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$\overline{\rm MS}$ vs. Pole Masses of Gauge Bosons: Electroweak Bosonic Two-Loop Corrections

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Abstract

The relationship between $\overline{\rm MS}$ and pole masses of the vector bosons Z and W is calculated at the two-loop level in the Standard Model. We only consider the purely bosonic contributions which represent a gauge invariant subclass of diagrams. All calculations were performed in the linear R_ξ gauge with three arbitrary gauge parameters utilizing the method of asymptotic expansions. The results are presented in analytic form as series in the small parameters $\sin^2\theta_W$ and the mass ratio m_Z^2/m_H^2. We also present the corresponding on-shell mass counter-terms for the massive gauge bosons, which will be needed for the calculation of observables at two-loops in the on-shell renormalization scheme.

1 Introduction

Precision Physics of the electroweak gauge bosons Z and W started about 12 years ago at the LEP storage ring with the ALEPH, DELPHI, L3 and OPAL experiments and ended just recently with the dismantling of the LEP installation. In particular the very accurate determination of the masses and the couplings to the fermions revealed unexpectedly rich information about the quantum correction of the Standard Model (SM) [1]. Calculations of higher order corrections thus gained increasing importance. At the one–loop level these calculations for the relevant "2 fermions into 2 fermions" processes were completed before LEP started operating in 1989 [2]. These SM predictions enabled an indirect determination of the top mass which culminated in the top discovery at the Tevatron. Now after the top mass has been fixed with rather good accuracy, the indirect bound to the Higgs mass, the

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last missing SM parameter, is the main goal. The knowledge of the actual value of the Higgs mass is extremely important because it determines how Higgs physics will look like at future colliders like the LHC or TESLA [3]. Since the sensitivity of SM predictions on the Higgs mass is weak the precise meaning of the indirect Higgs mass bounds depend crucially on the accuracy of the theoretical predictions. Fortunately a lot of important theoretical progress has been made in the last decade with the calculation of leading and some sub-leading two–loop effects [4]-[9]. However, no complete two–loop calculation could be achieved so far, because such calculations are hampered by the dramatic increase in complexity encountered in such calculations. How important the precise evaluation of radiative corrections is may be illustrated by the following fact: taking only the leading corrections, the shift $\Delta\alpha_{\rm em}$ in the fine structure constant and the quadratic top mass correction $\Delta\rho_{\rm top}$ in the relationship between neutral and charged current effective couplings, predictions are about 10σ off from the data for most of the precisely known observables like $\sin^2\theta^\ell_{\rm eff}$ or M_W [10, 11]. Thus the sub-leading effects are huge in relation to current experimental precision. Therefore the issue of sub-leading two–loop corrections has to be taken very seriously. They easily may obscure the interpretation of the indirect Higgs mass bound obtained from LEP experiments by using SM predictions which are incomplete at the two–loop level.

Although we are a long way from filling the gap, as a first step we calculate the full bosonic two-loop electroweak corrections to the on-shell masses of gauge bosons. In other words, we evaluate the two-loop renormalization constants, the relations between bare, $\overline{\rm MS}$ and on-shell masses, which are required as an essential part for the two-loop renormalization program in the on-shell scheme, with the fine structure constant α and the gauge boson masses M_W and M_Z as independent input parameters.

The programs developed for performing the present calculations can be applied without modifications for the calculation of the on-shell wave-function renormalization constants of the gauge bosons. Since these quantities are gauge dependent we don't reproduce corresponding results here. As yet we have not calculated any observable quantity. However, the new set of corrections will be important for future, complete electroweak two-loop calculations of physical observables. Also after the shutdown of LEP it is important to continue such calculations because the question how additional corrections affect the Higgs mass bound can be answered retrospectively, once given the precise LEP/SLC results. Full two-loop calculations would be indispensable in future in any case if a project like TESLA with the Giga$Z$ option would be realized.

The most recent essential progress here was achieved in the calculation of the top–quark contributions to the two-loop electroweak corrections. The corresponding contributions to the ρ-parameter were considered in [6], the one's to Δr, which determines the $M_W - M_Z$ relationship, in [7] and to the Z boson partial widths in [8][5]. Two different approaches—the asymptotic expansion method [12] and numerical integration[13] —have been used to perform these calculations. One of the important steps when performing these calculations is the two-loop renormalization of the gauge boson masses, which also is contributing to the $\sin^2\theta_W$ renormalization [14]-[16]. In the SM so far no complete analytical calculation of the

[5]Another important step forward was the calculation of the two–loop QED correction to the muon decay width [9].

two-loop renormalized propagator has been carried out [17]. The first available results were given for zero external momentum [5], when the original diagrams may be reduced to a set of bubble-type integrals with different masses [18]. In [19] the two-loop unrenormalized fermion corrections to the gauge boson propagator have been presented for off-shell momentum in the general linear R_ξ gauge. The results are presented in terms of scalar master integrals with several different mass scales, the masses of fermions and bosons. For the evaluation of these master diagrams analytical results [20, 21] or one-fold integral representations are available [22].

The main aim of the present paper is to present a calculation within the Standard Model of the two-loop bosonic contributions to the relationship between $\overline{\rm MS}$ and on-shell masses of the gauge bosons (W, Z). This relationship, alternatively, will be represented in terms of on-shell renormalization constants. We shall discuss in some detail the algorithm used for calculating two-loop electroweak corrections for on-shell quantities in an arbitrary gauge.

The paper is organized as follows. In Section 2 we briefly reconsider the definition of the pole mass of the massive gauge bosons within the Standard Model. The calculations have been performed with the help of computer programs which will be described in some detail in Section 3. In Section 4 we discuss the UV renormalization of the pole mass and the interrelation of our results with the standard renormalization group approach. In particular we make several cross-checks of the singular $1/\varepsilon^2$- and $1/\varepsilon$-terms. The numerical results for the finite parts are presented and discussed in Section 5. For further technical details and some useful formulae we refer to four appendices. In Appendix A we collect results for the one-loop propagator type diagrams. Special attention is given here to the ε-parts of the corresponding integrals which are needed for the two-loop calculation. Appendix B and C collect one-loop results in $d = 4$ and $d \neq 4$, respectively. They are included for completeness. In Appendix D we present the analytical coefficients which are the main results of our investigation.

2 Pole mass

The position of the pole s_P of the propagator of a massive gauge boson in a quantum field theory is a solution for p^2 at which the inverse of the connected full propagator equals zero, i.e.,

$$s_P - m^2 - \Pi(s_P, m^2, \cdots) = 0, \tag{2.1}$$

where $\Pi(p^2, \cdots)$ is the transversal part of the one-particle irreducible self-energy. The latter depends on all SM parameters but, in order to the keep notation simple, we have indicated explicitly only the dependence on the external momentum p and in some cases also m, where m is the mass of the particle under consideration. This can be either the bare mass m_0 or the renormalized mass defined in some particular renormalization scheme.

Generally, the pole s_P is located in the complex plane of p^2 and has a real and an imaginary part. We write

$$s_P \equiv M^2 - {\rm i}M\Gamma. \tag{2.2}$$

The real part of (2.2) defines M which we call the pole mass[6], while the imaginary part is related to the width Γ of the particle. This is the natural generalization of the physical mass of a stable particle, which is defined by the mass of its asymptotic scattering state.

For the remainder of the paper we will adopt the following notation: capital M always denotes the pole mass; lower case m stands for the renormalized mass in the $\overline{\rm MS}$ scheme, while m_0 denotes the bare mass. In addition we use e and g to denote the $U(1)_{\rm em}$ and $SU(2)_{\rm L}$ couplings of the SM in the $\overline{\rm MS}$ scheme.

