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A new hypergeometric representation of one-loop scalar integrals in d dimensions

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Abstract

A difference equation w.r.t. space-time dimension d for n-point one-loop integrals with arbitrary momenta and masses is introduced and a solution presented. The result can in general be written as multiple hypergeometric series with ratios of different Gram determinants as expansion variables. Detailed considerations for $2-, 3-$ and $4-$point functions are given. For the $2-$point function we reproduce a known result in terms of the Gauss hypergeometric function ${}_2F_1$. For the $3-$point function an expression in terms of ${}_2F_1$ and the Appell hypergeometric function F_1 is given. For the $4-$point function a new representation in terms of ${}_2F_1$, F_1 and the Lauricella-Saran functions F_S is obtained. For arbitrary $d = 4 - 2\varepsilon$, momenta and masses the $2-, 3-$ and $4-$point functions admit a simple one-fold integral representation. This representation will be useful for the calculation of contributions from the $\varepsilon-$ expansion needed in higher orders of perturbation theory. Physically interesting examples of $3-$ and $4-$point functions occurring in Bhabha scattering are investigated.

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1 Introduction

The calculation of radiative corrections in the electroweak Standard Model (SM) is especially demanding because of the many species of particles and fields and the large variety of interactions between them as well as the many mass and energy scales which typically show up in a high energy scattering process. Practical problems in the calculation and numerical evaluation of Feynman integrals were encountered at an early stage of SM calculations beyond the tree level, at the level of the one–loop integrals already. To a large extent, known analytic results and techniques have been reviewed, extended and discussed long time ago in [1]. In addition M. Veltman first had the idea to develop a library of numerical routines, the program `FormF` [2], which allowed to calculate the one–loop radiative corrections for $2 \to 2$ scattering processes in a numerically stable way. Unfortunately, this program was only running in CDC assembler, and used very high precision (more than 100 digits) to overcome the numerical instabilities that plagues the calculation of one-loop electroweak radiative correction. Later G.J. van Oldenborgh implemented the most relevant one–loop integrals as a normal `Fortran` library `FF` [3], which utilizes alternative algorithms which run in double precision `Fortran 77`. This program was very useful for many of the calculations which were needed for precision physics at LEP. Meanwhile a number of extensions to two– and more–loops have been developed. For example, the programs `SHELL2` [4] and `MINCER` [5] allow to calculate special types of integrals.

More recently we face new problems in view of the requirements for future colliders. For example, we certainly will need full electroweak two–loop calculations for $2 \to 2$ fermion processes as well as full one–loop $2 \to 4$ fermion processes for precision physics at a future e^+e^- linear collider like TESLA. In both cases the existing program libraries need important extensions.

In the first case, for the renormalization of two and more loops, one needs one–loop integrals to higher order in the ε–expansion ($d = 4 - 2\varepsilon$), while in the second case efficient algorithms are needed for the calculation of $5-$, $6-$ and higher– point functions, which at least on the level of the scalar integrals are ultraviolet finite in $d = 4$ dimensions. Usually, one may reduce them to a sum of $4-$ and lower–point functions, however with "unphysical" external kinematics. While `FF` serves all requirements to calculate $2 \to 2$ processes at one–loop, it does not work in general if we try to evaluate the expressions we obtain as a result from the reduction of higher point functions and we thus have to extend this tool appropriately.

Another important extension we need stems from the fact that unstable particles, like the massive gauge bosons W, Z, the Higgs and the top quark, show up as intermediate states in scattering matrix elements. Most interesting is their production near resonance, where ordinary perturbation theory breaks down and requires to work with finite width propagators not only for single resonant virtual particle lines but also within loops. Most obviously this shows up in the interplay between virtual and real photon emission in infrared sensitive quantities.

In this paper we propose a new method to deal with one–loop integrals which allows for extensions in several respects. The algorithm we propose works for arbitrary space–time dimensions, for complex masses and momenta and allows to find the appropriate physical domain of the analytic structure.

The paper is organized as follows: in Sect. 2 we recall the basic recursion relation and introduce the relevant kinematical quantities. Based on this recursion relation a difference equation and its solution is given in Sect. 3, which quite generally represents $n-$point functions in terms of series over $(n-1)-$point functions. An asymptotic method to determine the 'boundary term' is discussed. In the following sections, we give explicit representations: in Sect. 4 for the 2−point function, where also the approach itself is demonstrated in great detail; similarly in Sect. 5 for the 3−point function and in Sect.6 for the 4−point function. Explicit examples needed in Bhabha scattering are also presented. For the boundary term of a 4−point Bhabha diagram a differential equation is set up as alternative method for its determination.

2 General remarks on one–loop functions

Relations between Feynman integrals in different dimensions are known since quite a long time [6]. Recurrence relations for n-point one-loop integrals, required for our investigation, are given in [7], simplifying and extending the work of [8]. We consider scalar one-loop integrals depending on $n-1$ external momenta:

$$I_n^{(d)} = \int \frac{d^d q}{[i\pi^{d/2}]} \prod_{j=1}^{n} \frac{1}{c_j^{\nu_j}}, \tag{1}$$

where

$$c_j = (q-p_j)^2 - m_j^2 + i\epsilon. \tag{2}$$

The corresponding diagram and the convention for the momenta are given in Fig.1.

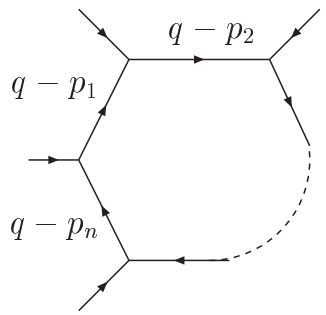

Fig. 1: One-loop diagram with n external legs.

A recurrence relation connecting integrals $I_n^{(d)}$ with different space-time dimensions was given in [7]

$$(d - \sum_{i=1}^{n} \nu_i + 1) G_{n-1} I_n^{(d+2)} = \left[2\Delta_n + \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- \right] I_n^{(d)}, \tag{3}$$

where $\partial_j \equiv \partial/\partial m_j^2$, and

$$\Delta_n \equiv \Delta_n(\{p_1,m_1\},\dots\{p_n,m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}, \tag{4}$$

$$Y_{ij} = -(p_i - p_j)^2 + m_i^2 + m_j^2, \tag{5}$$

$$G_{n-1} \equiv G_{n-1}(p_1,\dots,p_n) =$$

$$-2^n \begin{vmatrix} (p_1-p_n)(p_1-p_n) & (p_1-p_n)(p_2-p_n) & \dots & (p_1-p_n)(p_{n-1}-p_n) \\ (p_1-p_n)(p_2-p_n) & (p_2-p_n)(p_2-p_n) & \dots & (p_2-p_n)(p_{n-1}-p_n) \\ \vdots & \vdots & \ddots & \vdots \\ (p_1-p_n)(p_{n-1}-p_n) & (p_2-p_n)(p_{n-1}-p_n) & \dots & (p_{n-1}-p_n)(p_{n-1}-p_n) \end{vmatrix}, \tag{6}$$

$$G_0 \equiv -2,$$

p_i are combinations of external momenta flowing through i-th lines, respectively, and m_i is the mass of the i-th line. In the following we also quite often use the abbreviation $p_{ij}^2 = (p_i - p_j)^2$. By shifting the integration momentum we choose in general $p_n = 0$. Where no confusion can arise, we simply refer to the above functions as Δ_n, G_{n-1}. Considering integrals for $n = 2, 3, 4$ in the next sections we will use also an indexed notation for Δ_n and G_{n-1}

$$\lambda_{i_1 i_2 \dots i_n} = \Delta_n(\{p_{i_1}, m_{i_1}\}, \{p_{i_2}, m_{i_2}\}, \dots, \{p_{i_n}, m_{i_n}\}),$$

$$g_{i_1 i_2 \dots i_n} = G_{n-1}(p_{i_1}, p_{i_2}, \dots, p_{i_n}). \tag{7}$$

The indexed notation will be useful when considering integrals obtained from $I_n^{(d)}$ by contracting some lines. Rather frequently the results will depend on ratios of $\lambda_{i_1 i_2 \dots i_n}$ and $g_{i_1 i_2 \dots i_n}$ and therefore it is convenient to introduce the notation

$$r_{ij\dots k} = -\frac{\lambda_{ij\dots k}}{g_{ij\dots k}}. \tag{8}$$

With this definition a useful relation is:

$$r_{i_1 \dots i_{k-1} i_k i_{k+1} \dots i_n} - r_{i_1 \dots i_{k-1} i_{k+1} \dots i_n} = -\frac{(\partial_{i_k} \lambda_{i_1 \dots i_n})^2}{2 g_{i_1 \dots i_n}\ g_{i_1 \dots i_{k-1} i_{k+1} \dots i_n}}. \tag{9}$$

Using

$$\sum_{j=1}^{n} \partial_j \lambda_{i_1 \dots i_n} = -g_{i_1 \dots i_n} = -G_{n-1} \tag{10}$$

one shows that to all orders in ϵ

$$\lambda_{i_1 i_2 \dots i_n}(\{m_r^2 - i\epsilon\}) = \lambda_{i_1 i_2 \dots i_n}(\{m_r^2\}) + i g_{i_1 i_2 \dots i_n}\ \epsilon, \tag{11}$$

and therefore the ϵ prescription for r is rather simple (with the same ϵ for all masses)

$$r_{ij\ldots k}|_{m_j^2-i\epsilon} = r_{ij\ldots k}|_{m_j^2} - i\epsilon. \tag{12}$$

As we will see later, the results for one loop integrals will be expressed in terms of hypergeometric functions depending on ratios of different r's plus terms proportional to $r_{i\ldots j}^{d/2}$. Relation (12) makes the r's appear like masses, i.e. their $i\epsilon$ prescription is the same as for masses (their dimension is mass squared). This simple property is of extreme importance for the study of the analytic behavior of one loop integrals, in particular their analytic continuation to different kinematical regions for arbitrary space-time dimension. For brevity, we shall omit "causal" $i\epsilon$'s below.

