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Theoretical precision in estimates of the hadronic contributions to $(g-2)_\mu$ and $\alpha_{\rm QED}(M_Z)$

F. Jegerlehner[§]

DESY Zeuthen
Platanenallee 6, D–15738 Zeuthen, Germany

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Theoretical precision in estimates of the hadronic contributions to $(g-2)_\mu$ and $\alpha_{\rm QED}(M_Z)$*

F. Jegerlehner [a],

[a]DESY, Platanenallee 6, D-15738, Zeuthen, Germany

I review recent estimates of the non-perturbative hadronic vacuum polarization contributions. Since these at present can only be evaluated in terms of experimental data of limited precision, the related uncertainties pose a serious limitation in our ability to make precise predictions. Besides e^+e^-– annihilation data also τ decay spectra can help to get better predictions. Here, it is important to account for all possible iso-spin violations in τ–decay spectra, from which e^+e^- cross sections may be obtained by an iso-spin rotation. The observed 10% discrepancy in the region above the ρ may be understood as a so far unaccounted iso-spin breaking effect.

1. INTRODUCTION

Non-perturbative hadronic contributions affect electroweak precision observables mainly via the hadronic excitations in the photon vacuum polarization (charge screening) which leads to the energy dependence of the effective fine structure "constant" $\alpha(E)$. Of particular interest are $\alpha(M_Z)$ (precision physics at LEP/SLC) [1] and $a_\mu \equiv (g-2)_\mu/2$ which has been measured at the unbelievable precision of 0.7ppm at BNL [2]. Apart from the electroweak effects (leptons etc.) which are calculable in perturbation theory, a serious problem shows up for the strong interaction effects (hadrons/quarks etc.) for the calculation of which perturbation theory fails. Fortunately, general principles allow us to evaluate the problematic contributions via dispersion relations from experimental e^+e^-–annihilation data represented usually in terms of the cross section ratio

$$R_\gamma(s) \equiv \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)} \,. \qquad (1)$$

The impact is that the errors of the experimental cross section data are now a dominating factor for the theoretical uncertainties of electroweak Standard Model predictions. Therefore an art has developed of getting precise results from measurements of often very limited precision. The situation is also a big challenge for precision experiments on $\sigma(e^+e^- \to hadrons)$ as currently performed by KLOE at DAΦNE [3] and BABAR at PEP [4]. I should mention that the measurements of R are a difficult task: besides the needed particle identification and the background rejection one has to get

$$R^{\rm exp}(s) = \frac{N_{\rm had}\,(1+\delta_{\rm RC})}{N_{\rm norm}\,\varepsilon}\,\frac{\sigma_{\rm norm}(s)}{\sigma_{\mu\mu,\,0}(s)} \,, \qquad (2)$$

where $N_{\rm had}$ is the number of observed hadronic events, $N_{\rm norm}$ is the number of observed normalizing events, ε is the efficiency-acceptance product of hadronic events while $\delta_{\rm RC}$ are radiative corrections to hadron production. $\sigma_{\rm norm}(s)$ is the physical cross section for normalizing events (including all radiative corrections integrated over the acceptance used for the luminosity measurement) and $\sigma_{\mu\mu,\,0}(s) = 4\pi\alpha^2/3s$ is the normalization. In particular this shows that a precise measurement of R requires precise knowledge of the relevant radiative corrections.

For the normalization mostly the Bhabha process is utilized [or $\mu\mu$ itself in some cases]. In general, it is important to be aware of the fact that the effective fine structure constant $\alpha(\mu)$ enters radiative correction calculations with different scales μ in "had" and "norm" and thus must be taken into account appropriately.

Recent advances/issues in the evaluation of the hadronic vacuum polarization effects are based on the following results:

- Updated results from the precise measure-

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ments of the processes $e^+e^- \to \rho \to \pi^+\pi^-$, $e^+e^- \to \omega \to \pi^+\pi^-\pi^0$ and $e^+e^- \to \phi \to K_L K_S$ performed by the CMD-2 collaboration have appeared recently [5]. The update was necessary due to an overestimate of the integrated luminosity in the previous analysis which was published in 2002 [6]. A more progressive error estimate (improving on radiative corrections, in particular) allowed a reduction of the systematic error from 1.4% to 0.6 % . Also some other CMD-2 and SND data at energies $E < 1.4$ GeV have become available.

• In 2001 BES-II published their final R–data which, in the region 2.0 GeV to 5.0 GeV, allowed to reduce the previously huge systematic errors of about 20% to 7% [7].