In perturbation theory (2.1) is to be solved order by order. To two loops we have the following solution of (2.1)

$$s_P \;=\; m^2 + \Pi^{(1)}(m^2, m^2, \cdots) + \Pi^{(2)}(m^2, m^2, \cdots) + \Pi^{(1)}(m^2, m^2, \cdots)\Pi^{(1)\prime}(m^2, m^2, \cdots), \quad (2.3)$$

which yields the pole mass M^2 and the width Γ at this order. $\Pi^{(L)}$ is the bare ($m = m_0$) or $\overline{\rm MS}$ -renormalized (m the $\overline{\rm MS}$ -mass) L-loop contribution to Π, and the prime denotes the derivative with respect to p^2. In this way we need to evaluate propagator type diagrams and their derivatives at $p^2 = m^2$. Diagrammatically the self-energy contributions are shown in Fig. 1.

The (on-shell) mass counter terms δM^2 for the on-shell renormalization scheme are obtained by considering the real part of (2.3) upon identifying the r.h.s with the bare expression, setting $m^2 \equiv m_0^2 = M^2 + \delta M^2$ and solving for $\delta M^2 = (\delta M^2)^{(1)} + (\delta M^2)^{(2)}$ to two loops.

In this paper we show by explicit calculation at the two-loop level that the bosonic correction to the pole s_P (and hence to the on-shell mass counter-term) is a gauge invariant and infrared stable quantity[7]. The free propagator of a massive vector boson in the linear R_ξ gauge reads

$$D_{\mu\nu}(p) = \frac{i}{p^2 - m^2}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi m^2}\right), \quad (2.4)$$

where ξ is a gauge parameter. We decompose the vector boson self-energy $\Pi_{\mu\nu}(p^2, m^2, \cdots)$ into a transverse $\Pi(p^2)$ and a longitudinal $L(p^2)$ part

$$\Pi_{\mu\nu}(p^2, m^2, \cdots) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi(p^2) + \frac{p_\mu p_\nu}{p^2}L(p^2). \quad (2.5)$$

In (2.3) only the transverse part contributes and the dressed propagator reads

$$D_{\mu\nu}(p) = \frac{-i}{p^2 - m^2 - \Pi(p^2)}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{p_\mu p_\nu}{p^2}\;\cdots \quad (2.6)$$

The simple relation between the full propagator and the irreducible self-energy only holds if there is no mixing, like for the W-boson. In the neutral sector, because of $\gamma - Z$ mixing, we

[6]Throughout this paper we identified the terms pole mass and on-shell mass.

[7]From general considerations we know (see [16], for example), that after summing all diagrams, including the tadpoles, at a given order of perturbation theory the location of the pole of the propagator is a gauge invariant quantity. The gauge invariance of the 2-loop massless fermion correction to the pole mass of the gauge bosons was verified in [19].

cannot consider the Z and γ propagators separately. They form a 2×2 matrix propagator, so that (2.1) is modified into (see details in [16, 23])

$$s_P - m_Z^2 - \Pi_{ZZ}(s_P) - \frac{\Pi_{\gamma Z}^2(s_P)}{s_P - \Pi_{\gamma\gamma}(s_P)} = 0. \tag{2.7}$$

We note that the $\Pi_{\gamma Z}^2$ mixing term starts to contribute at the two-loop level. Obviously, we do not need to compute $\Pi_{\gamma\gamma}$ here since it starts to play a role only beyond the two-loop approximation. In the sequel we will denote the self-energies by Π_V ($V = W, Z$) with

$$\Pi_W(p^2, \cdots) = \Pi_{WW}(p^2, \cdots)$$

and

$$\Pi_Z(p^2, \cdots) = \Pi_{ZZ}(p^2, \cdots) + \frac{\Pi_{\gamma Z}^2(p^2, \cdots)}{p^2 - \Pi_{\gamma\gamma}(p^2, \cdots)} \; .$$

Thus, formally, the form (2.1) applies for both the W and the Z.

The non-zero imaginary part (width) (2.2) of the on-shell gauge boson self-energy appears as soon as the fermions are included. For the bosonic contributions alone the imaginary part of $\Pi(p^2)$ on the mass-shell is zero at the two-loop level (see below).

3 Program part

In order to find the relations between the pole masses M_Z^2, M_W^2 and the $\overline{\rm MS}$ masses m_Z^2, m_W^2 we have to compute the one- and two-loop self-energies for Z- and W-bosons at $p^2 = m_Z^2$ and $p^2 = m_W^2$, respectively. The complete set of topologies that occurs in this calculation is shown in Fig. 1. In order to be able to work with manifestly gauge parameter independent renormalization constants we have to include the Higgs tadpole diagrams.

While at one-loop order we have about 50 diagrams, in the two-loop approximation the number of diagrams is about 1000, which requires an automatized generation and evaluation of diagrams. We use **QGRAF** [24] to generate the diagrams and then the C-program **DIANA** [25] to produce for each diagram an input suitable for our **FORM** [26] packages[8].

For two-loop propagator type diagrams with several masses a complete set of recurrence relations is given in [27]. It allows us to reduce all tensor integrals to a small set of so-called master-integrals. However, the master-integrals which show up in the SM are not expressible in terms of known functions but may be written e.g. as one-fold integrals [22]. Instead of using these explicit formulae we resort to some approximations here, namely, we perform an appropriate series expansion in mass ratios[9]. Each coefficient of this series can be calculated analytically by means of the asymptotic expansion algorithm described in [12].

To keep control of gauge invariance we work in a R_ξ gauge with three different gauge parameters ξ_W, ξ_Z and ξ_γ. The corresponding free vector boson propagators (2.4) thus

[8] **DIANA** generates additional information, e.g. identifying symbols for the particles of the diagram and their masses, distribution of integration momenta, number of fermion loops etc.

[9] For diagrams with several different masses, there may exist several small parameters. In this case we apply different asymptotic expansions (see [28]) one after another.

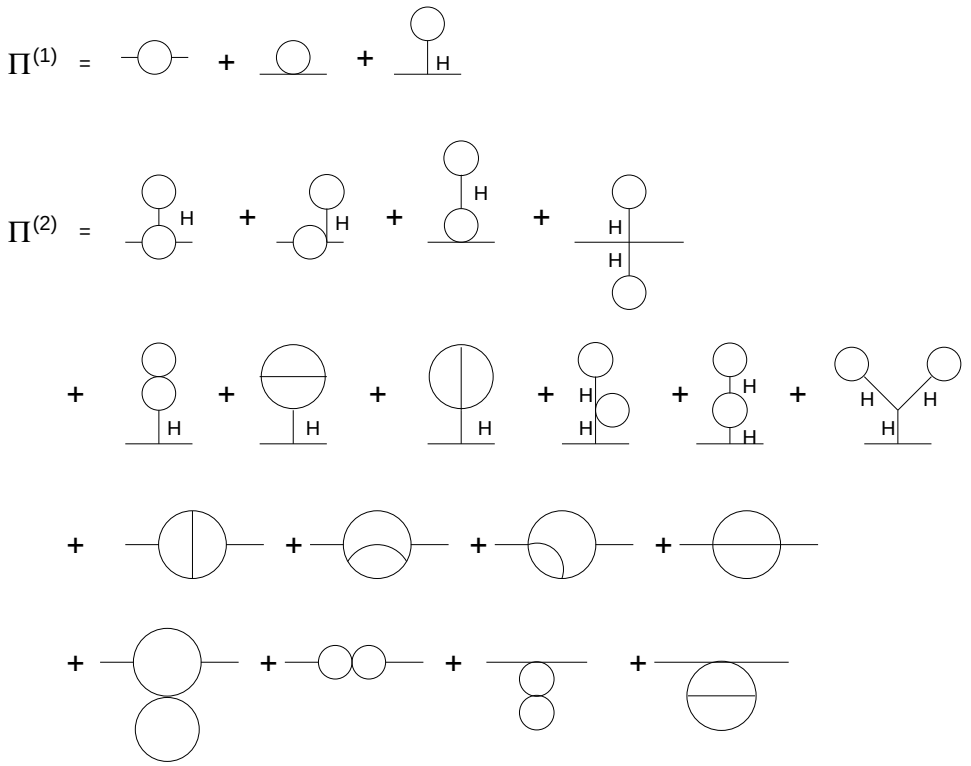


Figure 1: One- and two-loop contributions to the massive boson self-energies.