3 Solution of the difference equation

In this section we will derive an explicit solution of relation (3) for $\nu_i = 1$. This relation has the simpler form

$$(d-n+1)G_{n-1}I_n^{(d+2)} = \left[2\Delta_n + \sum_{k=1}^{n}(\partial_k\Delta_n)\mathbf{k}^-\right] I_n^{(d)}. \tag{13}$$

If we assume that, evaluating n -point functions, we already know $n-1$ point functions, then relation (13) represents an inhomogeneous first order difference equation with respect to d. Methods to solve this kind of equations are well described in the mathematical literature [9], [10]. By the redefinition

$$I_n^{(d)} = \frac{1}{\Gamma\left(\frac{d-n+1}{2}\right)}\left(\frac{\Delta_n}{G_{n-1}}\right)^{\frac{d}{2}}\overline{I}_n^{(d)} \tag{14}$$

we obtain the simpler equation

$$\overline{I}_n^{(d+2)} = \overline{I}_n^{(d)} + \frac{\Gamma\left(\frac{d-n+1}{2}\right)}{2\Delta_n}\left(\frac{G_{n-1}}{\Delta_n}\right)^{\frac{d}{2}}\sum_{k=1}^{n}(\partial_k\Delta_n)\mathbf{k}^- I_n^{(d)}. \tag{15}$$

Without loss of generality we can parameterize d as

$$d = 2l - 2\varepsilon, \tag{16}$$

where l is integer and ε small and possibly complex. Then the solution of the equation for $\overline{I}_n^{(d)}$ can be written as

$$\overline{I}_n^{(2l-2\varepsilon)} = \sum_{r=0}^{l}\frac{\Gamma\left(r-1-\varepsilon-\frac{n-1}{2}\right)}{2\Delta_n}\left(\frac{G_{n-1}}{\Delta_n}\right)^{r-1-\varepsilon}\sum_{k=1}^{n}(\partial_k\Delta_n)\mathbf{k}^- I_n^{(2r-2-2\varepsilon)} + \tilde{b}_n(\varepsilon), \tag{17}$$

where $\tilde{b}_n(\varepsilon) = \overline{I}_n^{(-2-2\varepsilon)}$ is an l independent constant. With $\sum_{r=0}^{l} = \sum_{r=0}^{\infty} - \sum_{r=l+1}^{\infty}$ and shifting the summation index in the second sum like $r \to r+l+1$, the solution (17) can be rewritten in the form

$$\overline{I}_n^{(2l-2\varepsilon)} = -\sum_{r=0}^{\infty}\frac{\Gamma\left(r+\frac{d-n+1}{2}\right)}{2\Delta_n}\left(\frac{G_{n-1}}{\Delta_n}\right)^{r+\frac{d}{2}}\sum_{k=1}^{n}(\partial_k\Delta_n)\mathbf{k}^- I_n^{(2r+d)} + \overline{b}_n(\varepsilon) \tag{18}$$

by redefining the l independent 'boundary term' $\tilde{b}_n$. The final result for $I_n^{(d)}$ then reads

$$I_n^{(d)} = b_n(\varepsilon) - \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{2\Delta_n}\right) \sum_{r=0}^{\infty} \left(\frac{d-n+1}{2}\right)_r \left(\frac{G_{n-1}}{\Delta_n}\right)^r \mathbf{k}^- I_n^{(d+2r)}. \tag{19}$$

As we will show in the next sections, b_n can be determined from the asymptotic behavior of $I_n^{(d)}$ for $d \to \infty$ or by setting up a differential equation for it. This term depends on the kinematic domain.

The series in the above solution converge in general if the expansion parameter does not exceed 1. If it does, one has to continue the series analytically. This can be done by different methods. If the result is already obtained in terms of known hypergeometric functions, e.g., then known formulae for their analytic continuation can be applied. Another method will be to modify the procedure of solving the difference equation. According to the general theory of difference equations, we should repeat the derivation in our case by using the following parameterization for d (changing the sign of l):

$$d = -2l - 2\varepsilon. \tag{20}$$

With this parameterization for the function

$$f_l = \overline{I}^{(-2l-2\varepsilon)}, \tag{21}$$

we obtain from Eq. (15)

$$f_{l+1} = f_l - \frac{\Gamma\left(\frac{-2l-2\varepsilon-n+1}{2}\right)}{2\Delta_n} \left(\frac{G_{n-1}}{\Delta_n}\right)^{-l-\varepsilon-1} \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- I_n^{(-2l-2-2\varepsilon)}. \tag{22}$$

Solving this equation we obtain for $I_n^{(d)}$

$$I_n^{(d)} = b_n(\varepsilon) - \frac{1}{(d-n-1)} \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{\Delta_n}\right) \sum_{r=0}^{\infty} \frac{1}{\left(\frac{3+n-d}{2}\right)_r} \left(-\frac{\Delta_n}{G_{n-1}}\right)^{r+1} \mathbf{k}^- I_n^{(d-2r-2)}. \tag{23}$$

where $(a)_r \equiv \Gamma(r+a)/\Gamma(a)$ is the Pochhammer symbol. Here we have for convenience introduced the same notation for the boundary term as before, keeping in mind, however, that the value may be different from the one in (19). In order to obtain convergent series for all the contributions in $\sum_k^n$, one can use different parameterizations for d in the various terms.

In general the situation is such that to obtain the multiple series is straightforward. The problem is rather to determine the boundary term. In the next chapter we discuss for this purpose the asymptotic behaviour for $d \to \infty$.

3.1 Asymptotic behavior of $I_n^{(d)}$ for $d \to \infty$

For arbitrary n the asymptotic value of $I_n^{(d)}$ as $d \to \infty$ can be found by utilizing the parametric representation for $I_n^{(d)}$, i.e., with

$$\frac{1}{c_1 c_2 \dots c_n} = \int_0^1 \dots \int_0^1 \frac{dx_1 \dots dx_{n-1} \quad \Gamma(n) \quad x_1^{n-2} x_2^{n-3} \dots x_{n-2}}{[c_1 x_1 \dots x_{n-1} + c_2 x_1 \dots x_{n-2}(1-x_{n-1}) + \dots + c_n(1-x_1)]^n}, \tag{24}$$

shifting q in order to remove the linear term and integrating over q by means of

$$\int \frac{d^dq}{[i\pi^{d/2}]}\frac{1}{(q^2-m_i^2)^\alpha}=(-1)^\alpha\frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)(m_i^2)^{\alpha-\frac{d}{2}}}, \tag{25}$$

any n-point function can be represented as a multiple parametric integral of the form:

$$I_n^{(d)}=\Gamma\left(n-\frac{d}{2}\right)\int_0^1\ldots\int_0^1 dx_1\ldots dx_{n-1}f(x_1,\ldots,x_{n-2})(h_n(\{p_j,m_s\},x_1,\ldots,x_{n-1}))^{\frac{d}{2}-n}. \tag{26}$$

In the analyticity domain of $I_n^{(d)}$ the polynomial $h_n(\{p_j,m_s\},x_1,\ldots,x_{n-1})\geq 0$ and $I_n^{(d)}$ is an integral of Laplace type. In the domain of nonanalyticity there are subdomains of the integration region where $h_n(\{p_j,m_s\},x_1,\ldots,x_{n-1})<0$ and due to that the integral $I_n^{(d)}$ gets an imaginary part. For this kinematical configuration $I_n^{(d)}$ may be represented as

$$\begin{aligned}I_n^{(d)}=\Gamma\left(n-\frac{d}{2}\right)\Bigg\{&\int\{dx\}\theta(h_n)f(\{x\})(h_n(\{p_j,m_s\},\{x\}))^{\frac{d}{2}-n}\\&+\cos\frac{\pi}{2}(d-2n)\sum_j\int_{\Omega_j^-}\{dx\}f(x_1,\ldots,x_{n-2})\,|h_n(\{p_j,m_s\},x_1,\ldots,x_{n-1})|^{\frac{d}{2}-n}\\&\pm i\sin\frac{\pi}{2}(d-2n)\sum_j\int_{\Omega_j^-}\{dx\}f(x_1,\ldots,x_{n-2})\,|h_n(\{p_j,m_s\},x_1,\ldots,x_{n-1})|^{\frac{d}{2}-n}\Bigg\}.\end{aligned} \tag{27}$$

Here Ω_j^- are subdomains of the integration region where $h_n<0$. The sign of the imaginary part has to be determined from the $i\epsilon$ prescription in h_n.

