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FIRCLA, one-loop correction to $e^+e^- \to \nu\overline{\nu}H$ and basis of Feynman integrals in higher dimensions[‡]

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FIRCLA, one-loop correction to $e^+e^- \to \nu\overline{\nu}H$ and basis of Feynman integrals in higher dimensions

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An approach for an effective computer evaluation of one-loop multi-leg diagrams is proposed. It's main feature is the combined use of several systems - DIANA, FORM and MAPLE. As an application we consider the one-loop correction to Higgs production in $e^+ e^- \to \nu\overline{\nu}H$, which is important for future e^+e^- colliders. To improve the stability of numerical evaluations a non-standard basis of integrals is introduced by transforming integrals to higher dimensions.

1. INTRODUCTION

Electroweak SM calculations of one-loop corrections to processes with five and more external legs are quite demanding. The number of diagrams to be evaluated typically is in the several thousands, integrands of the individual diagrams are in general very lengthy and in spite of large cancellations which take place the final result usually comes out to be not as compact as required for an efficient and reliable numerical evaluation. At the present time there is no computer algebra system which allows one to perform in an efficient way complete calculations of multi-leg one-loop corrections. For example, FORM is very efficient in doing Lorentz algebra but it can not handle ratios of Gram determinants depending on several momenta and masses. To overcome this kind of problems we use a combination of several systems and exploit their most advantageous features. We are testing the effectiveness of our strategy in the evaluation of the one-loop correction to the process $e^+ e^- \to \nu\overline{\nu}H$. In the last part of this note we also will discuss improvements possible by utilizing non-standard sets of master-integrals.

2. FIRCLA

To achieve a better performance we use a combination of several tools like DIANA [1] based on C and program libraries based on FORM and MAPLE which also act as interfaces between them. Since the main tool of evaluation is based on recurrence relations we called this collection of packages FIRCLA which stands for **F***eynman* **I***ntegral* **R***ecursive* **C***alcu***LA***tor*.

FIRCLA works as follows. After the process is specified to DIANA it invokes QGRAF [2], to generate all diagrams, then constructs input expressions suitable for use by FORM and provides additional information like types and masses of particles as well as the relations between kinematical variables. The output from DIANA is stored in a file which is used as an input by our FORM package. Additionally, DIANA may be utilized to produce pictures of all or particular diagrams as a Postscript file.

The FORM package performs the Lorentz algebra, takes traces of the Dirac γ-matrices, transforms products of spinors times γ matrices to a chosen basis of amplitudes, and utilizing the algorithm [3], reduces tensor integrals to a combination of scalar integrals, some of them with shifted space-time dimension. For each diagram the FORM package creates a file with expression for further processing by our MAPLE package. The expressions are written in MAPLE format. Each of these files also provides information about scalar products of external momenta and the masses of the particles.

The values of all scalar invariants are calculated in the FORM package. Relations between the momenta carried by the lines of the diagrams (p_i) and the external momenta (q_j) are evaluated by DIANA after generating a diagram and then

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transfered via FORM to the MAPLE package. In this way all useful information from DIANA can be transfered to the MAPLE package. The relations between momenta are needed, for example, to find which integrals are independent or equivalent. In the FORM output files, after the kinematic information an expression in MAPLE format follows. It has the form of a sum of integrals multiplied by polynomials of scalar products, masses and spinor amplitudes. Integrals are just the names of MAPLE procedures like:

$$\mathrm{npoint}(\prod_{\mathrm{j=1}}^{\mathrm{N}} \mathrm{P(j,m_j^2)}^{\nu_\mathrm{j}}, \mathrm{p}, \mathrm{data}) \qquad (1)$$

where $P(j,m_j^2) = (k_1 - p_j)^2 - m_j^2 + i\epsilon$ is an inverse scalar propagator, ν_j its power, N the number of different propagators, the parameter p specifies the shift of the space-time dimension $D = d + 2p$ and *data* is a set of substitutions for the momenta and scalar invariants.

In the MAPLE package integrals are evaluated separately by using the recurrence relations algorithm for the evaluation of one-loop integrals described in [3,4] (see also [5]).