exhibit the new masses $\sqrt{\xi_W} m_W$ and $\sqrt{\xi_Z} m_Z$ in the propagators. This complicates the calculation enormously both in Tarasov's algorithm as well as in the asymptotic expansion approach. With the first method the presence of new masses in both the reduction formulae and the Gram determinants leads to cumbersome expressions which are difficult to simplify. In the asymptotic expansion approach the unphysical parameters $\xi_W m_W^2$ and $\xi_Z m_Z^2$ define two new scales. Of course, in order to keep things manageable, we have to keep the number of scales as small as possible. This can be done by expanding diagrams about some fixed values of the gauge parameters. Three different regimes of expansion are feasible: a) $\xi \to \infty$, b) $\xi \to 0$ and c) $\xi \to 1$. We choose the last possibility expanding the original propagators at $\xi_i = 1$ in a Taylor series. For the purpose of checking the gauge invariance of our results it is sufficient to keep the first three terms of the expansion, so that the propagators of the vector bosons and associated Higgs scalar ghosts look like

$$
\begin{aligned}
D^V_{\mu\nu}(p) &= \frac{i}{p^2 - m_V^2} \left(-g_{\mu\nu} + (1-\xi_V) \frac{p_\mu p_\nu}{p^2 - m_V^2} - (1-\xi_V)^2 \frac{m_V^2 p_\mu p_\nu}{(p^2 - m_V^2)^2} + \ldots \right), \\
\Delta_V(p) &= \frac{i}{p^2 - m_V^2} \left(1 - (1-\xi_V) \frac{m_V^2}{p^2 - m_V^2} + (1-\xi_V)^2 \left(\frac{m_V^2}{p^2 - m_V^2} \right)^2 + \ldots \right), \qquad (3.8)
\end{aligned}
$$

where $V = W, Z$ and the dots stand for the terms of higher order in $(1 - \xi_V)$, which we

don't take into account. At the same time we do not have any problems with the photon propagator and use it in its usual form.

One drawback of our choice of the expansion about $\xi_i = 1$ is that possible unphysical thresholds do not show up in our expansion and hence will not produce the correct imaginary part for a given diagram. Examples are the thresholds $p^2 = 4\xi_Z M_Z^2$ ($Z \to \phi^0\phi^0$ production, ϕ^0 the neutral Higgs ghost) which is below the Z mass-shell $p^2 = M_Z^2$ when $\xi_Z < \frac{1}{4}$ or $p^2 = \xi_W M_W^2$($W^\pm \to \phi^\pm\gamma$ production, $\phi^\pm$ the charged Higgs ghosts) which is below the W mass-shell $p^2 = M_W^2$ when $\xi_W < 1$. In such cases, the above expansion does not preserve the analytical properties of diagrams exhibiting ghost particles. However, in any case unphysical degrees of freedom should not contribute to the physical width. While individual diagrams exhibit an imaginary part, in the sum of all diagrams it has to cancel. This cancellation is a consequence of the Slavnov-Taylor identities, which tell us how Higgs ghost, Faddeev-Popov ghosts and scalar components contained in the gauge boson fields decouple from observables like the physical width. Therefore, the above expansion indeed reproduces the correct gauge invariant result. At the one-loop level one can do an exact analytic calculation for the R_ξ gauge with an arbitrary value of the gauge parameter. This has been done long time ago [16] and, not surprisingly, the on-shell self-energies of the W- and Z-bosons are gauge invariant and do not exhibit any unphysical threshold (possible problems related to unphysical thresholds in the non-gauge invariant wave-function renormalization factor are also discussed in [16]). For the two-loop contribution, as we should, we get a gauge invariant on-shell limit which is real. Gauge cancellations are highly non-trivial and only happen if one is doing a perfectly consistent calculation. We have verified that for gauge parameters $\xi > 1$ the imaginary part of the W and Z self-energies in the bosonic sector up to two-loops is zero, by applying the Cutkowsky rules and inspecting all possible two and three particle intermediate states allowed by the SM Lagrangian. While for $\xi > 1$ the imaginary part is zero for each individual diagram, for small enough values of the gauge parameters a non-trivial cancellation must take place. An independent direct check of this is possible by considering the problem in the limit $\xi \to 0$, for example.

After we have made the expansion (3.8), the bosonic contributions we are considering only depend on the three different masses m_W^2, m_Z^2 and m_H^2. One natural small parameter is the weak mixing parameter $\sin^2\theta_W = 1 - m_W^2/m_Z^2 < 0.25$. We expand in this parameter and get rid in this way of m_W (or m_Z). For diagrams which contain Higgs boson lines[10] we apply an asymptotic expansion with respect to a heavy Higgs mass. Taking into account the most recent lower bound on the value of the Higgs boson mass we are dealing with an expansion parameter $m_Z^2/m_H^2 \lesssim 0.64$. This implies that we have to calculate quite a number of coefficient in the expansion in order to get a convincing result. We should mention that these expansions are well behaved because $p^2 = M_V^2$ in any case is below the possible thresholds. The convergence of such expansions can be easily checked for the one-loop case where the exact analytic result is available.

In our case we find four different prototype structures with Higgs lines inside of the loops. The large mass expansion has been performed with the help of the packages **TLAMM** [29]

[10]We have 280 and 357 two-loop one-particle irreducible diagrams for the Z- and W-boson, respectively.

and [30] once for Euclidean [31] and once for Minkowski space-time, respectively. The corresponding topologies and set of subgraphs are given in Fig. 2. The diagrams without

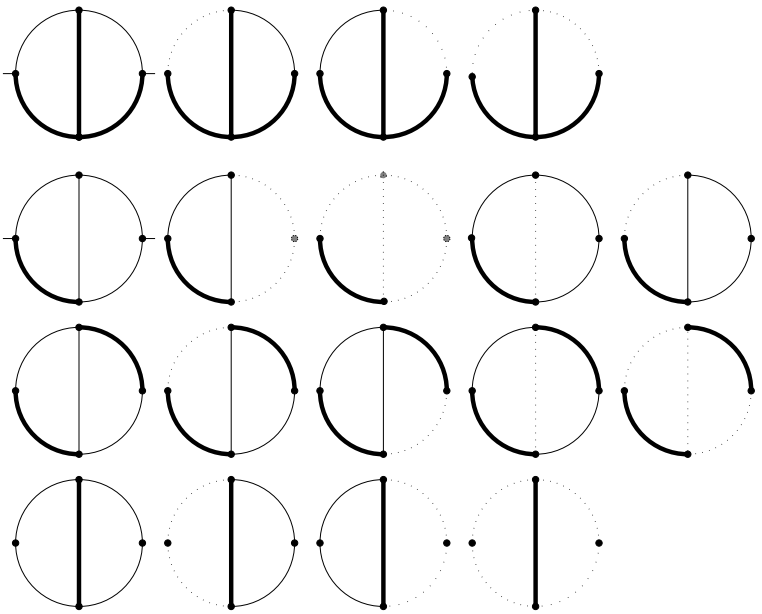


Figure 2: The prototype diagrams and their subgraphs contributing to the large mass expansion for two-loop diagrams with heavy propagators. Thick and thin lines correspond to the heavy- and light-mass (massless) particle propagators, respectively. Dotted lines indicate the lines omitted in the subgraph.