• After 1997 precise τ–spectral functions became available [8–10] which, to the extent that flavor $SU(2)_{\rm f}$ in the light hadron sector is a symmetry, allows to obtain the iso–vector part of the e^+e^-–cross section [11,12]. This possibility has first been exploited in the present context in [13].

• With increasing precision of the low energy data it more and more turned out that we are confronted with a serious obstacle to further progress: in the region just above the ω–resonance, the iso-spin rotated τ–data, after being corrected for the known iso-spin violating effects, do not agree with the e^+e^-–data at the 10% level [14]. Before the origin of this discrepancy is found it will be hard to make further progress in pinning down theoretical uncertainties.

• In this context iso-spin breaking effects in the relationship between the τ– and the e^+e^-–data have been extensively investigated in [15]. The question remains whether all possible iso-spin violating effects have been taken into account in which case the discrepancy would have to be attributed to experimental problems.

• A new bound $\delta a_\mu(0.6 - 2.0\text{GeV}) < 0.7 \times 10^{-10}$ [16] for the contributions of $\pi\pi\gamma, \pi\eta\gamma$ which include decay products from $\pi^0\gamma, \sigma\gamma, f\gamma, a_1\gamma$ yields a severe constraint on possible missing contributions reported elsewhere [17].

• New results for hadronic e^+e^- cross–sections are expected soon from KLOE, BABAR and BELLE. These experiments, running at fixed energies, are able to perform measurements via the radiative return method [18,3,4]. Preliminary results presented recently by KLOE seem to agree very well with the final CMD-2 e^+e^-–data.

• Last but not least an important change in the hadronic contribution to a_μ was the change in sign of the leading hadronic light–by–light contribution (π^0 exchange) [19].

• Progress was made also in calculating the radiative corrections to $\pi^+\pi^-$ production in energy scans, for inclusive measurements in radiative return [20] and in photon tagging [18] relevant at the meson (Φ,B) factories.

Some of these results have substantially influenced the precision of the evaluations of the vacuum polarization effects in $\alpha_{\rm QED}(M_Z)$ and $(g-2)_\mu$ since 1995 [21]. The present status is reviewed in the following.

2. EVALUATION OF $\alpha(M_Z)$

The photon vacuum polarization $\Pi'_\gamma(q^2)$ modifies the fine structure constant according to

$$\begin{aligned} \alpha(q^2) &= \frac{\alpha}{1-\Delta\alpha} \\ \Delta\alpha &= -\text{Re}\left(\Pi'_\gamma(q^2) - \Pi'_\gamma(0)\right), \end{aligned} \tag{3}$$

$\Pi'(q^2)$ =

and makes it running. The shift $\Delta\alpha$ is large due to the large change in scale going from zero momentum to the Z-mass scale $\mu = M_Z$ and due to the many species of fermions contributing.

The various contributions to the shift in the fine structure constant come from the leptons (lep = e, μ and τ), the 5 light quarks (u, b, s, c, and b and the corresponding hadrons = had) and from the top quark:

$$\Delta\alpha = \Delta\alpha_{\rm lep} + \Delta^{(5)}\alpha_{\rm had} + \Delta\alpha_{\rm top} + \cdots \tag{4}$$

Also W–pairs contribute at $q^2 > 2M_W^2$ (see [22, 23]). The leptonic contributions are calculable in perturbation theory where at leading order the free lepton loops yield

$$\begin{aligned} \Delta\alpha_{\rm lep}(s) &= \sum_{\ell=e,\mu,\tau} \frac{\alpha}{3\pi}\left[\ln\left(s/m_\ell^2\right) - \frac{5}{3} + O\left(m_\ell^2/s\right)\right] \\ &\simeq 0.03142 \text{ for } s = M_Z^2 , \end{aligned} \tag{5}$$

where $\beta_\ell = \sqrt{1-4m_\ell^2/s}$. This leading contribution is affected by small electromagnetic corrections only in the next to leading order. The leptonic contribution is actually known to three loops [24,25] at which it takes the value ($M_Z \sim 91.19$ GeV)[2]

