z^2 + (-2d^2 - 20 + 12d)\xi^2 z - (d-2)(d-6)\xi^2 \\
b_3 &= -56 + 24d - 2d^2 + (d^3 + 6 - 5d^2 + 5d)z - (d-2)(d^2 - 9d + 19)z^2 \\
&\quad +(d-2)(d-4)\xi + (d-3)(3d-10)\xi z + (d-2)\xi z^2 \\
b_4 &= 2(d-2)(d-4) - 4(d-2)(d-4)z + 2(d-2)(d-4)z^2 \\
b_5 &= -12 - 10d + 4d^2 + (-20 + 16d - 2d^2)z + 2(d-2)(d-5)z^2 \\
&\quad +(d-1)(2d^2 - 15d + 26)\xi + 2(d-1)(d-3)(d-4)\xi z + (d-1)(d-2)\xi z^2 \\
&\quad +2(d-2)(d-3)\xi^2 z + 2(d-3)(d-4)\xi^2 \\
b_6 &= -2d(d-6) + 2(d-2)(d-4)z \\
b_7 &= -2(7d-18)(d-6) - 10(d-2)^2 z \\
b_8 &= 36(7d-20)(d-3) + (-4680d + 576d^2 + 8336)z + (2288 + 408d^2 - 1928d)z^2 \\
&\quad +(472d - 592 - 96d^2)z^3 + 4(3d-8)(d-3)z^4 \\
b_9 &= -216 + 36d + (32d - 48)z + (-4d+8)z^2 \\
b_{10} &= -756d + 2160 + (8832 - 1644d)z + (4d-16)z^5 + (2160 - 1032d)z^2 \\
&\quad +(-1072 + 424d)z^3 + (224 - 68d)z^4 \\
b_{11} &= -3168d - 8d^5 + 2624d^2 + 132d^4 - 874d^3 + 768 - (d-2)(d^4 + 5d^3 - 92d^2 + 312d \\
&\quad -320)z + 4(3d-8)(d-4)(d^3 - 9d^2 + 25d - 18)\xi \\
&\quad +2(3d-8)(d-2)(d^3 - 7d^2 + 4d + 28)\xi z - (3d-8)(d-2)(d-4)(d^2 - 9d + 16)\xi^2 z \\
&\quad -2(3d-8)(d-3)(d-4)(d^2 - 9d + 16)\xi^2 \\
b_{12} &= 16(d-2)(d-3)(n_f - 1) + 8(d-2)^2(n_f - 1)z
\end{aligned}
$$

$$
\begin{aligned}
b_{13} &= 2(d-4)(5d^2-16d+6)-2(2d-7)(d-2)(d-4)z-2(d-1)(3d-10)(3d-8)\xi \\
&\quad -2(d-1)(d-2)(3d-8)\xi z+2(3d-8)(d-2)(d-3)\xi^2 z+4(3d-8)(d-3)^2\xi^2 \\
b_{14} &= 2160d-86d^4-1792d^2-384+5d^5+585d^3+(608d^2+960-109d^3+d^5+2d^4 \\
&\quad -1304d)z-2(d-2)(d-4)(d-6)(2d-7)z^2-2(3d-8)(d-4)(d^3-9d^2+25d-18)\xi \\
&\quad -2(3d-8)(d-3)(d^3-8d^2+14d+4)\xi z-2(d-6)(d-2)(d-4)\xi z^2 \\
&\quad +(3d-8)(d-3)(d-4)(d^2-9d+16)\xi^2 z+(3d-8)(d-3)(d-4)(d^2-9d+16)\xi^2 \\
b_{15} &= -16(n_f-1)(d-2)-16(n_f-1)(d-2)z \\
b_{16} &= -4(d-4)(3d^2-11d+3)+4(d-4)(d^2-9d+13)z+8(d-2)(d-4)z^2 \\
&\quad +2(d-1)(3d-10)(3d-8)\xi+16(d-1)(d-3)^2\xi z-2(d-1)(d-2)(d-4)\xi z^2 \\
&\quad -4(3d-8)(d-3)^2\xi^2 z-4(3d-8)(d-3)^2\xi^2 \\
b_{17} &= -(d-2)(d^2-10d+28)-(d-2)(d^2-5d+5)z-(d-2)(d-4)\xi+(-d+2)\xi z \\
b_{18} &= -2(d-5)(d-3)(d-4)+(-72d^3-4044d+988d^2+4664)z \\
&\quad +(-3676d+3392-140d^3+1288d^2)z^2+(-212d^2+24d^3+612d-616)z^3 \\
&\quad -2(d-5)(d-3)(d-4)z^4 \\
b_{19} &= -15+d+(2d-2)z-(d-1)(2d-7)\xi+(-d+1)\xi z+(3-d)\xi^2
\end{aligned}
$$