Higgs[11] are nothing but single scale massive diagrams, when all internal masses are equal to the external momentum or zero. Such diagrams can be calculated analytically. For this purpose we use the packages **ONSHELL2** [32] and another one written by O.V. [33] for the calculation of the set of the master integrals given in [34]. We find that only the following four prototypes are required (in terms of the notation used in [32]): **F11111**, **F11110**, **F01111** and **F01101**.

For independent verification of the input, the Feynman rules and the evaluation we performed calculations independently in Euclidean (M.Yu.K) and Minkowski (O.V.) space-time and got full agreement between them [12].

4 UV renormalization in the $\overline{\rm MS}$ scheme

Here we describe in more detail the renormalization procedure. It is well known that the pole mass in QED/QCD is a gauge independent and infrared stable quantity to all orders

[11] At the two-loop level we find 336 one-particle irreducible diagrams without a Higgs line for the Z-boson and 435 for the W-boson.

[12] The packages used in the calculations and the results can be found at the following URL address `http://www-zeuthen.desy.de/~kalmykov/pole/pole.html`.

of the loop expansion [35]. To renormalize the pole mass at the two-loop level requires to calculate the one-loop renormalization constants for all physical parameters (charge and masses), and the two-loop renormalization constant only for the mass itself. Not needed are the wave-function renormalizations or ghost (unphysical) sector renormalizations. The above mentioned basic properties of the pole mass are valid also in the SM. In order to obtain a gauge invariant result in the SM, however, we have to add in a proper way the tadpole contributions [16]. The tadpole terms are due to the vacuum expectation value (VEV) of the Higgs field, which does not vanish automatically. By a constant shift we can adjust the Higgs field to have vanishing VEV, however. Since here the Higgs field is integrated out in the path integral the result cannot depend on whether we perform a shift or not. Indeed, after we take into account all diagrams shown in Fig. 1 we find a gauge invariant result for the pole mass up to two loops in terms of the bare parameters. The tadpole contribution can be calculated either from tadpole diagrams[13] or from Ward identities, which connect the tadpoles with the one-particle irreducible self-energies of the pseudo-Goldstone bosons $\Pi_{\phi\phi}$ at zero momentum

$$\Pi^{\rm 1PI}_{\phi\phi}(0) + T = 0, \tag{4.1}$$

where $\phi = \phi^+, \phi^-, \phi^0$. We performed both types of calculations and obtained full agreement. Diagrams contributing to the Z-boson pole mass do not contain infrared singularities at the two-loop level. Infrared finiteness of the W-boson mass was proven in [36]. We also give an alternative proof of this statement for our case.

In our calculation dimensional regularization [37] is used, which allows one to regularize both UV and IR singularities by the same parameter ε related to the dimension of space-time by $d = 4 - 2\varepsilon$. We first perform the UV-renormalization within the $\overline{\rm MS}$ scheme in order to obtain finite results. In a next step we find the relation between the pole- and $\overline{\rm MS}$ parameters. We adopt the convention that the $\overline{\rm MS}$-parameters are defined by multiplying each L-loop integral by the factor $(\exp(\gamma)/4\pi)^{\varepsilon L}$. Each loop picks an additional factor $1/16\pi^2$.

4.1 One-loop charge renormalization

The bare charge e_0 and the $\overline{\rm MS}$ charge[14] e are related via $e_0 = \mu^\varepsilon e \left(1 + Z^{(1)}_{\overline{\rm MS}}/\varepsilon + O(e^4)\right)$, with the appropriate constant $Z^{(1)}_{\overline{\rm MS}}$. At the same time, the relation between the $\overline{\rm MS}$ and the on-shell charge reads

$$e = e_{\rm OS} \left(1 + z^{(1)}_{\rm OS} + O(e^4_{\rm OS})\right),$$

where $e_{\rm OS}$ is defined by the Thompson limit of Compton scattering. The electromagnetic Ward–Takahashi identity implies that some of the diagrams cancel, such that $z^{(1)}_{\rm OS}$ at the one-loop level can be written in terms of self-energies only [15]

$$z^{(1)}_{\rm OS} = -\frac{1}{2}\Pi^{(1)\prime}_{\gamma\gamma}(0) + \frac{\sin\theta_W}{\cos\theta_W}\frac{\Pi^{(1)}_{\gamma Z}(0)}{M_Z^2} - \frac{1}{\varepsilon}Z^{(1)}_{\overline{\rm MS}}\,. \tag{4.2}$$

[13] These are the two-loop bubble diagrams for the process H in terms of **QGRAF** notation.

[14] All $\overline{\rm MS}$-parameters, like e, g, g' and all renormalization constants, like $Z_{\overline{\rm MS}}$ or $z_{\rm OS}$ are μ-dependent quantities.

UV-finiteness of $z_{\rm OS}^{(1)}$ implies

$$e_0 = \mu^\varepsilon e \left(1 - \frac{1}{\varepsilon}\,\frac{7}{2}\,\frac{e^2}{16\pi^2}\right) . \tag{4.3}$$

This may be confirmed also by using a renormalization group analysis of the SM keeping the Yang–Mills and the Higgs sector only. From the relation

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2} , \tag{4.4}$$

where g' and g are the $U(1)$ and the $SU(2)$ gauge coupling constants, respectively, it is easy to deduce that

$$\beta_e = e^3 \left(\frac{\beta_g}{g^3} + \frac{\beta_{g'}}{g'^3}\right) = -\frac{7}{2}\frac{e^3}{16\pi^2} + O(e^5) . \tag{4.5}$$

The β-functions $\beta_{g,g'}^{(1)}$ may be calculated in the unbroken theory. They have been calculated in [38]. We see that the above result is in agreement with (4.3) if we take into account that $(\mu^2 d/d\mu^2)\, e = -\frac{\varepsilon}{2} e + \beta_e$.

4.2 Mass renormalization

We introduce the following notation for the mass renormalization constants

$$m_{V,0}^2 = m_V^2(\mu) \left(1 + \frac{g^2(\mu)}{16\pi^2\varepsilon} Z_V^{(1,1)} + \frac{g^4(\mu)}{(16\pi^2)^2\varepsilon} Z_V^{(2,1)} + \frac{g^4(\mu)}{(16\pi^2)^2\varepsilon^2} Z_V^{(2,2)}\right), \tag{4.6}$$

where V stands for any of the bosons Z, W or H. In addition to the masses we have one coupling constant as a free parameter of the SM which we have chosen above to be $e = g \sin\theta_W$. The one-loop mass counter-terms are well known [16]. For the purely bosonic contributions we have

$$Z_H^{(1,1)} = -\frac{3}{2} - \frac{3}{4}\frac{m_Z^2}{m_W^2} + \frac{3}{4}\frac{m_H^2}{m_W^2}, \tag{4.7}$$

$$Z_W^{(1,1)} = -\frac{3}{4}\frac{m_H^2}{m_W^2} - 3\frac{m_W^2}{m_H^2} - \frac{3}{2}\frac{m_Z^4}{m_H^2 m_W^2} + \frac{3}{4}\frac{m_Z^2}{m_W^2} - \frac{17}{3}, \tag{4.8}$$

$$Z_Z^{(1,1)} = -\frac{3}{4}\frac{m_H^2}{m_W^2} - 3\frac{m_W^2}{m_H^2} - \frac{3}{2}\frac{m_Z^4}{m_W^2 m_H^2} + \frac{11}{12}\frac{m_Z^2}{m_W^2} - 7\frac{m_W^2}{m_Z^2} + \frac{7}{6}. \tag{4.9}$$

All masses here are $\overline{\rm MS}$ masses and depend on the renormalization scale μ: $m_V^2 = m_V^2(\mu)$. It should be noted that unlike in the case of the couplings the mass renormalization constants cannot be calculated from the unbroken gauge theory. Here in any case the calculation of the Feynman diagrams in the Standard Model is required.