$$\Delta\alpha_{\rm lep}(M_Z^2) \simeq 314.98 \times 10^{-4}. \quad (6)$$

In contrast, the corresponding free quark loop contribution gets substantially modified by low energy strong interaction effects, which cannot be obtained by perturbative QCD (pQCD). As already mentioned, fortunately, one can evaluate this hadronic term $\Delta\alpha^{(5)}_{\rm had}$ from hadronic e^+e^--annihilation data by using a dispersion relation. The relevant once subtracted vacuum polarization amplitude (3) satisfies a convergent dispersion relation and correspondingly the shift of the fine structure constant α is given by

$$\Delta^{(5)}\alpha_{\rm had} = -\frac{\alpha s}{3\pi}\left(P\!\!\!\!\!\!\int_{4m_\pi^2}^{E_{\rm cut}^2} ds' \frac{R_\gamma^{\rm data}(s')}{s'(s'-s)} + P\!\!\!\!\!\!\int_{E_{\rm cut}^2}^{\infty} ds' \frac{R_\gamma^{\rm QCD}(s')}{s'(s'-s)}\right) \quad (7)$$

where

$$R_\gamma(s) = 12\pi {\rm Im}\Pi'_{\rm had}(s) \quad (8)$$

is given by (1). Accordingly, the one particle irreducible (1pi) blob

$\Pi'_{\rm had}(q^2) =$

which is our relevant building block, is given by diagrams which cannot be cut into two disconnected parts by cutting a single photon line. At low energies it exhibits intermediate states like $\pi^0\gamma, \rho, \omega, \phi, \cdots, \pi\pi, 3\pi, 4\pi, \cdots, \pi\pi\gamma, \pi\pi Z$, $\cdots, \pi\pi H, \cdots, KK, \cdots$ (at least one hadron plus any strong, electromagnetic or weak interaction contribution). The corresponding contributions are to be calculated via a dispersion relation from the imaginary parts which are given by the production of the corresponding intermediate states in e^+e^-–annihilation via virtual photons (at energies sufficiently below the point where $\gamma - Z$ interference comes into play).

A direct evaluation of the $R_\gamma(s)$–data up to $\sqrt{s} = E_{cut} = 5$ GeV and for the Υ resonance–region between 9.6 and 13 GeV and applying perturbative QCD from 5.0 to 9.6 GeV and for the high energy tail above 13 GeV at $M_Z = 91.19$ GeV yields[3]:

$$\begin{aligned}\Delta\alpha^{(5)}_{\rm had}(M_Z^2) &= 0.027690 \pm 0.000353 \quad (9)\\ \alpha^{-1}(M_Z^2) &= 128.922 \pm 0.049\,.\end{aligned}$$

The contributions from different energy ranges are shown in Tab. 1[4]. The Euclidean method, described in [30,31], allows to replace data for the Adler function by pQCD at space–like momenta > 2.5 GeV. This yields

$$\begin{aligned}\Delta\alpha^{(5)}_{\rm had}(M_Z^2) &= 0.027651 \pm 0.000173 \quad (10)\\ \alpha^{-1}(M_Z^2) &= 128.939 \pm 0.024\,.\end{aligned}$$

with a substantially reduced error. This estimate is on a sound theoretical basis and should not be confused with so called "theory driven" estimates, which utilize pQCD in a much less controlled manner. In future this approach would allow to evaluate $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ with an accuracy $\delta\Delta\alpha = 0.00007$ or 0.00005 provided future cross-section measurements allow to reduce the errors below $\delta\sigma \lesssim 1\%$ up to J/ψ or Υ, respectively. This assumes that in the meantime pQCD parameters will be known with much better precision as well. This reduction of the error by a factor about 5 is needed in order to satisfy future requirements for precision physics at a linear collider [32]. Our analysis is as close to the experimental results as possible by utilizing the trapezoidal rule together

[2]For $m_t \sim 174.3$ GeV we have $\Delta\alpha_{\rm top}(M_Z^2) \simeq -\frac{\alpha}{3\pi}\frac{4}{15}\frac{M_Z^2}{m_t^2} \simeq -6\times 10^{-5}$.

[3]pQCD for calculating $R(s)$, as worked out to high accuracy in Refs. [26]–[28], is used here only where it has been checked to work and converge well: in non–resonant regions at sufficiently high energies and sufficiently far from resonances and thresholds. I have further checked that results obtained with my own routines agree very well with the ones obtained via the recently published program `rhad-1.00` [29].