Our parametric integrals are of multiple Laplace type which in general can be written as

$$F(\lambda)=\int_\Omega f(x)\exp[\lambda S(x)]dx \tag{28}$$

(identifying $S(x)\equiv ln(|h(x)|)$ and $\lambda\equiv\frac{d}{2}-n$), where Ω is a bounded simply connected domain in k dimensional Euclidean space, $x=(x_1,\ldots x_k)$ and $S(x)$ is a real function. The functions $S(x)$ and $f(x)$ are sufficiently differentiable functions of their arguments throughout Ω. Supposing that a relative maximum of $S(x)$ in Ω is achieved at the interior point $x=\overline{x}$, [3] then at $\lambda\to\infty$ [11]

$$F(\lambda)\sim\exp[\lambda S(\overline{x})]\sum_{r=0}^{\infty}a_r\lambda^{-r-\frac{k}{2}} \tag{29}$$

(correspondingly this holds for a relative minimum). This expansion may be differentiated w.r.t. λ any number of times. The leading term of the expansion is

$$F(\lambda)=\exp[\lambda S(\overline{x})]\ (2\pi/\lambda)^{\frac{k}{2}}\ \frac{f(\overline{x})+O(\lambda^{-1})}{\sqrt{|\det S_{xx}(\overline{x})|}}, \tag{30}$$

[3] If the function h_n reaches its maximum value on the boundary of the integration region then $F(\lambda)\sim\exp[\lambda S(\overline{x})]\sum_{r=0}^{\infty}a_r\lambda^{-r-\frac{k+1}{2}}$.

where $\det S_{xx}$ is the Hessian determinant, the indices appearing as second derivatives. The behavior of $I_n^{(d)}$ for $d \to \infty$ can thus be obtained. An extremum is found from

$$\frac{\partial h_n}{\partial x_i} = 0, \tag{31}$$

and using the determinant representation of h_n, it is possible to show that [4]

$$\overline{x}_i = \frac{\sum_{k=1}^{n-i} \partial_k \Delta_n}{\sum_{j=1}^{n-i+1} \partial_j \Delta_n}, \tag{32}$$

and at this point

$$h_n(\{p_j, m_s\}, \overline{x}_1, \ldots, \overline{x}_{n-1}) = r_{1\ldots n}. \tag{33}$$

All $\overline{x}_i$ are inside the interior region, i.e.

$$0 \leq \overline{x}_i \leq 1, \quad (i = 1, \ldots, n-1) \tag{34}$$

if all derivatives $\partial_k \Delta_n$ have the same sign. Conversely, it is easy to see that, provided (34) is true, due to (10) all $\partial_k \Delta_n$ have the opposite sign as G_{n-1}: assuming $\sum_{k=1}^{n} \partial_k \Delta_n \geq 0 (\leq 0)$, from the left hand side of the inequalities (34) follows $\sum_{k=1}^{n-i} \partial_k \Delta_n \geq 0 (\leq 0), i = 1, \ldots, n-1$. Multiplying (34) with the positive (negative) denominators of $\overline{x}_i$, the right hand side yields $\partial_{n-i+1} \Delta_n \geq 0 (\leq 0), i = 1, \ldots, n-1$.

The general idea to determine the boundary term b_n from the asymptotic behavior as $d \to \infty$ is as follows: from (14) and (18) we see that for large d

$$b_n \sim r_{1\ldots n}^{d/2}. \tag{35}$$

Such a contribution can come from the asymptotic behavior of $I_n^{(d)}$ on the l.h.s. of (19) due to (29) and (33). This may happen, however, only if all $0 < \overline{x}_i < 1$, i.e. if for the specific kinematics under consideration an extremum occurs inside the integration region and an absolute maximum of $|h_n|$ does not occur on the border. Finally we can write in the region where the integrand of $I_n^{(d)}$ has an extremum inside the integration region

$$I_n^{(d)} = -(2\pi)^{\frac{n}{2}} \frac{\Gamma\left(1 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-n+1}{2}\right)} \frac{r_n^{\frac{d-n-1}{2}}}{\sqrt{\pi |G_{n-1}|}} - \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{2\Delta_n}\right) \sum_{r=0}^{\infty} \left(\frac{d-n+1}{2}\right)_r \times \left(\frac{G_{n-1}}{\Delta_n}\right)^r \mathbf{k}^- I_n^{(d+2r)}, \tag{36}$$

where $r_n = r_{1\ldots n}$ [5]. To determine the boundary term, convergent series are requested on the r.h.s. of (36). These series are convergent in general only in certain kinematical domains and only for these is the obtained boundary term the correct one. The method of analytic continuation to other domains has been indicated above. In fact, the summation with

[4] There are other solutions, but for them in general one of the $\overline{x}_i (i = 1, \ldots n-2)$ equals 0 such that in these cases no contribution to $I_n^{(d)}$ is obtained

[5] If the function h_n reaches its absolute maximum value on the boundary of the integration region then $I_n^{(d)}$ has no asymptotic behavior like $\sim r_{1\ldots n}^{d/2}$ and $b_n = 0$.

$l > 0$ and $l < 0$, respectively, for the analytic continuation turns out to be easier in general than the direct analytic continuation of hypergeometric functions. In the next section we demonstrate the procedure in all details for the 2−point function and also give an example for the summation with $l < 0$.

Analytic continuation of the (generalized) hypergeometric functions can also be useful to find the asymptotic behavior, like double logarithms, for certain diagrams in specific kinematic regions.

4 2-point function

We find it convenient to label the lines of $I_2^{(d)}$ as i, j because the integrals $I_2^{(d)}$ will be encountered in calculating $I_3^{(d)}$ (and $I_4^{(d)}$) as a result of contraction of different lines. Expression (19) for $I_2^{(d)}$ includes two one-fold sums over tadpole integrals $I_1^{(d)}$, given in (25). $I_1^{(d)}$ can also be obtained from (19) assuming that in dimensional regularization $I_0^{(d)} = 0$

$$I_1^{(d)}(m_i) = -\Gamma\left(1 - \frac{d}{2}\right)(m_i^2)^{\frac{d-2}{2}}. \tag{37}$$

For $n = 2$, substituting (37) into (19) gives (observe the different normalization of b_2 in comparison with (19)):

$$\begin{aligned}\frac{2\lambda_{ij} I_2^{(d)}}{\Gamma\left(1-\frac{d}{2}\right)} = b_2 \;&+\; \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(-\frac{m_j^2 G_1}{\lambda_{ij}}\right)^r \\ &+\; \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(-\frac{m_i^2 G_1}{\lambda_{ij}}\right)^r,\end{aligned} \tag{38}$$

where

$$G_1 = -4p_{ij}^2. \tag{39}$$

and the infinite series in (38) can be represented as hypergeometric functions, i.e.

$$\sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} z^r = {}_2F_1\left[\begin{matrix}1, \frac{d-1}{2}; \\ \frac{d}{2};\end{matrix}\; z\right] \tag{40}$$

The parametric formula for $I_2^{(d)}$ is a one-fold integral:

$$I_2^{(d)} = \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx_1 \; [p_{ij}^2 x_1^2 - x_1(p_{ij}^2 - m_i^2 + m_j^2) + m_j^2]^{\frac{d}{2}-2}. \tag{41}$$

The extremum of h_2 is located at

$$\overline{x}_1 = \frac{\partial_1 \lambda_{ij}}{\partial_1 \lambda_{ij} + \partial_2 \lambda_{ij}} = \frac{p_{ij}^2 - m_i^2 + m_j^2}{2p_{ij}^2}, \tag{42}$$

where

$$\lambda_{ij} = -(p_{ij}^2)^2 - m_i^4 - m_j^4 + 2p_{ij}^2 m_i^2 + 2p_{ij}^2 m_j^2 + 2m_i^2 m_j^2. \tag{43}$$

and

$$\partial_i \lambda_{ij} = 2(p_{ij}^2 - m_i^2 + m_j^2) \quad , \quad \partial_j \lambda_{ij} = 2(p_{ij}^2 + m_i^2 - m_j^2). \tag{44}$$

Since

$$\frac{\partial^2 h_2}{\partial x_1^2} = 2p_{ij}^2, \tag{45}$$

a maximum of h_2 inside the integration region exists for Euclidean momenta $p_{ij}^2 < -(m_j^2 - m_i^2)$ (without loss of generality we assume here $m_j \geq m_i$). A minimum exists 'inside' for $p_{ij}^2 > +(m_j^2 - m_i^2)$. In the first case we have $\partial_1 \lambda_{ij} < 0$ and $\partial_2 \lambda_{ij} < 0$ while in the second case we have $\partial_1 \lambda_{ij} > 0$ and $\partial_2 \lambda_{ij} > 0$, i.e. in both cases the two derivatives have the same sign.

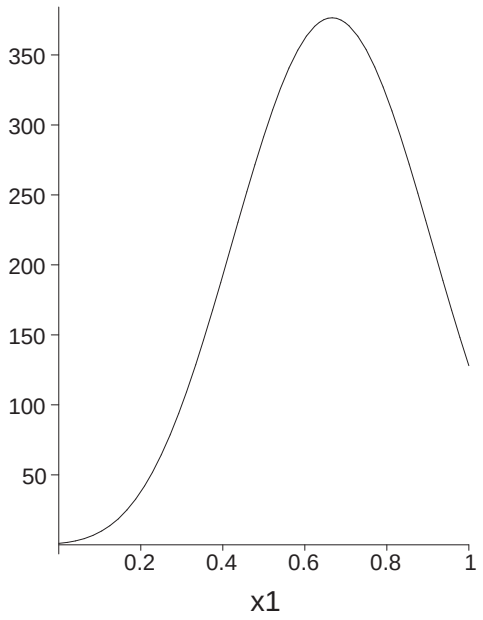

Fig. 2: Function $h_2^{d/2-2}$ at $p_{ij}^2 = -3$, $m_i^2 = 2, m_j^2 = 1$, $d = 18$.