Appendix D

From the small q^2 expansion, as described in appendix B, we get the following expressions in the limit $z = q^2/m^2 \to 0$ for the renormalized amplitudes $A_{\rm ren}$ and $B_{\rm ren}$.

$$
\begin{aligned}
A_0 = {} & C_F\frac{\alpha_s}{4\pi}\Big\{1+3L_\mu+\xi(1+L_\mu)+z(\frac{3}{2}+\frac{1}{2}\xi)+z^2(\frac{1}{2}+\frac{1}{6}\xi)\Big\} \\
& +(\frac{\alpha_s}{4\pi})^2\Big\{C_AC_F\Big[\frac{629}{24}+\frac{11}{2}\xi+\frac{5}{8}\xi^2-\frac{81}{2}S_2+L_\mu(\frac{313}{12}+\frac{7}{2}\xi+\frac{3}{4}\xi^2)+L_\mu^2(\frac{11}{2}+\frac{3}{4}\xi+\frac{1}{4}\xi^2) \\
& +\zeta_2(1-\frac{3}{2}\xi+\frac{1}{2}\xi^2)+z\Big(\frac{403}{24}+\frac{5}{4}\xi+\frac{5}{8}\xi^2-\frac{9}{4}S_2+L_\mu(\frac{11}{2}+\frac{3}{4}\xi+\frac{1}{4}\xi^2)+\zeta_2(1+\frac{1}{2}\xi)\Big) \\
& +z^2\Big(\frac{313}{36}+\frac{2}{3}\xi+\frac{1}{3}\xi^2-\frac{1}{4}S_2+L_\mu(\frac{11}{6}+\frac{1}{4}\xi+\frac{1}{12}\xi^2)+\frac{1}{6}\xi\zeta_2\Big)\Big] \\
& +C_F^2\Big[-19-10\xi+\xi^2+81S_2+L_\mu(-15-2\xi+\xi^2)+L_\mu^2(\frac{9}{2}+3\xi+\frac{1}{2}\xi^2)+\zeta_2(1+6\xi-\xi^2) \\
& +z\Big(-\frac{57}{4}+\xi+\frac{1}{4}\xi^2+\frac{9}{2}S_2+L_\mu(-\frac{9}{2}+\frac{1}{2}\xi^2)+3\zeta_2\Big)+z^2\Big(-\frac{295}{36}+\frac{1}{2}\xi+\frac{1}{12}\xi^2+\frac{1}{2}S_2 \\
& +L_\mu(-\frac{9}{2}-\xi+\frac{1}{6}\xi^2)+\frac{5}{3}\zeta_2\Big)\Big]+C_Ft\Big[-32-\frac{20}{3}n_f+162S_2-\frac{20}{3}n_fL_\mu-2n_fL_\mu^2 \\
& +4\zeta_2(1-n_f)+z\Big(-60+252S_2-\frac{16}{3}n_f-2n_fL_\mu\Big) \\
& +z^2\Big(-\frac{800}{9}+352S_2-\frac{25}{9}n_f-\frac{2}{3}n_fL_\mu\Big)\Big]\Big\}+O(z^3)
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
B_0 = {} & C_F\frac{\alpha_s}{4\pi}\xi\Big\{\frac{1}{2}+L_\mu+\frac{2}{3}z+\frac{1}{4}z^2\Big\} \\
& +(\frac{\alpha_s}{4\pi})^2\Big\{C_AC_F\Big[\frac{61}{8}+\frac{15}{4}\xi+\frac{1}{4}\xi^2-27S_2+L_\mu(\frac{25}{4}+\frac{11}{4}\xi+\frac{1}{2}\xi^2)+L_\mu^2(\frac{3}{4}\xi+\frac{1}{4}\xi^2) \\
& +\zeta_2(3-\frac{3}{2}\xi+\frac{1}{2}\xi^2)+z\Big(\frac{71}{12}+\frac{13}{12}\xi+\frac{1}{2}\xi^2-15S_2+\xi L_\mu+\frac{1}{3}\xi^2L_\mu+\zeta_2(\frac{11}{6}+\frac{1}{2}\xi)\Big)
\end{aligned}
$$