If the Gram determinants of an integral are different from zero then a set of three relations can be used: the relation for removing dots (a dot represents one power of momentum attached to the line) from lines reads

$$2\Delta_n \nu_j \mathbf{j}^+ I_n^{(d)} = \sum_{k=1}^{n} (1+\delta_{jk}) \left(\frac{\partial \Delta_n}{\partial Y_{jk}}\right) \times \left[d - \sum_{i=1}^{n} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_n^{(d)}. \qquad (2)$$

Another relation reduces the shift of the space-time dimension and the index of the j-th line simultaneously

$$G_{n-1}\nu_j \mathbf{j}^+ I_n^{(d+2)} = \left[(\partial_j \Delta_n) + \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n) \mathbf{k}^- \right] I_n^{(d)} . \qquad (3)$$

Integrals with shifted space-time dimension can be expressed in terms of integrals in generic dimension by applying the formula:

$$(d - \sum_{i=1}^{n} \nu_i + 1) G_{n-1} I_n^{(d+2)} = \left[2\Delta_n + \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- \right] I_n^{(d)} . \qquad (4)$$

In the above formulae the shift operators $\mathbf{j}^\pm$ etc. shift the indices $\nu_j \to \nu_j \pm 1$,

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix},$$

$$Y_{ij} = -(p_i - p_j)^2 + m_i^2 + m_j^2,$$

p_j are combinations of external momenta and m_j is the mass of the j-th line , $\partial_j \equiv \partial/\partial m_j^2$ and

$$G_{n-1} = -2^n \begin{vmatrix} p_1 p_1 & p_1 p_2 & \dots & p_1 p_{n-1} \\ p_1 p_2 & p_2 p_2 & \dots & p_2 p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 p_{n-1} & p_2 p_{n-1} & \dots & p_{n-1} p_{n-1} \end{vmatrix}.$$

When one of the determinants G_{n-1} and Δ_n is equal to zero it is possible to express integrals with n lines as a combination of integrals with $n-1$ lines. If $G_{n-1} = 0$ then the relation

$$I_n^{(d)} = -\sum_{k=1}^{n} \frac{(\partial_j \partial_k \Delta_n)}{(\partial_j \Delta_n)} \, \mathbf{k}^- I_n^{(d)} \qquad (5)$$

should be applied until one of the lines of the integral will be removed. For the integral with $n-1$ lines the Gram determinant may be different from zero and therefore the relations (2), (3) and (4) can be used if needed.

If $\Delta_n = 0$ one should apply the relation

$$(d - \sum_{i=1}^{n} \nu_i - 1) G_{n-1} I_n^{(d)} = \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d-2)},$$

until a line of the integral will be removed. As in the previous case if the Gram determinants for integrals with $n-1$ lines are different from zero one can use relations (2), (3) and (4). Since the space-time dimension of $n-1$-point integrals will be decreased by this reduction one should subsequently apply the relation

$$I_n^{(d)} \;=\; -\sum_{j=1}^{n} \nu_j \mathbf{j}^+ I_n^{(d+2)} \;\; , \qquad (6)$$

in order to increase the dimension back to $d = 4 - 2\varepsilon$.

Our MAPLE package reads the FORM output diagram by diagram, evaluates each integral from the input expression, adds results for the integrals and simplifies the sum. For each diagram the result is written in a separate file. The expression stored for each diagram is a combination of master-integrals and spinor-amplitudes multiplied by ratios of polynomials in scalar products of external momenta, masses and the dimension d. Summing up the results from all diagrams is performed by a separate program.

3. $O(\alpha)$ CORRECTION TO $e^+e^- \to \overline{\nu}\nu H$

In order to check the effectiveness of our method of evaluating Feynman diagrams we calculated the one-loop correction to Higgs production via

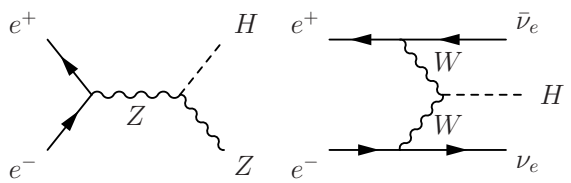
$$e^+(q_1)e^-(q_2) \to \overline{\nu}(q_3)\nu(q_5)H(q_4) \ .$$

This process will be important at future e^+e^- colliders. In the tree approximation it was considered in [6]. At one-loop order recently in [7].