For completeness we also give the imaginary part of $I_2^{(d)}$ on the cut, which we obtain from (27):

$$\text{Im } I_2^{(d)} = \pm\Gamma\left(2 - \frac{d}{2}\right) \sin\frac{\pi(d-2)}{2} (-p_{ij}^2)^{(\frac{d}{2}-2)} \int_{x_1^-}^{x_1^+} dx_1 ((x_1 - x_1^-)(x_1^+ - x_1))^{\frac{d}{2}-2}, \tag{46}$$

where

$$x_1^{\pm} = \frac{p_{ij}^2 - m_i^2 + m_j^2 \pm \sqrt{-\lambda_{ij}}}{2p_{ij}^2} \tag{47}$$

are solutions of the equation

$$h_2(\{p_{ij}, m_i, m_j\}, x_1) = 0. \tag{48}$$

For $p_{ij}^2 > (m_i + m_j)^2$ both $0 \leq x_1^{\pm} \leq 1$. Performing the integration in (46), we obtain

$$\text{Im } I_2^{(d)} = \frac{\mp \pi \Gamma\left(\frac{d-2}{2}\right)}{p_{ij}^{4-d} \Gamma(d-2)} \left(\frac{\sqrt{-\lambda_{ij}}}{p_{ij}^2}\right)^{d-3} \tag{49}$$

in agreement with the result [12] obtained by another method. Determining now b_2, we have to investigate the asymptotic behavior of $I_2^{(d)}$ for large d. For convenience we give the following expressions for r_{ij}:

$$r_{ij} = -\frac{\left[p_{ij}^2 - (m_i + m_j)^2\right]\left[p_{ij}^2 - (m_i - m_j)^2\right]}{4p_{ij}^2} \tag{50}$$

and

$$r_{ij} - m_j^2 = -\frac{(\partial_i \lambda_{ij})^2}{16 p_{ij}^2}, r_{ij} - m_i^2 = -\frac{(\partial_j \lambda_{ij})^2}{16 p_{ij}^2}. \tag{51}$$

We begin with $p_{ij}^2 < -(m_j^2 - m_i^2) < 0$. Here we have a maximum like the one shown in Fig. 2 with $0 < \overline{x}_1 < 1$. In this case $r_{ij} > m_i^2, m_j^2$ so that the hypergeometric series in (38) converge and a term $\sim r_{ij}^{d/2}$ comes from the asymptotic behavior of $I_2^{(d)}$. Evaluating this, we can write the result in the form

$$\begin{aligned}\frac{2\lambda_{ij} I_2^{(d)}}{\Gamma\left(1-\frac{d}{2}\right)} &= -\frac{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} r_{ij}^{\frac{d-2}{2}} \left[\frac{\partial_i \lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}} + \frac{\partial_j \lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right] \\ &+ \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(\frac{m_j^2}{r_{ij}}\right)^r + \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(\frac{m_i^2}{r_{ij}}\right)^r .\end{aligned} \tag{52}$$

For $-(m_j^2 - m_i^2) < p_{ij}^2 < 0$ we have again $r_{ij} > m_i^2, m_j^2$, but $\overline{x}_1 < 0$, so that there is no extremum inside the integration region and as a consequence no contribution $\sim r_{ij}^{d/2}$ comes from I_2^d and the boundary term is zero. In fact in this case the two contributions in the boundary term of (52) cancel and the formula is still valid.

Investigating as next $p_{ij}^2 > 0$ we can make the general observation that $r_{ij} < m_i^2, m_j^2$, but $r_{ij} < 0$ is possible . Thus, continuing further to $0 < p_{ij}^2 < (m_j - m_i)^2$ we have $-\infty < r_{ij} < 0$, but independent of its absolute value this does not cause problems since even if the series in (52) do not converge, for negative arguments the hypergeometric functions remain in their domain of analyticity (also the boundary term vanishes) and (52) is valid if written in the form

$$\begin{aligned}\frac{2\lambda_{ij}\ I_2^{(d)}}{\Gamma\left(1-\frac{d}{2}\right)} &= -\frac{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} r_{ij}^{\frac{d-2}{2}} \left[\frac{\partial_i \lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}} + \frac{\partial_j \lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right] \\ &+ \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}}\, {}_2F_1\left[\begin{matrix}1, \frac{d-1}{2}\,; \\ \frac{d}{2}\,;\end{matrix}\ \frac{m_j^2}{r_{ij}}\right] + \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}}\, {}_2F_1\left[\begin{matrix}1, \frac{d-1}{2}\,; \\ \frac{d}{2}\,;\end{matrix}\ \frac{m_i^2}{r_{ij}}\right] .\end{aligned} \tag{53}$$

By means of [13]

$$ {}_2F_1\left[\begin{array}{c} 1, \frac{d-1}{2} \\ \frac{d}{2} \end{array}; z\right] = (2-d)\, {}_2F_1\left[\begin{array}{c} 1, \frac{d-1}{2} \\ \frac{3}{2} \end{array}; 1-z\right] + \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \frac{z^{\frac{2-d}{2}}}{\sqrt{1-z}}, \qquad (54) $$
$$ |\arg(1-z)| < \pi $$

a compact expression is obtained from (53):

$$ \frac{\lambda_{ij}\, I_2^{(d)}}{\Gamma\left(2-\frac{d}{2}\right)} = \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}}\, {}_2F_1\left[\begin{array}{c} 1, \frac{d-1}{2} \\ \frac{3}{2} \end{array}; 1-\frac{m_j^2}{r_{ij}}\right] + \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}}\, {}_2F_1\left[\begin{array}{c} 1, \frac{d-1}{2} \\ \frac{3}{2} \end{array}; 1-\frac{m_i^2}{r_{ij}}\right], \qquad (55) $$

which applies as before for $r_{ij} > m_i^2, m_j^2$, i.e. the series for the hypergeometric functions do still converge, but beyond that it is a valid representation also when we proceed to $(m_j - m_i)^2 < p_{ij}^2 < (m_j + m_i)^2$ since $0 < r_{ij} < m_i^2, m_j^2$ in this case, i.e. the argument of the hypergeometric function gets negative and therefore is in the domain of analyticity.

Above the threshold, i.e. $(m_j + m_i)^2 < p_{ij}^2$, we have again $r_{ij} < 0$. Due to (50) for $p_{ij}^2 \to +\infty$, r_{ij} behaves as $r_{ij} \to -\infty$. Since in this case $0 < \overline{x}_1 < 1$ and h_2 has a minimum, apparently for some value of p_{ij}^2 we have $|r_{ij}| > m_i^2, m_j^2$ and the modulus of the argument of the hypergeometric function in (53) is < 1. Therefore, according to our approach, the boundary term given in (53) is also obtained in this kinematical domain.

A very useful transformation-formula, applied to (53), is

$$ {}_2F_1\left[\begin{array}{c} 1, \frac{d-1}{2} \\ \frac{d}{2} \end{array}; z\right] = \frac{1}{1-z}\, {}_2F_1\left[\begin{array}{c} 1, \frac{1}{2} \\ \frac{d}{2} \end{array}; \frac{z}{z-1}\right], \qquad (56) $$

i.e. for all $r_{ij} < 0$ the arguments of the hypergeometric functions can be transformed into values of modulus less than 1 such that the series converge.

To conclude, we have found for all $-\infty < p_{ij}^2 < +\infty$ a valid representation for $I_2^{(d)}$ and except for $0 < r_{ij} < m_i^2, m_j^2$ even convergent series. If we wish to obtain a convergent series in the latter case as well, we have to proceed as described in Sect. 3, (23), i.e. to perform the summation with $l < 0$. Without repeating the calculation in detail, we here just give the result:

$$ \frac{g_{ij} I_2^{(d)}}{\Gamma\left(2-\frac{d}{2}\right)} = -\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3-d}{2}\right)}{\Gamma\left(2-\frac{d}{2}\right)} \left[\frac{\partial_i \lambda_{ij}}{\sqrt{m_j^2 - r_{ij}}} + \frac{\partial_j \lambda_{ij}}{\sqrt{m_i^2 - r_{ij}}}\right] r_{ij}^{\frac{d-3}{2}} $$
$$ - \frac{\partial_i \lambda_{ij}}{(d-3)(m_j^2)^{\frac{4-d}{2}}}\, {}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2} \\ \frac{5-d}{2} \end{array}; \frac{r_{ij}}{m_j^2}\right] - \frac{\partial_j \lambda_{ij}}{(d-3)(m_i^2)^{\frac{4-d}{2}}}\, {}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2} \\ \frac{5-d}{2} \end{array}; \frac{r_{ij}}{m_i^2}\right]. \qquad (57) $$

Using

$$ {}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2} \\ \frac{5-d}{2} \end{array}; z\right] = (d-3)\, {}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2} \\ \frac{3}{2} \end{array}; 1-z\right] + \frac{\Gamma\left(\frac{5-d}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4-d}{2}\right)} \frac{z^{\frac{d-3}{2}}}{\sqrt{1-z}} $$
$$ |\arg(1-z)| < \pi. \qquad (58) $$

yields from (57) the already known result [14]:

$$\frac{g_{ij}\ I_2^{(d)}}{\Gamma\left(2-\frac{d}{2}\right)} = -\frac{\partial_i\lambda_{ij}}{(m_j^2)^{\frac{4-d}{2}}}\,{}_2F_1\left[\begin{matrix}1, \frac{4-d}{2};\\ \frac{3}{2};\end{matrix}\ 1-\frac{r_{ij}}{m_j^2}\right] - \frac{\partial_j\lambda_{ij}}{(m_i^2)^{\frac{4-d}{2}}}\,{}_2F_1\left[\begin{matrix}1, \frac{4-d}{2};\\ \frac{3}{2};\end{matrix}\ 1-\frac{r_{ij}}{m_i^2}\right]. \tag{59}$$

Having thus obtained different representations for $I_2^{(d)}$ by different ways of solving the difference equation (see Sect. 3), we can also show that they agree: by means of

$${}_2F_1\left[\begin{matrix}1, \frac{d-1}{2};\\ \frac{3}{2};\end{matrix}\ 1-z\right] = z^{-1}\,{}_2F_1\left[\begin{matrix}1, \frac{4-d}{2};\\ \frac{3}{2};\end{matrix}\ 1-\frac{1}{z}\right] \tag{60}$$

(59) can be obtained from (55).