$$+z^2\Big(\frac{1409}{432}+\frac{5}{6}\xi+\frac{17}{48}\xi^2-\frac{133}{12}S_2+L_\mu(\frac{3}{8}\xi+\frac{1}{8}\xi^2)+\zeta_2(1+\frac{1}{6}\xi)\Big)\Big]$$
$$+C_F^2\Big[\frac{1}{8}-8\xi+\frac{1}{2}\xi^2+54S_2+L_\mu(-\frac{3}{2}-6\xi+\frac{1}{2}\xi^2)+\frac{1}{2}\xi^2L_\mu^2+\zeta_2(-6+6\xi-\xi^2)$$
$$+z\Big(-\frac{49}{12}-\frac{11}{6}\xi-\frac{1}{12}\xi^2+30S_2+L_\mu(-4\xi+\frac{2}{3}\xi^2)-\frac{4}{3}\zeta_2\Big)+z^2\Big(-\frac{599}{216}-\frac{9}{4}\xi+\frac{1}{12}\xi^2+\frac{133}{6}S_2$$
$$+L_\mu(-3\xi+\frac{1}{4}\xi^2)-\zeta_2\Big)\Big]+C_Ft\Big[-24-\frac{5}{2}n_f+108S_2-2n_fL_\mu+z\Big(-\frac{128}{3}-\frac{4}{3}n_f+168S_2\Big)$$
$$+z^2\Big(-\frac{1628}{27}-\frac{1}{2}n_f+\frac{700}{3}S_2\Big)\Big]\Big\}+O(z^3). \qquad (35)$$

Appendix E

Using the asymptotic formulae discussed in appendix B, we obtain the following expressions in the limit $z=q^2/m^2\to\infty$ for the renormalized amplitudes $A_{\rm ren}$ and $B_{\rm ren}$ ($L_q=\ln(-q^2/\mu^2)$)

$$A_\infty=\frac{\alpha_s}{4\pi}C_F\left[(4-3L_q+\xi(2-L_q))+\frac{1}{z}(1+L_q+L_\mu)(3+\xi)-\frac{1}{2z^2}(3+\xi)\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2C_F\Big\{C_A\Big[\frac{1531}{24}-3(7+\xi)\zeta_3+10\xi+\frac{15}{8}\xi^2-(\frac{445}{12}+5\xi+\frac{5}{4}\xi^2)L_q$$
$$+(\frac{11}{2}+\frac{3}{4}\xi+\frac{1}{4}\xi^2)L_q^2+\frac{1}{z}\Big(\frac{349}{12}+\frac{5}{2}\xi+\frac{3}{4}\xi^2+3(3+\xi)\zeta_3+(\frac{181}{12}+2\xi+\frac{1}{4}\xi^2)L_q$$
$$-(11+\frac{3}{2}\xi+\frac{1}{2}\xi^2)L_q^2+(\frac{313}{12}+\frac{7}{2}\xi+\frac{3}{4}\xi^2-(11+\frac{3}{2}\xi+\frac{1}{2}\xi^2)L_q)L_\mu\Big)$$
$$+\frac{1}{z^2}\Big(-\frac{14}{3}+\frac{1}{2}\xi-\frac{1}{4}\xi^2+(23+3\xi+\frac{1}{2}\xi^2)L_q+\frac{3}{4}(9+\xi)L_q^2$$
$$+(\frac{35}{2}+\frac{9}{4}\xi+\frac{1}{4}\xi^2+\frac{3}{2}(9+\xi)L_q)L_\mu+\frac{3}{4}(9+\xi)L_\mu^2\Big)\Big]+t\Big[(-\frac{52}{3}+\frac{32}{3}L_q-2L_q^2)n_f$$
$$+\frac{1}{z}\Big(-\frac{8}{3}L_qn_f-24L_q+4L_q^2n_f-\frac{20}{3}n_f-(24-4L_qn_f+\frac{20}{3}n_f)L_\mu\Big)$$
$$+\frac{1}{z^2}\Big(48-4(n_f+9)L_q+\frac{7}{3}n_f-2(18+n_f)L_\mu\Big)\Big]$$
$$+C_F\Big[13+12\zeta_3+8\xi+\xi^2-2(\xi^2+5\xi+6)L_q+(\frac{9}{2}+3\xi+\frac{1}{2}\xi^2)L_q^2$$
$$+\frac{1}{z}\Big(-16+4\xi+4\xi^2+12\zeta_3+6(3+\xi)L_q-(\xi+3)^2L_q^2$$
$$+(27+12\xi+\xi^2-\xi^2L_q+9L_q)L_\mu+6(3+\xi)L_\mu^2\Big)+\frac{1}{z^2}\Big(-\frac{69}{4}-\frac{17}{2}\xi+\frac{5}{4}\xi^2$$
$$-(3-6\xi-4\xi^2)L_q-6L_q^2-(\frac{51}{2}+3\xi-\frac{7}{2}\xi^2+12L_q)L_\mu-6L_\mu^2\Big)\Big]\Big\}+o(\frac{1}{z^2}). \qquad (36)$$