Let us comment about the somewhat unusual looking dependence of our $\overline{\rm MS}$ renormalization constants on the particle masses. Since masses are induced by the Higgs mechanism we have the mass coupling relations

$$m_W = g\,\frac{v}{2}\,, \quad m_Z = \sqrt{g^2 + g'^2}\,\frac{v}{2}\,, \quad m_H = \sqrt{2\lambda}\,v$$

where v is the Higgs VEV and λ the Higgs self-coupling. One peculiarity of the "spontaneous symmetry breaking" and the related mixing of states, like the $\gamma - Z$ mixing, leads to the non-polynomial nature of the perturbation expansion in the SM. Due to the mixing, the actual Z coupling reads $\sqrt{g^2 + g'^2} = g/\cos\theta_W$ etc. In the dimensionless mass ratios the factors v^2 drop out and we have in fact just ratios of couplings. To a large extent this is a trivial consequence of factorizing out powers of g^2 which cancels against such factors which appear in the denominators of the $Z_V^{(1,1)}$'s. However, there are also true inverse powers of the Higgs self-coupling present. They originate in the tadpoles which we need taking into account for the sake of gauge invariance.

Our results for the two-loop mass renormalization constants are as follows:

$$\begin{aligned} Z_W^{(2,1)} &= \frac{63}{64}\frac{m_H^4}{m_W^4} - \frac{3}{4}\frac{m_H^2}{m_W^2} - \frac{3}{8}\frac{m_H^2 m_Z^2}{m_W^4} - \frac{301}{192}\frac{m_Z^4}{m_W^4} + \frac{17}{12}\frac{m_Z^2}{m_W^2} - \frac{53}{3} \\ &\quad + \frac{31}{12}\frac{m_Z^4}{m_H^2 m_W^2} - \frac{17}{2}\frac{m_Z^2}{m_H^2} - \frac{176}{3}\frac{m_W^2}{m_H^2} + \frac{59}{24}\frac{m_Z^6}{m_H^2 m_W^4}, \qquad (4.10) \\ Z_W^{(2,2)} &= -\frac{9}{32}\frac{m_H^4}{m_W^4} + \frac{43}{8}\frac{m_H^2}{m_W^2} + \frac{65}{32}\frac{m_Z^4}{m_W^4} - \frac{35}{8}\frac{m_Z^2}{m_W^2} + \frac{334}{9} + \frac{27}{4}\frac{m_Z^4}{m_H^2 m_W^2} \\ &\quad + 6\frac{m_Z^2}{m_H^2} + 34\frac{m_W^2}{m_H^2} - \frac{5}{2}\frac{m_Z^6}{m_H^2 m_W^4} + 9\frac{m_W^4}{m_H^4} + 9\frac{m_Z^4}{m_H^4} + \frac{9}{4}\frac{m_Z^8}{m_H^4 m_W^4}, \qquad (4.11) \end{aligned}$$

$$\begin{aligned} Z_Z^{(2,1)} &= \frac{63}{64}\frac{m_H^4}{m_W^4} - \frac{3}{4}\frac{m_H^2}{m_W^2} - \frac{3}{8}\frac{m_H^2 m_Z^2}{m_W^4} + \frac{11}{3} + \frac{7}{6}\frac{m_Z^2}{m_W^2} - \frac{64}{3}\frac{m_W^2}{m_Z^2} - \frac{253}{192}\frac{m_Z^4}{m_W^4} \\ &\quad - \frac{176}{3}\frac{m_W^2}{m_H^2} - \frac{17}{2}\frac{m_Z^2}{m_H^2} + \frac{31}{12}\frac{m_Z^4}{m_H^2 m_W^2} + \frac{59}{24}\frac{m_Z^6}{m_H^2 m_W^4}, \qquad (4.12) \\ Z_Z^{(2,2)} &= -\frac{9}{32}\frac{m_H^4}{m_W^4} + \frac{1}{4}\frac{m_H^2}{m_W^2} - \frac{1}{8}\frac{m_H^2 m_Z^2}{m_W^4} + \frac{21}{4}\frac{m_H^2}{m_Z^2} - \frac{55}{6} + \frac{11}{12}\frac{m_Z^2}{m_W^2} + \frac{629}{288}\frac{m_Z^4}{m_W^4} \\ &\quad + \frac{245}{6}\frac{m_W^2}{m_Z^2} + \frac{27}{2}\frac{m_W^2}{m_H^2} + 16\frac{m_Z^2}{m_H^2} + 21\frac{m_W^4}{m_Z^2 m_H^2} - \frac{7}{2}\frac{m_Z^4}{m_H^2 m_W^2} - \frac{11}{4}\frac{m_Z^6}{m_H^2 m_W^4} \\ &\quad + 9\frac{m_W^4}{m_H^4} + 9\frac{m_Z^4}{m_H^4} + \frac{9}{4}\frac{m_Z^8}{m_H^4 m_W^4}. \qquad (4.13) \end{aligned}$$

Again the higher order pole terms $1/\varepsilon^2$ can be checked by means of the appropriate renormalization group equation. Let us write the relation $m_{V,0}^2 = m_V^2\left(1 + \sum_n Z_V^{(n)}/\varepsilon^n\right)$ which connects bare and renormalized masses and introduces the anomalous dimension of the mass $\gamma_V = (\mu^2\, d/d\mu^2)\ln m_V^2$. Taking into account that $(\mu^2 d/d\mu^2)\, m_{V,0}^2 = 0$ and repeating the calculations given in [39] we find

$$\gamma_V = \frac{1}{2} g \frac{\partial}{\partial g} Z_V^{(1)}, \qquad (4.14)$$

$$\left(\gamma_V + \beta_g \frac{\partial}{\partial g} + \sum_i \gamma_i m_i^2 \frac{\partial}{\partial m_i^2}\right) Z_V^{(n)} = \frac{1}{2} g \frac{\partial}{\partial g} Z_V^{(n+1)}. \qquad (4.15)$$

For each function like γ_V or $Z_V^{(n)}$ we may perform a loop expansion $\gamma_V = \sum_{k=1}^{\infty}(g^2/16\pi^2)^k\gamma_V^{(k)}$ or $Z_V^{(n)} = \sum_{k=n}^{\infty}(g^2/16\pi^2)^k Z_V^{(n,k)}$ and (4.15) can be written for each loop correction separately. In particular, for $n = 1$, we have

$$\gamma_V^{(1)} = Z_V^{(1,1)},$$
$$\gamma_V^{(2)} = 2Z_V^{(2,1)},$$

such that the coefficient of the $n = 2$ poles can be checked via

$$\left(Z_V^{(1,1)}\right)^2 + \frac{16\pi^2}{g^2} 2\frac{\beta_g^{(1)} Z_V^{(1,1)}}{g} + \sum_i Z_{m_i}^{(1,1)} m_i^2 \frac{\partial}{\partial m_i^2} Z_V^{(1,1)} = 2Z_V^{(2,2)} \; . \tag{4.16}$$

The value of $\beta_g^{(1)} = -\frac{43}{12}\frac{g^3}{16\pi^2}$ may be calculated from the relation

$$\beta_g^{(1)} \sin\theta_W = \frac{1}{2} g \frac{\cos^2\theta_W}{\sin\theta_W} \left(\frac{g^2}{16\pi^2}\right) \left(\gamma_W^{(1)} - \gamma_Z^{(1)}\right) + \beta_e^{(1)}, \tag{4.17}$$

where $\beta_e^{(1)}$ is given in (4.5), and $\cos^2\theta_W = m_W^2/m_Z^2$.