For future applications we also give the ε expansion of ${}_2F_1$ occurring in (56) (see [15] formula A.3 in the Appendix):

$$\begin{aligned}{}_2F_1\left[\begin{matrix}1, \frac{1}{2};\\ \frac{d}{2};\end{matrix}\ x\right] &= \frac{(1-\varepsilon)(1-y^2)}{(1-2\varepsilon)y}\left[\frac{y}{1-y} + \ln(1-y)\varepsilon - \Big(\mathrm{Li}_2(y) + \ln^2(1-y)\Big)\varepsilon^2\right.\\ &\left. + \left(2S_{12}(y) + 2\ln(1-y)\mathrm{Li}_2(y) - \mathrm{Li}_3(y) + \frac{2}{3}\ln^3(1-y)\right)\varepsilon^3\right] + O(\varepsilon^4),\end{aligned} \tag{61}$$

where

$$y = \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}. \tag{62}$$

The ε expansion of ${}_2F_1$ in (58) can be obtained from

$${}_2F_1\left[\begin{matrix}1, 2-\frac{d}{2};\\ \frac{3}{2};\end{matrix}\ 1-z\right] = \frac{1}{z(2-d)}\,{}_2F_1\left[\begin{matrix}1, \frac{d-1}{2};\\ \frac{d}{2};\end{matrix}\ \frac{1}{z}\right] + \frac{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}\frac{z^{\frac{d-3}{2}}}{\sqrt{z-1}}. \tag{63}$$

More information on the ε expansion of hypergeometric functions is given in [16].

5 3-point function

In complete analogy to (38) we proceed for the 3-point function. In this case we have to sum according to (19) 2-point functions, which in Sect. 4 were represented in terms of hypergeometric functions ${}_2F_1$. In order to simplify the summation, we eliminate d in the first argument of ${}_2F_1$ by means of

$${}_2F_1\left[\begin{matrix}a, b;\\ c;\end{matrix}\ z\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix}a, c-b;\\ c;\end{matrix}\ \frac{z}{z-1}\right]. \tag{64}$$

Dissolving the result into double series, it can be written as Appell hypergeometric functions $F_3((d-2)/2, 1, 1, 1/2, d/2; x, y)$ defined as

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}\ m!\ n!}x^m y^n, \tag{65}$$

which in turn can be reduced to F_1 by means of [13]

$$F_3(\alpha,\alpha',\beta,\beta',\alpha+\alpha';x,y) = (1-y)^{-\beta'} F_1\left(\alpha,\beta,\beta',\alpha+\alpha';x,\frac{y}{y-1}\right), \tag{66}$$

F_1 being defined as

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}\ m!\ n!} x^m y^n. \tag{67}$$

In the next step the boundary term in (19) has to be determined by means of the asymptotic method described in Sect. 3.1 . The polynomial h_3 (see (27) for $n=3$) needed to determine a possible extremum 'inside' the integration region, is given by

$$\begin{aligned} h_3 = &-x_1x_2(1-x_1)p_{ik}^2 - x_1^2x_2(1-x_2)p_{ij}^2 - x_1(1-x_1)(1-x_2)p_{jk}^2 \\ &+ x_1x_2m_i^2 + x_1(1-x_2)m_j^2 + (1-x_1)m_k^2. \end{aligned} \tag{68}$$

It has an extremum at

$$\overline{x}_1 = -\frac{(\partial_i\lambda_{ijk}+\partial_j\lambda_{ijk})}{G_2}, \quad \overline{x}_2 = \frac{\partial_i\lambda_{ijk}}{\partial_i\lambda_{ijk}+\partial_j\lambda_{ijk}}, \tag{69}$$

provided the determinant

$$D = \frac{\partial^2 h_3}{\partial x_1\partial x_1}\frac{\partial^2 h_3}{\partial x_2\partial x_2} - \left(\frac{\partial^2 h_3}{\partial x_1\partial x_2}\right)^2 = -\frac{(\partial_i\lambda_{ijk}+\partial_j\lambda_{ijk})^2}{2G_2} > 0. \tag{70}$$

Further, the following second derivatives determine the type of the extremum:

$$\frac{\partial^2 h_3}{\partial x_1\partial x_1} = \frac{2G_2^2(m_k^2 - r_{ijk})}{(\partial_i\lambda_{ijk}+\partial_j\lambda_{ijk})^2}, \tag{71}$$

$$\frac{\partial^2 h_3}{\partial x_2\partial x_2} = \frac{2p_{ij}^2(\partial_i\lambda_{ijk}+\partial_j\lambda_{ijk})^2}{G_2^2}. \tag{72}$$

If both are >0, h_3 has a local minimum, if both are <0, h_3 has a local maximum. Thus we see that a maximum will be achieved if

$$G_2 < 0, \quad r_{ijk} > m_i^2, m_j^2, m_k^2, \quad p_{ij}^2 < 0. \tag{73}$$

Here the explicit appearance of m_k in (71) and of p_{ij}^2 in (72) are due to the particular choice of (68), i.e. the choice of the Feynman parameters. In general no preference of any choice of parameters can occur, hence the above condition for r_{ijk} and p_{ij}^2.

The result for $I_3^{(d)}$ finally reads [17]

$$\frac{\lambda_{ijk}}{\Gamma\left(2-\frac{d}{2}\right)} I_3^{(d)} = b_3 + \theta_{ijk}\ \partial_k\lambda_{ijk} + \theta_{kij}\ \partial_j\lambda_{ijk} + \theta_{jki}\ \partial_i\lambda_{ijk}, \tag{74}$$

where

$$b_3 = 2^{\frac{3}{2}}\pi\ \sqrt{-g_{ijk}}\ r_{ijk}^{\frac{d-2}{2}}, \tag{75}$$

(see (36)) provided an extremum of h_3 ($G_2 < 0$) occurs inside the integration region of the Feynman parameters. Otherwise $b_3 = 0$. For $\lambda_{ij} \neq 0$ we have

$$\begin{aligned}
\lambda_{ij}\ \theta_{ijk} &= -\left[\frac{\partial_i\lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}}+\frac{\partial_j\lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right] r_{ij}^{\frac{d-2}{2}}\ \frac{\sqrt{\pi}\ \Gamma\left(\frac{d-2}{2}\right)}{4\Gamma\left(\frac{d-1}{2}\right)}\ {}_2F_1\left[\begin{array}{l} 1,\frac{d-2}{2}\,; \\ \frac{d-1}{2}\,; \end{array}\ \frac{r_{ij}}{r_{ijk}}\right] \\
&+ \frac{(m_i^2)^{\frac{d-2}{2}}}{2(d-2)}\frac{\partial_j\lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};\frac{m_i^2}{r_{ijk}},\frac{m_i^2}{r_{ij}}\right) \\
&+ \frac{(m_j^2)^{\frac{d-2}{2}}}{2(d-2)}\frac{\partial_i\lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}}F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};\frac{m_j^2}{r_{ijk}},\frac{m_j^2}{r_{ij}}\right). \qquad (76)
\end{aligned}$$

To our knowledge there exists no simpler hypergeometric representation of the 3−point function for d dimensions in the literature. For another approach see e.g. [18]. The function F_1 admits a simple integral representation:

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-xu)^\beta(1-yu)^{\beta'}}\ du, \qquad (77)$$

which in our case is

$$F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};x,y\right) = \frac{d-2}{2}\int_0^1 \frac{u^{\frac{d-4}{2}}}{(1-xu)\sqrt{1-yu}}du. \qquad (78)$$

For further information on F_1, we mention for its analytic continuation [19] and concerning methods for its numerical evaluation [20]. For λ_{ij} arbitrary, using the transformation formula

$$\begin{aligned}
F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};x,y\right) &= \frac{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}y^{\frac{2-d}{2}}\ {}_2F_1\left[\begin{array}{l} 1,\frac{d-2}{2}\,; \\ \frac{d-1}{2}\,; \end{array}\ \frac{x}{y}\right] \\
&+ \frac{(2-d)\ \sqrt{1-y}}{(1-x)y}F_1\left(1,2-\frac{d}{2},1,\frac{3}{2};1-\frac{1}{y},\frac{x(1-y)}{y(1-x)}\right) \qquad (79)
\end{aligned}$$

the following compact form for θ_{ijk} is obtained:

$$\begin{aligned}
g_{ij}\ \theta_{ijk} = &+ \partial_j\lambda_{ij}\frac{(m_i^2)^{\frac{d-4}{2}}}{2\left(1-\frac{m_i^2}{r_{ijk}}\right)}\ F_1\left(1,2-\frac{d}{2},1,\frac{3}{2};1-\frac{r_{ij}}{m_i^2},\frac{r_{ij}-m_i^2}{r_{ijk}-m_i^2}\right) \\
&+ \partial_i\lambda_{ij}\frac{(m_j^2)^{\frac{d-4}{2}}}{2\left(1-\frac{m_j^2}{r_{ijk}}\right)}\ F_1\left(1,2-\frac{d}{2},1,\frac{3}{2};1-\frac{r_{ij}}{m_j^2},\frac{r_{ij}-m_j^2}{r_{ijk}-m_j^2}\right). \qquad (80)
\end{aligned}$$

where again F_1 has a simple integral representation

$$F_1\left(1,2-\frac{d}{2},1,\frac{3}{2};x,y\right) = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{1-u}}\frac{(1-xu)^{\frac{d-4}{2}}}{(1-yu)}. \qquad (81)$$