$$B_\infty=\frac{\alpha_s}{4\pi}C_F\xi\left[1-L_q+\frac{2}{z}+\frac{1}{z^2}(\frac{1}{2}+L_q+L_\mu)\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2C_F\Big\{C_A\Big[\frac{41}{4}+\frac{13}{2}\xi+\frac{9}{8}\xi^2-3(1+\xi)\zeta_3-(\frac{25}{4}+\frac{7}{2}\xi+\frac{3}{4}\xi^2)L_q+\frac{1}{4}(3+\xi)\xi L_q^2)$$
$$+\frac{1}{z}\Big(\frac{31}{2}+\frac{9}{2}\xi+\xi^2+3(\xi-3)\zeta_3-\xi(3+\xi)L_q\Big)$$

$$
\begin{aligned}
&+\frac{1}{z^2}\left(-\frac{27}{2}+\frac{19}{4}\xi+\frac{3}{4}\xi^2+24\zeta_3-(\frac{17}{4}-\frac{7}{2}\xi-\frac{1}{4}\xi^2)L_q-(\frac{9}{4}+\frac{3}{4}\xi+\frac{1}{2}\xi^2)L_q^2\right.\\
&\left.+(-\frac{17}{4}-\frac{1}{2}L_q\xi^2-\frac{9}{2}L_q+\frac{17}{4}\xi+\frac{1}{2}\xi^2)L_\mu+\frac{3}{4}(\xi-3)L_\mu^2\right)\Bigg]\\
&+t\left[(2L_q-\frac{7}{2})n_f+\frac{1}{z}\,(18-12L_q-4n_f-12L_\mu)\right.\\
&\left.+\frac{1}{z^2}\left(\frac{332}{9}-n_f-2L_qn_f+\frac{8}{3}L_q+4L_q^2+(\frac{8}{3}+8L_q-2n_f)L_\mu+4L_\mu^2\right)\right]\\
&+C_F\left[-\frac{5}{8}-L_q\xi^2+\frac{3}{2}L_q+\frac{1}{2}\xi^2L_q^2+\frac{1}{z}\left(-12+10\xi+4\xi^2-2\xi(6+\xi)L_q\right)\right.\\
&+\frac{1}{z^2}\left(\frac{11}{4}+\frac{15}{2}\xi+\frac{11}{4}\xi^2-24\zeta_3+(19\xi+3\xi^2-\frac{9}{2})L_q-\xi(9+\xi)L_q^2\right.\\
&\left.\left.-(\frac{9}{2}-19\xi-\frac{7}{2}\xi^2+\xi(6+\xi)L_q)L_\mu-3\xi L_\mu^2\right)\right]\Bigg\}+o(\frac{1}{z^2}).
\end{aligned}
\tag{37}
$$

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