An independent verification of the $1/\varepsilon$ terms can be obtained from the relationship $e^2 = g^2 \sin^2\theta_W$, which is valid for bare and $\overline{\rm MS}$ renormalized quantities. Differentiating the renormalized quantities with respect to $\ln\mu^2$ we find

$$e^4 \left(\frac{\beta_g}{g^3} + \frac{\beta_{g'}}{g'^3}\right) = g\beta_g \sin^2\theta_W - \frac{1}{2} g^2 \left(\gamma_W - \gamma_Z\right) \cos^2\theta_W, \tag{4.18}$$

or for the two-loop case

$$g^2 \frac{\beta_{g'}^{(2)}}{g'^3} \sin^4\theta_W - \frac{\beta_g^{(2)}}{g} \sin^2\theta_W \cos^2\theta_W = -\left(\frac{g^2}{16\pi^2}\right)^2 \left(Z_W^{(2,1)} - Z_Z^{(2,1)}\right) \cos^2\theta_W. \tag{4.19}$$

The two-loop β-functions for g and g' are given in [40] and read

$$\begin{aligned} \beta_{g'} &= \frac{1}{12}\frac{g'^3}{16\pi^2} + \frac{1}{4}\frac{g'^5}{(16\pi^2)^2} + \frac{3}{4}\frac{g'^3 g^2}{(16\pi^2)^2} \; , \\ \beta_g &= -\frac{43}{12}\frac{g^3}{16\pi^2} - \frac{259}{12}\frac{g^5}{(16\pi^2)^2} + \frac{1}{4}\frac{g^3 g'^2}{(16\pi^2)^2} \; . \end{aligned} \tag{4.20}$$

The fact that after UV renormalization we get a finite result confirms the infrared finiteness of the bosonic contribution to the pole mass.

Let us write now the RG equation for the effective Fermi constant G_F. G_F is usually defined as a low energy constant in one-to-one correspondence with the muon lifetime. However, if we consider physics at higher energies a parameterization in terms of low energy constants may lead to large radiative corrections. Much in the same way as the fine structure constant α often is replaced by the effective running fine structure constant $\alpha(\mu)$ we

expect that G_F should be replaced by an effective version of it at higher energies. Unlike in the case of α, however, because of the smallness of the light fermion Yukawa couplings, G_F starts to run effectively only at scales beyond the W–pair production threshold (see below). The bare relations are

$$m^2_{W,0} = \frac{g_0^2 v_0^2}{4}, \tag{4.21}$$

$$\frac{1}{v_0^2} = \sqrt{2}\, G_{F,0}. \tag{4.22}$$

Both these relations are valid also for $\overline{\rm MS}$ renormalized parameters, so that their differentiation with respect to $\ln \mu^2$ gives rise to the relation

$$\gamma_W = 2\frac{\beta_g}{g} - \gamma_{G_F}, \tag{4.23}$$

where we introduced the anomalous dimension of the Fermi constant $\gamma_{G_F} = (\mu^2\, d/d\mu^2) \ln G_F$. Its loop expansion looks like $\gamma_{G_F} = \gamma^{(1)}_{G_F} + \gamma^{(2)}_{G_F} + \cdots$. At the one-loop level we have

$$\begin{aligned}
\gamma^{(1)}_{G_F} &= 2\frac{\beta^{(1)}_g}{g} - Z_W^{(1,1)} \\
&= \frac{g^2}{16\pi^2}\frac{1}{4m_W^2}\left\{\frac{2}{m_H^2}\left(3\,(2m_W^4 + m_Z^4) + m_H^4 - 4\sum_f m_f^4\right)\right. \\
&- \left.\left(3\,(2m_W^2 + m_Z^2) - m_H^2 - 2\sum_f m_f^2\right)\right\}
\end{aligned} \tag{4.24}$$

and we used the results of Ref. [16] for the fermion contributions. The first term proportional $1/m_H^2$ is the contribution from the tadpoles. The appearance of the tadpole terms is somewhat mysterious, since we know that in renormalized observables tadpoles drop out. Here they seem to contribute to the renormalization group evolution of the Fermi constant. In any case the tadpoles are present in the relationship between the bare and the renormalized parameters. At the two-loop level, our results allows us to write the bosonic corrections only. They are given by

$$\begin{aligned}
\gamma^{(2),\rm boson}_{G_F} &= 2\frac{\beta^{(2)}_g}{g} - 2Z_W^{(2,1)} \\
&= -\frac{2g^4}{(16\pi^2)^2}\left\{\frac{63}{64}\frac{m_H^4}{m_W^4} - \frac{3}{4}\frac{m_H^2}{m_W^2} - \frac{3}{8}\frac{m_H^2 m_Z^2}{m_W^4} - \frac{301}{192}\frac{m_Z^4}{m_W^4} + \frac{7}{6}\frac{m_Z^2}{m_W^2} + \frac{25}{6}\right. \\
&\qquad \left. + \frac{31}{12}\frac{m_Z^4}{m_H^2 m_W^2} - \frac{17}{2}\frac{m_Z^2}{m_H^2} - \frac{176}{3}\frac{m_W^2}{m_H^2} + \frac{59}{24}\frac{m_Z^6}{m_H^2 m_W^4}\right\}.
\end{aligned} \tag{4.25}$$

The equations (4.24) and (4.25) are written in $\overline{\rm MS}$ scheme. As usual in this scheme, in solving the renormalization group equation

$$\mu^2 \frac{d}{d\mu^2}\, G_F(\mu) = G_F(\mu)\, \gamma_{G_F}$$

the decoupling of the heavy particles has to be performed "by hand". This means that for low values of the energy scale μ, when $\mu < m_H, m_W, m_Z$, the bosonic terms on the r.h.s. are equal to zero while the light fermion contributions proportional to $G_F m_f^2$ are tiny. Consequently, below the W mass, the effective Fermi constant does practically not change with scale. Obviously, the running of G_F only starts at about $\mu \sim m_Z$, when the scale of a process exceeds the masses of the bosons. Also the top quark will contribute once we have passed its threshold.

5 Results, discussion and conclusion

After UV renormalization the pole mass (see (2.3))

$$M_V^2 = m_V^2 + \hat{\Pi}_V^{(1)} + \hat{\Pi}_V^{(2)} + \hat{\Pi}_V^{(1)}\hat{\Pi}_V^{(1)\prime} \tag{5.1}$$

is represented in terms of finite $\overline{\rm MS}$ renormalized amplitudes

$$\hat{\Pi}_V^{(i)} = \Pi_V^{(i)}(p^2, m_V^2, \cdots)\Big|_{p^2=m_V^2,\,\text{F.P.}} \ .$$

F.P. denotes the $\overline{\rm MS}$ finite part prescription. The calculation of the one-loop $\overline{\rm MS}$ renormalized amplitude is well known. We get it by rewriting the bare expression in terms of $\overline{\rm MS}$ parameters according to (4.6) and (C.1)

$$\begin{aligned}\hat{\Pi}_V^{(1)} &\equiv \lim_{\varepsilon\to 0}\left(m_{0,V}^2 - m_V^2 + m_{0,V}^2\frac{g_0^2}{16\pi^2}X_{0,V}^{(1)}\right)\\ &= m_V^2(\mu)\frac{e^2}{16\pi^2\sin^2\theta_W}\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}Z_V^{(1,1)} + X_{0,V}^{(1)}\right) = m_V^2(\mu)\frac{e^2}{16\pi^2\sin^2\theta_W}X_V^{(1)} \ . \end{aligned}\tag{5.2}$$

We restrict ourselves to a consideration of the bosonic part $X_{0,V}^{(1),\text{boson}}$. The renormalization of the off-shell self-energy functions $\Pi_{0,V}^{(1)\prime}$ and $\Pi_{0,V}^{(2)}$ is more complicated. Besides the renormalization of the physical parameters, it would require us to perform order-by-order, the wave-function renormalization as well as the renormalization of the ghost sector, in particular of the unphysical gauge parameters. However, the $\overline{\rm MS}$ renormalization of the combination $\Pi_{0,V}^{(2)} + \Pi_{0,V}^{(1)}\Pi_{0,V}^{(1)\prime}$ it much simpler. The subtraction of sub-divergencies in this case is reduced to the one-loop renormalization of the charge and the physical masses only, while the wave-function renormalization or the renormalization of the unphysical sector is not needed. Accordingly, as a genuine two-loop counter-term, only the mass renormalization occurs.