To find $I_3^{(d)}$ up to first order in $\varepsilon = (4-d)/2$, three terms in the expansion of this function are needed

$$
\begin{aligned}
F_1\left(1, 2-\frac{d}{2}, 1, \frac{3}{2}; x, y\right) &= -\frac{2B}{1-B^2}\left\{\ln B\right. \\
&+ \varepsilon\left[\mathrm{Li}_2(1-AB) + \mathrm{Li}_2\left(1-\frac{B}{A}\right) - 2\mathrm{Li}_2(1-B) + \frac{1}{2}\ln^2 A\right] \\
&+\varepsilon^2\left[\mathrm{Li}_3\left(\frac{A(1-AB)}{A-B}\right) - \mathrm{Li}_3\left(\frac{A(A-B)}{1-AB}\right) + 2\mathrm{Li}_3\left(\frac{A(1-B)}{1-AB}\right)\right. \\
&\quad - 2\mathrm{Li}_3\left(\frac{A(1-B)}{A-B}\right) + 2\mathrm{Li}_3\left(\frac{1-B}{A-B}\right) - 2\mathrm{Li}_3\left(\frac{1-B}{1-AB}\right) \\
&+2\left[\mathrm{Li}_2\left(\frac{A(A-B)}{1-AB}\right) - \mathrm{Li}_2\left(\frac{A(1-B)}{1-AB}\right) + \mathrm{Li}_2\left(\frac{1-B}{A-B}\right) - \mathrm{Li}_2(-A)\right]\ln(A) \\
&+\left[\frac{1}{2}\ln^2(A) - \zeta(2)\right]\ln\left(\frac{B-A}{1-AB}\right) - \frac{1}{6}\ln^3\left(\frac{B-A}{1-AB}\right) \\
&\left.\left.+\frac{1}{2}\ln(A)\ln^2\left(\frac{B-A}{1-AB}\right)\right] + O(\varepsilon^3)\right\}. \qquad (82)
\end{aligned}
$$

where

$$
A = \frac{\sqrt{1-\frac{1}{x}}-1}{\sqrt{1-\frac{1}{x}}+1}, \quad B = \frac{\sqrt{1-\frac{1}{y}}-1}{\sqrt{1-\frac{1}{y}}+1}. \qquad (83)
$$

This expansion will be of interest for an evaluation of one-loop counterterms inserted into vertex functions, required in two-loop calculations.

The general result(74) for the 3-point function with arbitrary masses and external momenta is finally a bit lengthy. To demonstrate the usefulness of our result, we choose as a particular case the on-shell 3-point function occurring in Bhabha scattering: $J_G^{(d)}$ with $m_1^2 = m_2^2 = 0$, $m_3^2 = m^2$ and $p_1^2 = p_2^2 = m^2$. For $0 < p_{12}^2 < 4m^2$ we have $G_2 < 0$, i.e. an extremum (minimum) exists, but $\overline{x}_1 = \frac{4m^2}{4m^2 - p_{12}^2} > 1$, such that it is not 'inside'. Therefore $b_3 = 0$ in this case.

Evaluating the three remaining terms in (74), we put $i, j, k = 1, 2, 3$, i.e. we have to calculate $\theta_{123}, \theta_{312}$ and θ_{231}. With $\lambda_{12} = -p_{12}^2$ and $\lambda_{31} = \lambda_{23} = 0$ we see that for the two latter cases we have to use (80). In the first case only the hypergeometric function ${}_2F_1$ remains:

$$
\theta_{123} = -\frac{\sqrt{\pi}\ \Gamma\left(\frac{d-2}{2}\right)}{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)}(-p_{12}^2)^{\frac{d-4}{2}}\ {}_2F_1\left[\begin{array}{c} 1, \frac{d-2}{2}\,; \\ \frac{d-1}{2}\,; \end{array}\ 1 - \frac{p_{12}^2}{4m^2}\right] \qquad (84)
$$

For θ_{312} and θ_{231}, using (80), only one of the F_1 functions remains, respectively, since $\partial_3\lambda_{31} = 0$ and $\partial_3\lambda_{23} = 0$. Moreover the two last contributions in (74) are the same. Evaluating thus θ_{312} we obtain:

$$
\theta_{312} = -\frac{1}{2}(m^2)^{\frac{d-4}{2}}\frac{p_{12}^2}{4m^2}F_1\left(1, 1, 2-\frac{d}{2}, \frac{3}{2}; 1 - \frac{p_{12}^2}{4m^2}, 1\right). \qquad (85)
$$

With $\lambda_{123} = -2m^2(p_{12}^2)^2$, $\partial_3\lambda_{123} = -2p_{12}^2$, $\partial_1\lambda_{123} = \partial_2\lambda_{123} = 4m^2p_{12}^2$, and

$$
F_1(a, b, b', c, w, 1) = \frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')}\ {}_2F_1\left[\begin{array}{c} a, b\,; \\ c-b'\,; \end{array}\ w\right] \qquad (86)
$$

the final result is

$$\frac{J_G}{\Gamma\left(2-\frac{d}{2}\right)} = \frac{(m^2)^{\frac{d}{2}-3}}{2(d-3)}\,{}_2F_1\left[\begin{array}{c}1,1\,;\\ \frac{d-1}{2}\,;\end{array}\,1-\frac{p^2}{4m^2}\right] - \frac{\sqrt{\pi}\Gamma\left(\frac{d-2}{2}\right)(-p^2)^{\frac{d-4}{2}}}{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)m^2}\,{}_2F_1\left[\begin{array}{c}1,\frac{d-2}{2}\,;\\ \frac{d-1}{2}\,;\end{array}\,1-\frac{p^2}{4m^2}\right]. \tag{87}$$

The result for J_G at $d=4$ was given in [21].

6 4-point function

The calculation of the 4−point function follows the same scheme as has been described in detail in the foregoing sections. Concerning the boundary term, it is in principle not difficult to find a relative extremum in a cube, but to find a proper formulation of all conditions (as has been done in the case of the 3−point function, see (70) - (73)) in terms of 'standard' expressions like G_3 etc. turns out to be tedious. However, due to the discussion after (34), one excludes an extremum 'inside' if not all derivatives $\partial_k\Delta_n$ have the same sign. In the case of the 4−point Bhabha diagrams, nevertheless, we demonstrate an alternative approach to determine the boundary term.

We label the lines of the box integral in natural manner as i,j,k,l . As one particular case we select an $I_3^{(d)}$ obtained from $I_4^{(d)}$ by shrinking line l and take into account at first only the first F_1 term in (76):

$$I_{4,F_1}^{(d)} = -\left(\frac{\partial_l\lambda_{ijkl}}{2\lambda_{ijkl}}\right)\sum_{r=0}^{\infty}\left(\frac{d-3}{2}\right)_r\left(\frac{g_{ijkl}}{\lambda_{ijkl}}\right)^r I_{3,F_1}^{(d+2r)}. \tag{88}$$

In analogy to (64) for the 3−point function, to simplify the summation, we get rid of d in the first argument of F_1 by means of the transformation formula [13]

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha,\beta,\beta',\gamma;\frac{x}{x-1},\frac{y}{y-1}\right). \tag{89}$$

After this simplification we arrive at

$$\begin{aligned} I_{4,F_1}^{(d)} &= \frac{1}{8}\left(\frac{\partial_l\lambda_{ijkl}}{\lambda_{ijkl}}\right)\left(\frac{\partial_k\lambda_{ijk}}{\lambda_{ijk}}\right)\left(\frac{\partial_j\lambda_{ij}}{\lambda_{ij}}\right)\frac{(m_i^2)^{\frac{d-2}{2}}r_{ijk}}{r_{ijk}-m_i^2}\frac{r_{ij}}{r_{ij}-m_i^2}\Gamma\left(1-\frac{d}{2}\right) \\ &\sum_{r,m,n=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_r(1)_{m+n}(1)_r(1)_m\left(\frac{1}{2}\right)_n}{\left(\frac{d}{2}\right)_{r+m+n}\,m!n!r!}\left(\frac{m_i^2}{r_{ijkl}}\right)^r\left(\frac{m_i^2}{m_i^2-r_{ijk}}\right)^m\left(\frac{m_i^2}{m_i^2-r_{ij}}\right)^n \\ &= \frac{1}{8}\left(\frac{\partial_l\lambda_{ijkl}}{\lambda_{ijkl}}\right)\left(\frac{\partial_k\lambda_{ijk}}{\lambda_{ijk}}\right)\left(\frac{\partial_j\lambda_{ij}}{\lambda_{ij}}\right)\frac{(m_i^2)^{\frac{d-2}{2}}r_{ijk}}{r_{ijk}-m_i^2}\frac{r_{ij}}{r_{ij}-m_i^2}\Gamma\left(1-\frac{d}{2}\right) \\ &\times F_S\left(\frac{d-3}{2},1,1;1,1,\frac{1}{2};\frac{d}{2},\frac{d}{2},\frac{d}{2};\frac{m_i^2}{r_{ijkl}},\frac{m_i^2}{m_i^2-r_{ijk}},\frac{m_i^2}{m_i^2-r_{ij}}\right), \end{aligned} \tag{90}$$

where the Lauricella-Saran function F_S is given in terms of a triple hypergeometric series [22], [23]:

$$\begin{aligned} &F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_1,\gamma_1;x,y,z) \\ &\quad=\sum_{r,m,n=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_{m+n}(\beta_1)_r(\beta_2)_m(\beta_3)_n}{(\gamma_1)_{r+m+n}\;r!m!n!}x^ry^mz^n. \end{aligned} \tag{91}$$