[4]Table 1 also specifies largely details of the error handling. The different energy ranges mark typical generations of experiments within which systematic errors are considered to be 100% correlated, while all errors are treated as independent for all entries of the table.

final state	energy range (GeV)	$\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ (stat) (syst)	$\Delta\alpha^{(5)}_{\rm had}(-s_0)$ (stat) (syst)
χPT	(0.28, 0.32)	0.04 (0.00) (0.00)	0.03 (0.00) (0.00)
ρ	(0.28, 0.81)	26.16 (0.24) (0.27)	24.26 (0.23) (0.25)
ω	(0.42, 0.81)	3.02 (0.04) (0.08)	2.75 (0.03) (0.07)
ϕ	(1.00, 1.04)	4.74 (0.07) (0.11)	4.07 (0.06) (0.09)
J/ψ		11.50 (0.56) (0.61)	4.06 (0.19) (0.19)
Υ		1.27 (0.05) (0.07)	0.07 (0.00) (0.00)
had	(0.81, 1.40)	12.92 (0.13) (0.52)	11.05 (0.11) (0.43)
had	(1.40, 3.10)	27.13 (0.11) (0.60)	15.75 (0.06) (0.37)
had	(3.10, 3.60)	5.31 (0.11) (0.10)	1.90 (0.04) (0.04)
had	(3.60, 9.46)	51.49 (0.25) (3.00)	8.41 (0.04) (0.44)
had	(9.46,13.00)	18.59 (0.25) (1.36)	0.90 (0.01) (0.07)
pert	(13.0,∞)	115.59 (0.00) (0.12)	1.09 (0.00) (0.00)
data	(0.28,13.00)	162.14 (0.74) (3.46)	73.21 (0.33) (0.80)
total		277.73 (0.74) (3.46)	74.30 (0.33) (0.80)

Table 1
Results for $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)^{\rm data}$ and $\Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data}$ ($\sqrt{s_0} = 2.5$ GeV).

with PDG rules for taking weighted averages between different experiments as described in detail in [21].

The most important ingredient of our analysis are the e^+e^-–data which we described in detail in [21] (see also [13]) and the new data which have become available since then [33]. The distribution of hadronic contributions to $\Delta\alpha^{\rm had}$ in the e^+e^-–data based approach in shown in Fig. 1.

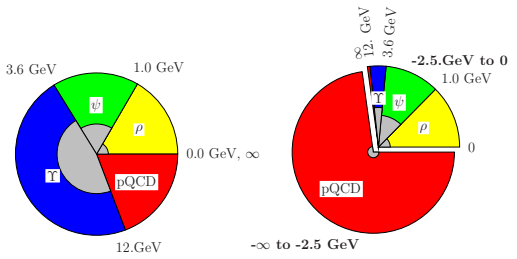


Figure 1. Comparison of the distribution of contributions and errors (shaded areas scaled up by 10) in the standard (left) and the Adler function based approach (right), respectively.

Our results are in good agreement with other recent analyzes

$$\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = \begin{cases} 0.027680 \pm 0.000360 & [34] \\ 0.027690 \pm 0.000180 & [35] \end{cases} \qquad (11)$$

The corresponding values for the effective fine structure constant are:

$$\alpha^{-1}(M_Z^2) = \begin{cases} 128.935 \pm 0.049 & [34] \\ 128.933 \pm 0.025 & [35] \end{cases} \qquad (12)$$

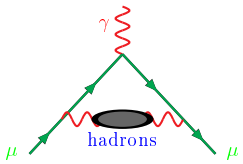
The numerical agreement does not necessarily mean agreement between the different approaches.

3. EVALUATION OF $a_\mu \equiv (g-2)_\mu/2$

The anomalous magnetic moment of the muon a_μ provides one of the most precise tests of the quantum field theory structure of QED and indirectly at a deeper level also of the electroweak SM. The precision measurement of a_μ is a very specific test of the magnetic helicity flip transition $\bar\psi_L \sigma_{\mu\nu} F^{\mu\nu} \psi_R$, a dimension 5 operator which is forbidden for any species of fermions at the tree level of any renormalizable theory. In the SM it is thus a finite prediction which can be tested unambiguously to the extent that we are able to calculate it with the necessary accuracy. For the perturbative part of the SM an impressive precision has been reached. Excitingly the new experimental result from Brookhaven [2] which reached a substantial improvement in precision shows a 1.9[0.7] σ deviation from the theoretical prediction: $\left|a_\mu^{\rm exp} - a_\mu^{\rm the}\right| = 221(113)[074(104)] \times$

10^{-11}, depending on whether one trusts more in an e^+e^-–data[τ–data] based evaluation of the hadronic vacuum polarization contribution [14].