At e^+e^- linear colliders operating in the 300–800 GeV energy range, the main production mechanisms for SM-like Higgs particles are

$$\begin{aligned} e^+e^- &\to (Z) \to Z + H \\ e^+e^- &\to \bar{\nu}\ \nu\ (WW) \to \bar{\nu}\ \nu\ + H \\ e^+e^- &\to e^+e^-(ZZ) \to e^+e^- + H \\ e^+e^- &\to (\gamma, Z) \to t\bar{t} + H \end{aligned}$$

i.e., "Higgs-strahlung", WW-fusion (see figure), ZZ-fusion, and "radiation off-top", respectively. The first two processes



are the dominant ones, in particular at energies above 500 GeV.

The Higgs-strahlung cross section for large s goes like $\sim 1/s$ and dominates at low energies. In contrast, the WW-fusion mechanism exhibits a cross section growing like $\sim \log(s/M_H^2)$ and therefore dominates at high energies. At $\sqrt{s} \sim 500$ GeV, the two processes have approximately the same cross sections. The relevance of one-loop corrections was considered in [8].

In order to generate all required diagrams a 'technical SM model' with two leptons doublets (in order to distinguish external leptons from leptons in loops) and one quarks doublet was considered. The contributions of the missing fermions have been obtained by adding up the corresponding results with the appropriate masses.

We put $m_e = 0$ but keep all other masses $m_\mu,\ m_b\ m_W,\ m_Z,\ m_t,\ m_H$ different from zero. The results for the pentagon diagrams depend on 5 scalar invariants. All calculations were done for arbitrary values of the gauge parameters $\xi_W,\ \xi_Z,\ \xi_\gamma$. In the specified model 326 diagrams contribute to the one-loop correction. Out of these 15 are pentagon type diagrams.

The results have been stored for each diagram separately in a file. Its size is $\sim$ 100 MB ($\sim 5MB$ in MAPLE format). The expression for each diagram has the form

$$D_k = \sum_{i,j} A_{i,j} I_i O_j \ ,$$

where I_i are master integrals and O_j are 24 spinor amplitudes.

After summing all diagrams we find that the result is gauge invariant as it must be. Each pentagon diagram separately gives a gauge invariant scalar 5-point integral - 15 in total. We observed a huge reduction in the number of scalar n-point (n=4,3,2) integrals D_0, C_0,B_0. For example, from 240 integrals C_0 only 79 remain in the final answer. It should be mentioned that at the initial stage before taking into account the symmetries of the integrals the total amount of master integrals C_0 with different values of momenta and masses was more than 500. A special routine has been added to the MAPLE package which recognizes the symmetries of integrals. By permuting masses and momenta into lexicographical order, taking into account momentum conservation, one obtains a set of independent standard integrals and hence a much more compact representation.

The final gauge invariant result is stored in a

file of about 1.2 MB in MAPLE format. Thus, it is still rather large for further numerical evaluations. Working out a more compact representation of the result, mainly by optimizing ratios of huge polynomials depending on several variables is one of the problems.

To obtain the complete radiative correction to the observable differential cross section we still have to add the bremstrahlung correction to our result. This work is in progress.

One observation we made is that in the final result the pentagon diagrams yield terms of the form

$$\frac{1}{(d-4)} I^{(d)}_{11111} \; . \tag{7}$$

In fact, second rank n-point tensor integrals produce terms of the form $\frac{1}{(d-n+1)} I^{(d)}_{11\ldots1}$ with a coefficient which is singular at $d=4$ for $n=5$. In our case, these terms originate from 2nd rank tensor integrals like

$$\int d^d k_1 \frac{(\overline{U}\ldots\hat{k_1}\ldots U)(\overline{V}\ldots\hat{k_1}\ldots V)}{c_1 c_2 c_3 c_4 c_5} \; . \tag{8}$$