The full two-loop $\overline{\rm MS}$ renormalized amplitude can be written in the form

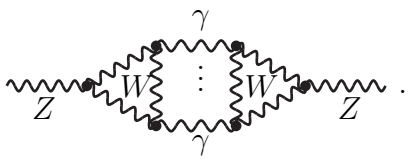
$$\begin{aligned}&\hat{\Pi}_V^{(2)} + \hat{\Pi}_V^{(1)}\hat{\Pi}_V^{(1)\prime} \equiv \left\{\Pi_{0,V}^{(2)} + \Pi_{0,V}^{(1)}\Pi_{0,V}^{(1)\prime}\right\}_{MS} = \lim_{\varepsilon\to 0}\Bigg(\Pi_{0,V}^{(2)} + \Pi_{0,V}^{(1)}\Pi_{0,V}^{(1)\prime}\\ &+m_V^2(\mu)\frac{1}{\varepsilon}\left(\frac{e^2}{16\pi^2\sin^2\theta_W}\right)^2\left[Z_V^{(1,1)} + \left[\frac{\Delta g^2}{g^2}\right] + \sum_j Z_{m_j^2}^{(1,1)}\frac{\partial}{\partial m_j^2}\right]X_{0,V}^{(1)}\\ &+m_V^2(\mu)\left(\frac{e^2}{16\pi^2\sin^2\theta_W}\right)^2\left[\frac{1}{\varepsilon}Z_V^{(2,1)} + \frac{1}{\varepsilon^2}Z_V^{(2,2)}\right]\Bigg)\end{aligned}\tag{5.3}$$

where the sum runs over all species of particles $j = Z$, W, H and

$$\left[\frac{\Delta g^2}{g^2}\right] = \frac{\cos^2\theta_W}{\sin^2\theta_W}\left(Z_W^{(1,1)} - Z_Z^{(1,1)}\right) - 7\sin^2\theta_W \ .$$

The functions $Z_V^{(i,j)}, X_V^{(1)}$ and $X_{0,V}^{(1)}$ are defined in (4.7)-(4.9), (4.10)-(4.13), (B.1) and (C.1), respectively. Again, throughout, we take into account only the bosonic corrections. The r.h.s. of (5.3) is given by the bare two-loop contributions, the two-loop contribution obtained by expanding the bare parameters in the one-loop amplitude (subtraction of the sub-divergences) and the genuine two-loop subtractions. The second type of contribution follows from writing $\Pi^{(1)} = m_0^2\left(\frac{g_0^2}{16\pi^2}\right) X_0^{(1)}$ and by utilizing (4.6). We would like to stress, that each of the one-loop bare amplitudes $\Pi_{0,V}^{(1)}$ and $\Pi_{0,V}^{(1)\prime}$ has to be expanded up to linear terms in ε. Together with the singular $1/\varepsilon$ terms the contributions linear in ε yield additional finite contributions in the limit $\varepsilon \to 0$. In the product $\Pi_{0,V}^{(1)}\Pi_{0,V}^{(1)\prime}$ the parameters may be identified with the $\overline{\rm MS}$ renormalized quantities, but all functions, A_0 and B_0 have to be taken in $d = 4 - 2\varepsilon$ dimensions and must be expanded up to terms linear in ε (see Appendix A).

As mentioned earlier, for the purely bosonic contributions alone the imaginary part of $\Pi(p^2)$ on the mass-shell is zero at the two-loop level. This is due to the fact that in the bosonic sector we have the physical masses $m_\gamma = 0$, M_Z, M_W and M_H and by inspection of the possible two and three particle intermediate states one observes that all physical thresholds lie above the mass shells of the W and Z bosons, i.e., the self-energies of the massive gauge bosons develop an imaginary part only at $p^2 > M_V^2$ (to two loops in the SM). On kinematical grounds imaginary parts could show up from the Higgs or Faddeev-Popov ghosts, which have square masses $\xi_V M_V^2$, for small values of the gauge parameter. However, as we have verified, the two-loop on-shell self-energies are gauge independent. This implies that ghost contributions have to cancel and hence cannot contribute to the imaginary part. Thus $s_P = M_V^2$ in our case. In higher orders for the Z–propagator one gets an imaginary part as soon as $p^2 > 0$, from diagrams like



For the W–propagator an imaginary part is only possible for $p^2 > M_W^2$, because charge conservation requires at least one W in any physical intermediate state.

For the massive gauge bosons Z and W we write

$$\frac{M_V^2}{m_V^2} = 1 + \left(\frac{e^2}{16\pi^2\sin^2\theta_W}\right) X_V^{(1)} + \left(\frac{e^2}{16\pi^2\sin^2\theta_W}\right)^2 X_V^{(2)} \ , \tag{5.4}$$

where both e and $\sin\theta_W$ are to be taken in the $\overline{\rm MS}$ scheme.

The one-loop coefficients $X_V^{(1)}$ for Z, W and H are known of course as exact results. We write them down for completeness in Appendix B. For the coefficients $X_V^{(2)}$ we make an expansion and perform them as double series: in $\sin^2\theta_W$ and in the mass ratio m_V^2/m_H^2. We have calculated the first six terms of the expansion with respect to $\sin^2\theta_W$ and the first six terms with respect to mass ratio m_V^2/m_H^2. The analytical values of these coefficients are presented in Appendix D. These represent our main result.

Sometimes in massive multi-loop calculations the so-called modified $\overline{\rm MS}$ scheme ($\overline{\rm MMS}$) is used [41]. The difference between $\overline{\rm MS}$ and $\overline{\rm MMS}$ is that in the former scheme each loop is multiplied by $(e^\gamma/4\pi)^\varepsilon$ while in the latter the normalization factor is $1/(4\pi)^\varepsilon/\Gamma(1+\varepsilon)$, which yields a difference at the two-loop level. It has been shown that in QCD both schemes reproduce the same formula for the mass relation analogous to (5.1) [42]. We have checked that the same holds true for the pole masses in the Standard Model.