Again contributions from virtual creation and reabsorption of strongly interacting particles cannot be computed with the help of pQCD and cause serious problems. Fortunately the major such contribution again enters via the photon vacuum polarization which can be calculated along the lines discussed for the effective charge. The contribution is described by the diagram



and is represented by the integral

$$a_\mu^{\rm had} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left(\int\limits_{4m_\pi^2}^{E_{\rm cut}^2} ds\, \frac{R_\gamma^{\rm data}(s)\, \hat{K}(s)}{s^2} + \int\limits_{E_{\rm cut}^2}^{\infty} ds\, \frac{R_\gamma^{\rm pQCD}(s)\, \hat{K}(s)}{s^2} \right) \quad (13)$$

which is similar to the integral (7), however with a different kernel $K(s)$ which may conveniently be written in terms of the variable

$$x = \frac{1-\beta_\mu}{1+\beta_\mu}\ , \quad \beta_\mu = \sqrt{1-4m_\mu^2/s}$$

and is given by

$$K(s) = \frac{x^2}{2}\,(2-x^2) + \frac{(1+x^2)(1+x)^2}{x^2} \left(\ln(1+x) - x + \frac{x^2}{2}\right) + \frac{(1+x)}{(1-x)}\, x^2 \ln(x). \quad (14)$$

The integral (13) is written in terms of the rescaled function

$$\hat{K}(s) = \frac{3s}{m_\mu^2} K(s)$$

which is bounded: it increases monotonically from 0.63 at threshold $s = 4m_\pi^2$ to 1 at ∞. Note the extra $1/s$–enhancement of contributions from low energies in a_μ as compared to $\Delta\alpha$. The relative importance of various regions is illustrated in Tab. 2 and Fig. 2. The update of the results [21], including the recent data from CMD-2 and BES-II yields

final state	range (GeV)	δa_μ (stat) (syst)
χPT	(0.28, 0.32)	2.14 (0.02) (0.03)
ρ	(0.28, 0.81)	429.02 (4.95) (5.59)
ω	(0.42, 0.81)	37.99 (0.46) (1.03)
ϕ	(1.00, 1.04)	36.07 (0.50) (0.83)
J/ψ		8.74 (0.41) (0.40)
Υ		0.11 (0.00) (0.01)
had	(0.81, 1.40)	105.17 (1.18) (3.29)
had	(1.40, 3.10)	56.33 (0.21) (1.47)
had	(3.10, 3.60)	4.06 (0.08) (0.08)
had	(3.60, 9.46)	14.43 (0.07) (0.75)
had	(9.46,13.00)	1.30 (0.02) (0.10)
pQCD	(13.0,∞)	1.53 (0.00) (0.00)
data	(0.28,13.00)	693.22 (5.15) (6.83)
total		694.75 (5.15) (6.83)

Table 2
Results for $\delta a_\mu^{\rm data}$.

$$a_\mu^{\rm had} = (694.75 \pm 8.56) \times 10^{-10}\ . \quad (15)$$

The most recent BNL $(g_\mu - 2)$ measurement [2] gives (world average)

$$a_\mu^{\rm exp} = (11659203\ \pm\ 8\ ) \times 10^{-10}$$

which compares with the theoretical prediction[5]

$$a_\mu^{\rm the} = (11659169.6 \pm\ 9.4) \times 10^{-10} \quad ({\rm SM})\ .$$

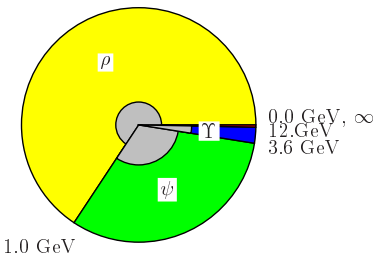


Figure 2. The distribution of contributions and errors (shaded areas scaled up by 10) for $a_\mu^{\rm had}$.

[5]Recent new results concern the hadronic light-by-light contribution [36] and the $O(\alpha^4)$ QED contribution to a_e [37].

The new analysis [14] is "data–driven" like [21, 13] and confirms a substantial discrepancy between e^+e^-– and τ–data. The τ–based result agrees with the corresponding result of [13]. The status is illustrated in Fig. 3. We refer to Ref. [38] for a recent review and possible implications.