The tensor integral here can be written in terms of scalar integrals with shifted dimension:

$$\begin{aligned}
\int d^d k_1 \frac{k_{1\mu} k_{1\nu}}{c_1 c_2 c_3 c_4 c_5} &= -\frac{1}{2} g_{\mu\nu} I^{(d+2)}_{11111} \\
&+ 2 p_{1\mu} p_{1\nu} I^{(d+4)}_{31111} + 2 I^{(d+4)}_{13111} p_{2\mu} p_{2\nu} \\
&+ 2 I^{(d+4)}_{11311} p_{3\mu} p_{3\nu} + 2 I^{(d+4)}_{11131} p_{4\mu} p_{4\nu} \\
&+ I^{(d+4)}_{22111} \{p_1, p_2\}_{\mu\nu} + I^{(d+4)}_{21211} \{p_1, p_3\}_{\mu\nu} \\
&+ I^{(d+4)}_{21121} \{p_1, p_4\}_{\mu\nu} + I^{(d+4)}_{12211} \{p_2, p_3\}_{\mu\nu} \\
&+ I^{(d+4)}_{12121} \{p_2, p_4\}_{\mu\nu} + I^{(d+4)}_{11221} \{p_3, p_4\}_{\mu\nu}
\end{aligned}$$

where $\{p_1, p_2\}_{\mu\nu} = p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu}$.

Different tensor structures give contributions to different of the abovementioned spinor amplitudes. By using the recurrence relations one gets

$$I^{(d+2)}_{11111} = \frac{2\Delta_5}{(d-4) G_4} I^{(d)}_{11111} + \sum_{k=1}^{5} \frac{\partial_k \Delta_5}{(d-4) G_4} \mathbf{k}^- I^{(d)}_5 \; .$$

The integral at the left hand side is UV and IR finite, however, in front of the integrals at the right hand side the spurious pole $1/\varepsilon$ appears. So it looks like we have to evaluate the ε term in the expansion of the pentagon integrals. There are no such problems for the 3- and 4-point functions.

In fact multiplying (8) by the Born term, summing over polarizations and taking the traces removes the problematic $1/(d-4)$ terms. However, if we would attempt to calculate the amplitude numerically before squaring it, we would have to work out first it's ε-expansion. A possibility to avoid this problem is to use a different basis of master-integrals in higher dimensions.

4. MASTER-INTEGRALS IN HIGHER DIMENSIONS

The idea to express one-loop tensor integrals in terms of integrals with shifted dimension was proposed in [9] and later in [5]. Recurrence relations which allow us to reduce any one-loop integral with shifted dimension to a combination of integrals in generic dimension were given in [3] and [4]. These were extended to massless integrals with both Gram determinants zero in [10].

In [11] it was discovered that one-loop integrals in higher dimensions provide better numerical stability.

A simple formula expressing any n-point integral in terms of integrals in higher dimensions was given in [4]. It reads

$$I^{(d)}_n = \frac{(d-n+1) G_{n-1}}{2\Delta_n} I^{(d+2)}_n - \sum_{k=1}^{n} \frac{(\partial_k \Delta_n)}{2\Delta_n} \mathbf{k}^- I^{(d)}_n \; .$$

The improved stability of numerical integrations of integrals in higher dimensions can be seen from their integral representation

$$I^{(d)}_n = \Gamma\left(n - \frac{d}{2}\right) \int_0^1 dx_1 \ldots \int_0^1 dx_{n-1} \; J_n h_n^{(d/2-n)},$$

where

$$\begin{aligned}
J_2 &= 1 \; , \; \cdots \; , \quad J_n = x_{n-2} x_{n-3}^2 \ldots x_1^{n-2} \; , \\
h_2 &= p_{12}^2 x_1^2 + m_2^2 - x_1 (p_{12}^2 - m_1^2 + m_2^2) \; , \\
h_3 &= -x_1 x_2 (1-x_1) p_{13}^2 - x_1^2 x_2 (1-x_2) p_{12}^2 \\
&\quad - x_1 (1-x_1)(1-x_2) p_{23}^2 + x_1 x_2 m_1^2 \\
&\quad + x_1 (1-x_2) m_2^2 + (1-x_1) m_3^2 \; .
\end{aligned}$$