In our case the following integral representation of F_S is useful [24]

$$\frac{\Gamma(\alpha_1)\Gamma(\gamma_1-\alpha_1)}{\Gamma(\gamma_1)}F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1;x,y,z) =$$
$$\int_0^1 \frac{t^{\gamma_1-\alpha_1-1}(1-t)^{\alpha_1-1}}{(1-x+tx)^{\beta_1}}F_1(\alpha_2,\beta_2,\beta_3;\gamma_1-\alpha_1;ty,tz)dt, \tag{92}$$

which, specified for the above parameters, yields

$$F_S\left(\frac{d-3}{2},1,1;1,1,\frac{1}{2};\frac{d}{2},\frac{d}{2},\frac{d}{2};x,y,z\right) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{3}{2}\right)}$$
$$\times\int_0^1 \frac{dt\ \sqrt{t}(1-t)^{\frac{d-5}{2}}}{(1-x+tx)}F_1\left(1,1,\frac{1}{2},\frac{3}{2};ty,tz\right). \tag{93}$$

By means of

$$F_1\left(a,b,b',b+b';w,z\right) = (1-z)^{-a}\,{}_2F_1\left[\begin{array}{c} a,b\,; \\ b+b'\,; \end{array}\ \frac{w-z}{1-z}\right] \tag{94}$$

F_1 in the integrand reduces to

$$F_1\left(1,1,\frac{1}{2},\frac{3}{2};ty,tz\right) = \frac{1}{1-tz}\,{}_2F_1\left[\begin{array}{c} 1,1\,; \\ \frac{3}{2}\,; \end{array}\ \frac{t(y-z)}{1-tz}\right] = \frac{\arcsin\sqrt{\frac{t(y-z)}{1-tz}}}{\sqrt{t(y-z)(1-ty)}} \tag{95}$$

and finally we obtain

$$F_S\left(\frac{d-3}{2},1,1;1,1,\frac{1}{2};\frac{d}{2},\frac{d}{2},\frac{d}{2};x,y,z\right)$$
$$= \frac{\Gamma\left(\frac{d}{2}\right)\ (y-z)^{-\frac{1}{2}}}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{3}{2}\right)}\int_0^1 \frac{\arcsin\sqrt{\frac{(y-z)t}{1-tz}}(1-t)^{\frac{d-5}{2}}}{(1-x+tx)\sqrt{1-ty}}\ dt. \tag{96}$$

Secondly the contribution to $I_4^{(d)}$ from ${}_2F_1$ in (76) can be evaluated in the same manner as in Sect. 5, where $I_3^{(d)}$ was calculated from $I_2^{(d)}$. The result reads:

$$I^{(d)}_{4,{}_2F_1} = \frac{\sqrt{\pi}}{8}\left(\frac{\partial_l\lambda_{ijkl}}{\lambda_{ijkl}}\right)\left(\frac{\partial_k\lambda_{ijk}}{\lambda_{ijk}}\right)\frac{(r_{ij})^{\frac{d-2}{2}}}{\lambda_{ij}\sqrt{1-\frac{r_{ij}}{r_{ijk}}}}\frac{\Gamma\left(\frac{4-d}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}$$
$$\times\left[\frac{\partial_i\lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}}+\frac{\partial_j\lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right]F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};\frac{r_{ij}}{r_{ijkl}},\frac{r_{ij}}{r_{ijk}}\right). \tag{97}$$

In complete analogy to (74) we now write the result for the 4-point function as

$$\frac{\lambda_{ijkl}}{\Gamma\left(2-\frac{d}{2}\right)}I_4^{(d)} \;=\; b_4$$
$$+\quad \phi_{ijkl}\ \partial_l\lambda_{ijkl}+\phi_{lijk}\ \partial_k\lambda_{ijkl}+\phi_{klij}\ \partial_j\lambda_{ijkl}+\phi_{jkli}\ \partial_i\lambda_{ijkl}, \tag{98}$$

where

$$\phi_{ijkl} = \Bigg\{ -\frac{\pi\ \sqrt{2}}{\sqrt{-g_{ijk}}}\ r_{ijk}^{\frac{d-4}{2}}\ {}_2F_1\Bigg[\begin{matrix}1, \frac{d-3}{2}\,; \\ \frac{d-2}{2}\,; \end{matrix}\ \frac{r_{ijk}}{r_{ijkl}}\Bigg]$$

$$+\frac{1}{8}\left(\frac{\partial_k \lambda_{ijk}}{\lambda_{ijk}}\right)\frac{1}{\lambda_{ij}}\Bigg[\frac{\sqrt{\pi}\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}\frac{r_{ij}^{\frac{d-2}{2}}}{\sqrt{1-\frac{r_{ij}}{r_{ijk}}}}\left(\frac{\partial_i \lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}}+\frac{\partial_j \lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right)$$

$$\times\ F_1\left(\frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; \frac{r_{ij}}{r_{ijkl}}, \frac{r_{ij}}{r_{ijk}}\right)$$

$$-\frac{2(m_i^2)^{\frac{d-2}{2}}}{d-2}(\partial_j \lambda_{ij})\frac{r_{ijk}}{r_{ijk}-m_i^2}\frac{r_{ij}}{r_{ij}-m_i^2}\ F_S\left(\frac{m_i^2}{r_{ijkl}}, \frac{m_i^2}{m_i^2-r_{ijk}}, \frac{m_i^2}{m_i^2-r_{ij}}\right)$$

$$-\frac{2(m_j^2)^{\frac{d-2}{2}}}{d-2}(\partial_i \lambda_{ij})\frac{r_{ijk}}{r_{ijk}-m_j^2}\frac{r_{ij}}{r_{ij}-m_j^2}\ F_S\left(\frac{m_j^2}{r_{ijkl}}, \frac{m_j^2}{m_j^2-r_{ijk}}, \frac{m_j^2}{m_j^2-r_{ij}}\right)\Bigg]\Bigg\}$$

$$+\{i,j,k \quad\to\quad k,i,j\}+\{i,j,k \quad\to\quad j,k,i\} \tag{99}$$

The first term on the r.h.s. of (99) comes from b_3 of (74), but is supposed to be 0 if the 3−point function is evaluated in a kinematical domain, where no boundary term occurs. The boundary term b_4 we give for completeness:

$$b_4 = 8\pi^{\frac{3}{2}}\ \frac{\Gamma\left(\frac{d}{2}\right)}{(d-2)\Gamma\left(\frac{d-3}{2}\right)}\sqrt{-g_{ijkl}}\ r_{ijkl}^{\frac{d-3}{2}} \tag{100}$$

with the same reasoning as for (75). So far this result demonstrates our general method also for the 4-point function.

To give an explicit and complete example we consider now the case with the following kinematical variables

$$p_1^2 = m^2, \quad p_2^2 = s, \quad p_3^2 = m^2, \quad p_1p_2 = p_2p_3 = \frac{1}{2}s, \quad p_1p_3 = m^2 - \frac{1}{2}t,$$
$$m_1^2 = m_3^2 = 0, \quad m_2^2 = m_4^2 = m^2, \tag{101}$$

which corresponds to the (scalar) box diagram with two photons in the s-channel, occurring in Bhabha scattering. Apart from (87) we also need the 3-point function $J_F^{(d)}$ with $m_1^2 = m_2^2 = m^2$, $m_3^2 = 0$ and $p_1^2 = p_2^2 = m^2$, which is

$$J_F^{(d)} = \frac{\Gamma\left(2-\frac{d}{2}\right)}{2\ m^{6-d}}\ {}_2F_1\Bigg[\begin{matrix}1, 3-\frac{d}{2}\,; \\ \frac{3}{2}\,; \end{matrix}\ \frac{q^2}{4m^2}\Bigg]. \tag{102}$$