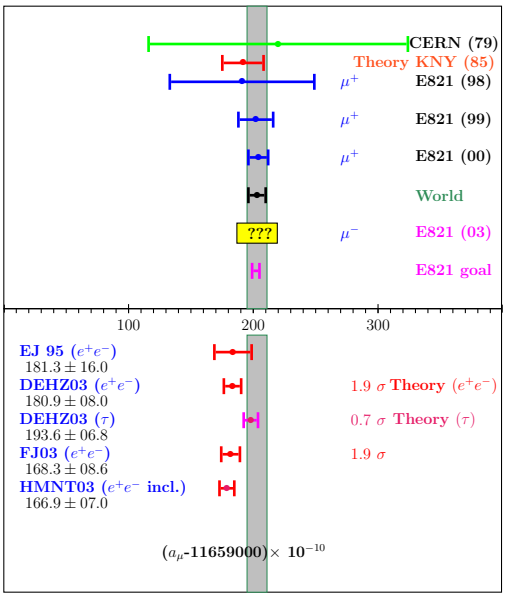


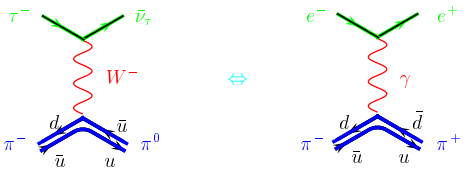
Figure 3. Experimental (upper part) and theoretical (lower part) status of a_μ.

4. e^+e^- CROSS-SECTIONS VIA τ–DECAY SPECTRAL FUNCTIONS

A substantial improvement of the evaluation of $a_\mu^{\rm had}$ would be possible, by including the τ–data, provided one would understand iso–spin violating effects sufficiently well [15]. This has been pioneered by Ref. [13]. Here one utilizes the fact that the vector–current hadronic τ–decay spectral functions are related to the iso–vector part of the e^+e^-–annihilation cross–section via an isospin rotation:

$$\tau^- \to X^- \nu_\tau \quad \leftrightarrow \quad e^+ e \to X^0$$

where X^- and X^0 are related hadronic states. The precise relationship may be derived by comparing diagrams like:



which for the e^+e^- case translates into

$$\sigma_{\pi\pi}^{(0)} \equiv \sigma_0(e^+e^- \to \pi^+\pi^-) = \frac{4\pi\alpha^2}{s} \times v_0(s) \quad (16)$$

and for the τ case into

$$\frac{1}{\Gamma}\frac{d\Gamma}{ds}(\tau^- \to \pi^-\pi^0\nu_\tau) = \frac{6\pi|V_{ud}|^2 S_{EW}}{m_\tau^2} \times v_-(s) \times \frac{B(\tau^- \to \nu_\tau\, e^-\, \bar\nu_e)}{B(\tau^- \to \nu_\tau\, \pi^-\pi^0)} \left(1 - \frac{s}{m_\tau^2}\right)\left(1 + \frac{2s}{m_\tau^2}\right) \quad (17)$$

where $|V_{ud}| = 0.9752 \pm 0.0007$ [39] denotes the CKM weak mixing matrix element and $S_{\rm EW(new)} = 1.0233 \pm 0.0006$ [$S_{\rm EW(old)} = 1.0194$] accounts for electroweak radiative corrections [40–42,15]. The spectral functions are obtained from the corresponding invariant mass distributions. The $B(i)$'s are branching ratios. SU(2) symmetry (CVC) would imply

$$v_-(s) = v_0(s) \ . \quad (18)$$

The spectral functions $v_i(s)$ are related to the pion form factors $F_\pi^i(s)$ by

$$v_i(s) = \frac{\beta_i^3(s)}{12\pi}|F_\pi^i(s)|^2 \ ; \quad (i = 0, -) \quad (19)$$

where $\beta_i^3(s)$ is the pion velocity. The difference in phase space of the pion pairs gives rise to the relative factor $\beta^3_{\pi^-\pi^0}/\beta^3_{\pi^-\pi^+}$.

With the precision of the validity of CVC, thus the τ–data allow us to improve the $I = 1$ part of the e^+e^- cross–section which by itself is not a directly measurable quantity. It mainly improves the knowledge of the $\pi^+\pi^-$ channel (ρ–resonance contribution) which is dominating in $a_\mu^{\rm had}$ (72%). After taking into account the known iso-spin breaking effects [15] the τ–data show substantial discrepancies in comparison with the e^+e^-–data (about 10% just above the ρ–resonance). This issue can certainly be settled by the radiative return experiments with KLOE [3] at LNF/Frascati and with BABAR [4] at SLAC. In fact preliminary results from KLOE are close to the CMD-2 results. At present one obtains incompatible

predictions for $a_\mu^{\rm had}$ based on e^+e^-–data or on τ–data (see Fig. 3).