If we transform integrals to the dimension $D = d + 2n - 2$ (assuming that $d = 4 - 2\varepsilon$) then the expansion of $I_n^{(d+2n-2)}$ for small $\varepsilon = 2 - d/2$ reads

$$I_n^{(d+2n-2)} = \Gamma\left(1+\varepsilon\right)\left[-\frac{s_n}{\varepsilon} - s_n - R_n^1 + O(\varepsilon)\right],$$

where

$$s_n = \int_0^1 \ldots \int_0^1 \{dx\}\; J_n h_n = \frac{-\sum p_{ij}^2}{(n+1)!} + \frac{1}{n!}\sum_{j=1}^{n} m_j^2,$$

$$R_n^1 = \int_0^1 \ldots \int_0^1 \{dx\}\; J_n h_n \ln h_n.$$

If we transform the integrals to dimension $D = d+2n-4$ then the expansion of $I_n^{(d+2n-4)}$ at small ε reads

$$I_n^{(d+2n-4)} = \Gamma\left(1+\varepsilon\right)\left[-\frac{1+\varepsilon}{(n-1)!\varepsilon} - R_n^0 + O(\varepsilon)\right],$$

where

$$R_n^0 = \int_0^1 \ldots \int_0^1 \{dx\}\; J_n \ln h_n.$$

We see that in the integrals $R_n^{0,1}$ there are no polynomials in the denominator and this is why these integrals in higher dimensions are more suitable for a direct numerical evaluation.

As an example we give the relations for some integrals in the logarithmic basis. In the latter for $n > 2$ we express the integrals $I_n^{(d)}$ in terms of integrals $I_n^{(d+2n-4)}$:

$$2\lambda_{123} I_3^{(d)} = (d-2) g_{123} I_3^{(d+2)} - \partial_1 \lambda_{123} I_2^{(d)}(23)$$
$$-\partial_2 \lambda_{123} I_2^{(d)}(13) - \partial_3 \lambda_{123} I_2^{(d)}(12),$$

$$4\lambda_{1234}^2 \lambda_{123}\lambda_{124}\lambda_{134}\lambda_{234} I_4^{(d)} = (d-1)(d-3)$$
$$\times g_{123}^2 \lambda_{123}\lambda_{124}\lambda_{134}\lambda_{234} I_4^{(d+4)}$$
$$+ f_{1234}^{(3)} + f_{2134}^{(3)} + f_{3124}^{(3)} + f_{4123}^{(3)} + f_{1234}^{(2)}$$
$$+ f_{3124}^{(2)} + f_{4123}^{(2)} + f_{3214}^{(2)} + f_{4213}^{(2)} + f_{4312}^{(2)}$$

where

$$f_{i_1 i_2 i_3 i_4}^{(2)} = (\partial_{i_1} \lambda_{i_1 i_2 i_3 i_4} \partial_{i_2} \lambda_{i_2 i_3 i_4} \lambda_{i_1 i_3 i_4}$$
$$+ \partial_{i_2} \lambda_{i_1 i_2 i_3 i_4} \partial_{i_1} \lambda_{i_1 i_3 i_4} \lambda_{i_2 i_3 i_4})$$
$$\times \lambda_{i_1 i_2 i_3 i_4} \lambda_{i_1 i_2 i_3} \lambda_{i_1 i_2 i_4} I_2^{(d)}(i_3 i_4)$$

$$f_{i_1 i_2 i_3 i_4}^{(3)} = -\partial_{i_1} \lambda_{i_1 i_2 i_3 i_4} ((d-2) g_{i_2 i_3 i_4} \lambda_{i_1 i_2 i_3 i_4}$$
$$+ (d-3) g_{i_1 i_2 i_3 i_4} \lambda_{i_2 i_3 i_4})$$
$$\times \lambda_{i_1 i_2 i_3} \lambda_{i_1 i_2 i_4} \lambda_{i_1 i_3 i_4} I_3^{(d+2)}(i_2 i_3 i_4).$$

where the following notation is used:

$$\lambda_{i_1 \ldots i_n} = \Delta_n, \qquad g_{i_1 \ldots i_n} = G_{n-1}\ .$$

In these expression we see multiple occurrence of Gram determinants and their derivatives. This fact may lead for some problems for kinematical regions where Gram determinants are close to zero. However outside these regions we expect good numerical stability of these integrals. In fact there is a correspondence between the integrals in higher dimensions and the method advocated in [12] for a direct numerical calculation of loop integrals. For the one-loop case, (3) and (4) are the explicite solutions of the relations which are derived in [12] by exploiting the Bernstein-Tkachov theorem.

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