With these two 3-point functions we can set up the difference equation for the 4-point function with kinematics given in (101), the solution of which finally reads (the indices of I_{1111} being the powers of the scalar propagators, see also (13))

$$I_{1111}^{(d)} = \frac{b_4}{z^{d/2}\Gamma\left(\frac{d-3}{2}\right)}$$

$$
\begin{aligned}
&-\frac{4m^{d-4}}{t(s-4m^2)}\Gamma\left(2-\frac{d}{2}\right)F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};\frac{s}{s-4m^2},-m^2z\right)\\
&+\frac{4m^{d-4}}{(d-3)t(s-4m^2)}\Gamma\left(2-\frac{d}{2}\right)F^{1;2;1}_{1;1;0}\left[\begin{matrix}\frac{d-3}{2}: & \frac{d-3}{2}, & 1; & 1;\\ \frac{d-1}{2}: & & \frac{d-2}{2}; & -;\end{matrix}\ -m^2z,1-\frac{4m^2}{t}\right]\\
&-\frac{\sqrt{\pi}(-t)^{\frac{d-4}{2}}}{2^{d-4}m\sqrt{t}}\ \frac{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}{(s-4m^2)\Gamma\left(\frac{d-1}{2}\right)}F_1\left(\frac{d-3}{2},1,\frac{1}{2};\frac{d-1}{2};\frac{tz}{4},1-\frac{t}{4m^2}\right)
\end{aligned}
\tag{103}
$$

with $z=\frac{4u}{t(4m^2-s)}$ and s,t,u the usual Mandelstam variables. Here the generalized hypergeometric functions are

$$
F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right)=\quad\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}}\ \frac{\left(\frac{1}{2}\right)_s}{(1)_s}\ x^ry^s, \tag{104}
$$

$$
F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};x,y\right)=\quad\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{3}{2}\right)_r\left(\frac{d-2}{2}\right)_s}\ x^ry^s \tag{105}
$$

and the Kampé de Fériet function [25]

$$
F^{1;2;1}_{1;1;0}\left[\begin{matrix}\frac{d-3}{2}: & \frac{d-3}{2}, & 1; & 1;\\ \frac{d-1}{2}: & & \frac{d-2}{2}; & -;\end{matrix}\ x,y\right]=\quad\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}}\ \frac{\left(\frac{d-3}{2}\right)_r}{\left(\frac{d-2}{2}\right)_r}\ x^ry^s=\phi(x,y). \tag{106}
$$

b_4 is the boundary term, which we have to determine in the following. First of all we calculate the derivatives $\partial_k\Delta_4$:

$$
\partial_1\Delta_4=\partial_3\Delta_4=2st(4m^2-s), \tag{107}
$$

and

$$
\partial_2\Delta_4=\partial_4\Delta_4=-2st^2. \tag{108}
$$

These two sets of derivatives obviously have opposite sign for physical values of the kinematical variables s and t (and also under their exchange for the crossed diagram). Thus, according to the discussion after (34), we conclude that the boundary term is zero.

Nevertheless it is of interest to investigate an alternative approach to determine the boundary term: instead of the asymptotic method described in Sect. 3 we set up a differential equation for b_4 with respect to m^2 . An expression for the derivative of $I^{(d)}_{1111}$ with respect to m^2 can be obtained by differentiating the parametric representation of $I^{(d)}_{1111}$

$$
I^{(d)}_{1111}=\frac{1}{i^{d/2}}\int_0^\infty\ldots\int_0^\infty\prod_{j=1}^{4}\frac{d\alpha_j}{D^{\frac{d}{2}}}\ \ \exp\left[i\frac{Q}{D}-i\sum_{k=1}^{4}\alpha_km_k^2\right] \tag{109}
$$

where

$$
\begin{aligned}
Q&=\alpha_1\alpha_2p_{12}^2+\alpha_1\alpha_3p_{13}^2+\alpha_1\alpha_4p_1^2+\alpha_2\alpha_3p_{23}^2+\alpha_2\alpha_4p_2^2+\alpha_3\alpha_4p_3^2,\\
D&=\alpha_1+\alpha_2+\alpha_3+\alpha_4.
\end{aligned}
\tag{110}
$$

Substituting the kinematical variables (101) gives

$$Q - D\sum_{k=1}^{4}\alpha_k m_k = \alpha_1\alpha_3 t + \alpha_2\alpha_4 s - (\alpha_2+\alpha_4)^2 m^2. \tag{111}$$

Differentiating (109) we obtain

$$\frac{\partial I^{(d)}_{1111}}{\partial m^2} = -2I^{(d+2)}_{1311} - 2I^{(d+2)}_{1212} - 2I^{(d+2)}_{1113}. \tag{112}$$

Using recursions with respect to the indices and dimension as described in [7], the integrals on the right hand side can be reduced to master integrals with the result

$$\begin{aligned}\frac{\partial I^{(d)}_{1111}}{\partial m^2} = \frac{1}{u(4m^2-s)}\Big\{&2[(4-d)t-u]I^{(d)}_{1111} - 8(d-3)I^{(d)}_F \\ &+ \frac{2(d-2)}{m^2}I^{(d)}_1(m^2) - \frac{2u(d-3)}{tm^2}I^{(d)}_G - (d-4)\frac{(u+4m^2)}{m^2}J^{(d)}_G\Big\},\end{aligned} \tag{113}$$

where $I^{(d)}_F$ and $I^{(d)}_G$ are 2-point functions

$$I^{(d)}_G = I^{(d)}_2\big|_{m_1=0,m_2=0} = \frac{\sqrt{\pi}}{(-p^2)^{(2-\frac{d}{2})}}\,\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{2^{d-3}\,\Gamma\left(\frac{d-1}{2}\right)}, \tag{114}$$

$$I^{(d)}_F = I^{(d)}_2\big|_{m_1=m,m_2=m} = (m^2)^{(\frac{d}{2}-2)}\,\Gamma\left(2-\frac{d}{2}\right)\,{}_2F_1\left[\begin{matrix}1, 2-\frac{d}{2};\\ \frac{3}{2};\end{matrix}\ \frac{p^2}{4m^2}\right]. \tag{115}$$

Differentiating (103) w.r.t. m^2 and equating the two expressions for the derivative $\frac{\partial I^{(d)}_{1111}}{\partial m^2}$, substituting (103) into the r.h.s. of (113) yields a differential equation for b_4. To derive this equation we used

$$\begin{aligned}
F_1 &= F_1\left(\frac{d-2}{2}, 1, \frac{1}{2}, \frac{d}{2}; x, y\right), \\
F_2 &= F_2\left(\frac{d-3}{2}, 1, 1, \frac{3}{2}, \frac{d-2}{2}; x, y\right), \\
2(x-y)x\frac{dF_1}{dx} &= [x(2-d)+y(d-3)]F_1 - y(d-3)F_1\left(\frac{d-2}{2}, 0, \frac{1}{2}, \frac{d}{2}; x, y\right) \\
&+ \quad x(1-y)(d-2)F_1\left(\frac{d}{2}, 1, \frac{1}{2}, \frac{d}{2}; x, y\right), \\
2(x-y)x\frac{dF_1}{dy} &= yF_1 + y(d-3)F_1\left(\frac{d-2}{2}, 0, \frac{1}{2}, \frac{d}{2}; x, y\right) \\
&- \quad y(1-x)(d-2)F_1\left(\frac{d}{2}, 1, \frac{1}{2}, \frac{d}{2}; x, y\right), \\
F_1(\alpha, 0, \beta', \gamma; x, y) &= {}_2F_1(\alpha, \beta', \gamma, y), \\
F_1(\alpha, \beta, \beta', \alpha; x, y) &= \frac{1}{(1-x)^\beta(1-y)^{\beta'}}.
\end{aligned} \tag{116}$$

For the Kampé de Fériet series the following relation was used

$$2(y-x)x\frac{\partial\phi(x,y)}{\partial x} = [y-(d-3)(y-x)]\phi(x,y)$$
$$-x(d-3)_2F_1\left(1,\frac{d-3}{2},\frac{d-2}{2},x\right)+(d-4)y\ {}_2F_1\left(1,\frac{d-3}{2},\frac{d-1}{2},y\right). \quad (117)$$

For the F_2 function we used the relation:

$$\begin{aligned}-8m^2stu\frac{\partial F_2}{\partial x} &= -8m^2(4m^2(t+u)-us)u\frac{\partial F_2}{\partial y}\\ &+t(4m^2-s)[(4m^2(t+u)-us)(d-4)-4m^2u]F_2\\ &+tu(s-8m^2)(s-4m^2)\ {}_2F_1\left(1,\frac{d-3}{2},\frac{d-2}{2},-m^2z\right)\\ &-(d-4)\ t^2(s-4m^2)^2\ {}_2F_1\left(1,3-\frac{d}{2},\frac{3}{2},\frac{s}{4m^2}\right) \quad (118)\end{aligned}$$

Finally the following differential equation was obtained for b_4:

$$\frac{\partial b_4}{\partial m^2} = \frac{2(u-4t)}{u(s-4m^2)}b_4. \quad (119)$$

As initial value we chose $b_4(m^2=0)=0$, as we found explicitely and thus $b_4=0$ for also for $m^2\neq 0$.

7 Conclusion

For many two-loop problems in the electroweak theory it is necessary to calculate one-loop diagrams in arbitrary dimension, e.g. to obtain the ε expansion when inserting counterterms. For this problem we offer a general solution for $2-$, $3-$ and $4-$point functions for arbitrary masses and kinematics. The solution is obtained in terms of generalized hypergeometric functions plus a 'boundary term'. To determine the latter, two different methods have been applied: an asymptotic expansion in the dimension d and alternatively a first order differential equation. Our results (for arbitrary kinematics) seem fairly lengthy, but they are finally expressed in terms of functions ${}_2F_1$,F_1 and F_S, which are quite accessible analytically as well as numerically. In particular we point out the simple integral representations (78) and (96), which can easily be evaluated for arbitrary kinematics, and thus will become of particular importance for the evaluation of $5-$point functions - as mentioned in the introduction. For diagrams occurring in Bhabha scattering we have given explicit results for arbitrary dimension of the needed one-loop diagrams. The obtained results will serve in further calculations as useful tools.

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