5. ISO-SPIN BREAKING CORRECTIONS IN τ VS. e^+e^-

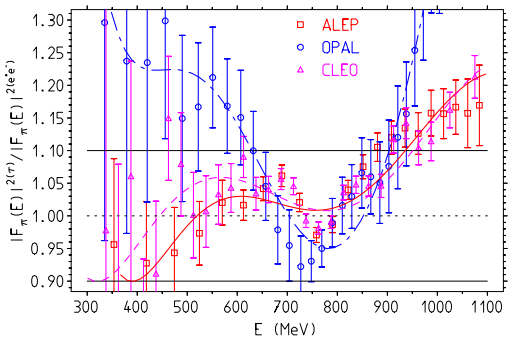


Figure 4. The ratio between τ–data sets from ALEPH, OPAL and CLEO and the $I = 1$ part of the CMD-2 fit of the e^+e^-–data. The curves which should guide the eye are fits of the ratios using 8th order Tschebycheff polynomials.

Before a precise comparison is possible all kind of iso-spin breaking effects have to be taken into account. As mentioned earlier, this has been investigated in [15] for the most relevant $\pi\pi$ channel. For the τ version of the pion form factor, following from (17) and (19), we perform the iso–spin breaking corrections

$$r_{\rm IB}(s) = \frac{1}{G_{\rm EM}(s)} \frac{\beta^3_{\pi^-\pi^+}}{\beta^3_{\pi^-\pi^0}} \frac{S_{\rm EW(old)}}{S_{\rm EW(new)}} \qquad (20)$$

with $G_{\rm EM}(s)$ (from [15]) accounting for the QED corrections of the τ–decays. Final state radiation (FSR) is modeled by scalar QED. The resulting bare form factor $|F_\pi|^{2\ I=1}_{(\tau)}(s)$ compares to the bare (vacuum polarization and FSR subtracted) form factor $|F_\pi|^{2\ I=1}_{(e^+e^-)}(s)$, which can be obtained from the measured pion form factor $|F_\pi|^{2\ \rm exp}_{(e^+e^-)}(s)$ by subtracting the $\rho-\omega$ mixing effects:

$$|F_\pi|^{2\ I=1}(s) = |F_\pi|^{2\ \rm exp}(s)/|\left(1+\frac{\epsilon s}{(s_\omega - s)}\right)|^2 \qquad (21)$$

with $s_\omega = (M_\omega - \frac{i}{2}\Gamma_\omega)^2$, ϵ determined by a fit to the data: $\epsilon = 0.00172$. In Fig. 4 we display $|F_\pi|^{2\ I=1}_{(\tau)}(s)/|F_\pi|^{2\ I=1}_{(e^+e^-)}(s)$ which shows large deviations from the CVC line represented by unity.

These above corrections were applied also in [14] and revealed that they were not sufficient to remove the unexpectedly large discrepancy (see [14] for details). The only large effect I am aware of (of order 10%) which is in the game of the comparison is a possible shift of the invariant mass of the pion-pairs in the ρ resonance region. An idea one gets if one is looking at the experimental ρ–mass values, shown in the particle data tables [39] ("dipole shape"). If the energy calibration of the $\pi\pi$–system would be too low in e^+e^- measurements or to high in τ measurements by 1% one could easily get a 10% decrease or increase in the tail, respectively. Since the $\rho^\pm - \rho^0$ mass difference as well as the difference in the widths $\Gamma^{\pm,0}(\rho \to \pi\pi, \pi\pi\gamma)$ are neither experimentally nor theoretically established, corresponding iso-spin violations cannot be corrected for appropriately. Note that the subtraction of the large and strongly energy dependent vacuum polarization effects necessary for the e^+e^-–data, which seems to worsen the problem, is properly treated in the analysis.

6. EVALUATION OF a_μ VIA THE ADLER FUNCTION

In Ref. [30] it has been shown how one can obtain a better control on the validity of pQCD by utilizing analyticity and looking at the problem in the t–channel (Euclidean field theory approach). It has been found that "data" may be safely replaced by pQCD at $\sqrt{-t} \geq 2.5$GeV. An application to the calculation of the running fine structure constant has been discussed in [31,33]. Here we consider the application to the calculation of $a_\mu^{\rm had}$. Starting point is the basic integral representation

$$a_\mu^{\rm had} = \frac{\alpha}{\pi} \int_0^\infty \frac{ds}{s} \int_0^1 dx \frac{x^2\,(1-x)}{x^2 + (1-x)\,s/m_\mu^2} \frac{\alpha}{3\pi} R(s). \qquad (22)$$

If we first integrate over x we find the well known standard representation as an integral along the cut of the vacuum polarization amplitude in the time–like region, while an interchange of the order of integrations yields an integral over the hadronic

shift of the fine structure constant in the space–like domain [43]:

$$a_\mu^{\rm had} = \frac{\alpha}{\pi} \int\limits_0^1 dx\,(1-x)\,\Delta\alpha^{\rm had}\left(-Q^2(x)\right) \qquad (23)$$

where $Q^2(x) \equiv \frac{x^2}{1-x} m_\mu^2$ is the space–like square momentum–transfer or

$$x = \frac{Q^2}{2m_\mu^2}\left(\sqrt{1+\frac{4m_\mu^2}{Q^2}}-1\right) \quad .$$

In this approach we (i) calculate the Adler function from the e^+e^-–data and pQCD for the tail above 13 GeV, (ii) calculate the shift $\Delta\alpha^{\rm had}$ in the Euclidean region with or without an additional cut in the t–channel at 2.5 GeV and (iii) calculate $a_\mu^{\rm had}$ via (23).

Alternatively, by performing a partial integration in (23) one finds

$$a_\mu^{\rm had} = \frac{\alpha}{\pi} m_\mu^2 \int\limits_0^1 dx x(2-x)\left(D(Q^2(x))/Q^2(x)\right) (24)$$

by means of which the number of integrations may be reduced by one. The evaluation in both forms provides a good stability test of the numerical integrations involved.

Utilizing the most recent e^+e^-–data we obtain a result which agrees with the values obtained by the direct evaluation also in the error. Not too surprisingly, as is well known, the contribution to $a_\mu^{\rm had}$ is dominated by the low energy e^+e^-–data below 1 GeV; here the replacement of data by pQCD does not reduce the uncertainty. The reason is hat the pQCD contribution replacing the Euclidean Adler function at $\sqrt{-t} >$ 2.5GeV shows a substantial uncertainty due to the uncertainty of the charm mass $m_c(m_c) =$ 1.15...1.35 GeV. The uncertainty in the strong coupling constant $\alpha_s(M_Z^2) = 0.120 \pm 0.003$ is small and is not the dominating effect. In contrast to other authors which use pQCD for estimating $R(s)$ directly, we do not obtain a reduction of the error. Of course our cut at 2.5 GeV, which we think is all we can justify, is more conservative than the 1.8 GeV in the time–like region anticipated elsewhere. Thus the best value we can obtain from presently available e^+e^-–data alone is the result (15). The Euclidean method of calculating $a_\mu^{\rm had}$ will only be useful once the QCD parameters will be determined much more accurately. For recent progress in this direction we refer to [44–47].

7. SUMMARY AND OUTLOOK

Evaluations of the hadronic vacuum polarization effects based on e^+e^-–data agree reasonably well between different groups, especially for the "conservative" analyzes which rely directly on the experimental data. The accuracy typically is 1.3%. Future precision physics at a high luminosity linear collider would require an improvement of the precision in $\alpha(M_Z)$ by a factor of about 5 at least. The "theory-driven" analyzes which replace data by perturbatively calculated R–ratios obtain much smaller errors (about a factor 2) in general. Obviously much more reliable is the Adler function "monitored" evaluation, which utilizes pQCD only for the smooth Adler function which is much better accessible to perturbation theory at sufficiently large space-like momenta ($|Q| > 2.5$GeV). The reason why this approach is not so popular is the fact that the method is much more elaborate. Necessary improvements are possible only by measurements at the 1% level of the hadronic cross-sections up to J/ψ or better up to the Υ. Plans for future experiments in this direction exist at many places (Novosibirsk, Frascati, SLAC, Beijing, Cornell and KEK). The τ–data are potentially important for improving the $a_\mu^{\rm had}$ evaluation. However, they can only be utilized after appropriate iso-spin corrections. It is likely that the observed τ vs e^+e^- disagreement is a so far unaccounted iso-spin breaking effect, due to the difference in the physical mass and width of the charged $\rho^\pm$ observed in τ–decays and of the neutral ρ^0 seen in e^+e^- annihilation.

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