Very often the inverse of (5.1) is required. To that end we have to express all $\overline{\rm MS}$ parameters in terms of on-shell ones. Thus

$$\begin{aligned} m_V^2 &= M_V^2 - \hat\Pi_V^{(1)} - \left\{\Pi_{0,V}^{(2)} + \Pi_{0,V}^{(1)}\Pi_{0,V}^{(1)\prime}\right\}_{MS} \\ &\quad - \sum_j (\Delta m_j^2)^{(1)} \frac{\partial}{\partial m_j^2}\hat\Pi_V^{(1)} - (\Delta e)^{(1)}\frac{\partial}{\partial e}\hat\Pi_V^{(1)}\Big|_{m_j^2=M_j^2,\, e=e_{\rm OS}}, \end{aligned} \tag{5.5}$$

where the sum runs over all species of particles $j = Z$, W, H and

$$(\Delta m_j^2)^{(1)} = -{\rm Re}\hat\Pi_j^{(1)}\Big|_{m_j^2=M_j^2,\, e=e_{\rm OS}} \equiv -M_V^2\frac{e_{\rm OS}^2}{16\pi^2\sin^2\theta_W}X_V^{(1)}\Big|_{m_j^2=M_j^2}$$

stands for the self-energy of the jth particle at $p^2 = m_j^2$ in the $\overline{\rm MS}$ scheme and parameters replaced by the on-shell ones. Note that in the above relation we had to perform a change from the $\overline{\rm MS}$ to the on-shell scheme also for the electric charge

$$e(\mu^2) = e_{\rm OS}\left[1 + \frac{e_{\rm OS}^2}{16\pi^2}\left(\frac{7}{2}\ln\left(\frac{M_W^2}{\mu^2}\right) - \frac{1}{3}\right)\right] \tag{5.6}$$

with $e_{\rm OS}^2/4\pi = \alpha \sim 1/137$. Accordingly, since $\hat\Pi^{(1)}$ depends on e by an overall factor e^2 only,

$$(\Delta e)^{(1)}\frac{\partial}{\partial e}\hat\Pi_V^{(1)} = \frac{e^2}{16\pi^2}\left[7\ln\left(\frac{M_W^2}{\mu^2}\right) - \frac{2}{3}\right]\,\hat\Pi_V^{(1)}\,.$$

By identifying $m_V^2 = m_{V,0}^2 = M_V^2 + \delta M_V^2$ (5.5) is the relationship appropriate to obtain the on-shell gauge-boson mass counter-terms δM_V^2 :

$$\begin{aligned} \delta M_V^2 &= -{\rm Re}\Bigg[\check\Pi_{V,0}^{(1)} + \check\Pi_{V,0}^{(2)} + \check\Pi_{V,0}^{(1)}\check\Pi_{V,0}^{(1)\prime} \\ &\qquad + \sum_j (\delta M_j^2)^{(1)}\frac{\partial}{\partial m_{j,0}^2}\check\Pi_{V,0}^{(1)} + (\delta e)^{(1)}\frac{\partial}{\partial e_0}\check\Pi_{V,0}^{(1)}\Bigg]\Bigg|_{m_{j,0}^2=M_j^2,\, e_0=e_{\rm OS}} \\ &= \left(Z_{\overline{\rm MS}}\cdot Z_{\rm OS} - 1\right)M_V^2 \end{aligned} \tag{5.7}$$

in terms of the original bare on-shell amplitudes

$$\check{\Pi}^{(i)}_{V,0} = \Pi^{(i)}_{V,0}(p^2, m^2_{V,0}, \cdots)\Big|_{p^2=M_V^2, m^2_{j,0}=M_j^2, e_0=e_{\rm OS}}$$

and the bare on-shell counter-terms δM_j^2 and δe. The second equality gives δM_V^2 in terms of the singular factor $Z_{\overline{\rm MS}} = m^2_{V,0}/m^2_V(\mu)$ given in (4.6) and the finite factor $Z_{\rm OS} = m_V^2/M_V^2$ given in (5.5). These will be needed in two-loop calculations of observables in the on-shell scheme.

We now turn to a discussion of our results which we obtained for the relationship (5.5). For our numerical analysis we used the following values of the pole masses: $M_W = 80.419$ GeV and $M_Z = 91.188$ GeV and $\alpha = 1/137.036$. We first investigate numerically the relationship between the $\overline{\rm MS}$–mass $m_V(\mu)$ and the pole-mass M_V. In Figs. 3 and 4 we plot $\Delta_Z \equiv m_Z^2(M_Z)/M_Z^2 - 1$ as function of the Higgs mass M_H for intermediate and heavier Higgs masses, respectively. For the one-loop corrections the exact analytical functions are evaluated while for the two-loop results we utilize all coefficients of our expansion. Analogously, in Figs. 5 and 6 the Higgs mass dependence of $\Delta_W \equiv m_W^2(M_W)/M_W^2 - 1$ is depicted for the same ranges of the Higgs mass. As we can see, for a "light" Higgs of mass less than about 200 GeV the two-loop corrections are small as compared to the one-loop ones. However, at a Higgs mass of about 220 GeV the absolute value of the two-loop correction is of the same size as the one-loop result, such that the two-loop corrections start to play an essential role.

Our analysis shows (see Figs. 4 and 6) that the perturbative expansion looses its meaning for large Higgs masses (strong coupling regime). In the relationship between the $\overline{\rm MS}$ – and the pole–masses corrections exceed the 50% level around 880 GeV (for $\mu = M_V$). The size of the corrections depend on the choice of the renormalization scale μ.

Since our results have been obtained by an expansion in m_V^2/m_H^2 (i.e., for a heavy Higgs), one of the main questions which remains to be considered is the validity of our results as we approach lighter Higgs masses. Note that we are dealing with an asymptotic expansion (which means the radius of convergence is zero) and thus we can answer the question about where the expansion is reliable only empirically; at least as long as an exact result is not available for a direct comparison. Since the structure of the one- and two-loop corrections for both massive gauge bosons is very similar, we are going to investigate in the following the problems of convergence of our results for the Z–boson only. For the W–boson the convergence is better because of the smaller value of the W–mass. Obviously our series–expansion breaks down for a "light" Higgs, when $m_H \geq m_Z$. Unexpectedly, we find that the two-loop corrections remain numerically small for values of the Higgs mass even down to 100 GeV (below about 0.2%). However, we observe a steep raise of the correction which signals that the expansion becomes unreliable below about 130 GeV[15]. We also should keep in mind that, on the level of the size of the two-loop corrections, which are very small in the region around 150 GeV for $\mu \sim m_Z$, the latter depend substantially on the choice of the $\overline{\rm MS}$

[15]We have checked the convergence of the corresponding expansion at the one-loop level, where the exact result is available. Deviations start to show up below about 145 GeV and grow to about 30% at 100 GeV. The best approximation in this case is based on the first 3 or 4 coefficients. In this case, beyond the first few terms, the series starts to diverge in a way typical for an asymptotic expansion

renormalization–scale μ, which causes an essentially constant shift in the quantities shown in the Figures which follow.

In Fig. 7 we show the dependences of the two-loop corrections to $\Delta_Z = m_Z^2(M_Z^2)/M_Z^2 - 1$ as a function of the Higgs mass and the number of coefficients of the expansion used for their evaluation. For a "light" Higgs the difference between the result, including six coefficients of the expansion, and the results obtained by including only the first three of them (leading, next-to-leading and next-to-next-to-leading) are numerically small. However, the convergence of the series is not satisfactory. The coefficients grow fast beyond the first four terms and the correction starts growing fast. For a heavy Higgs with a mass of more than 300 GeV the convergence is much better, so that we omit the corresponding plot. We only mention, that there is no essential differences between the full result and the next-to-leading one. Similarly, the dependence of the two-loop corrections on the number of coefficients of the expansion with respect to $\sin^2\theta_W$ for a "light" Higgs is illustrated in Fig. 8. Here unlike in the previous "light" case, the first coefficient is relatively large already, while the higher terms of the expansion do not alter the result in a significant manner. Again, we observe that results become unreliable below about 130 GeV.

Finally we analyze the Higgs mass dependence of $\sin^2\theta_W$. The relation between the $\overline{\rm MS}$ weak mixing parameter and its version in terms of the pole masses reads

$$\begin{aligned}\sin^2\theta_W &= 1 - \frac{m_W^2}{m_Z^2} = 1 - \frac{M_W^2}{M_Z^2}\left(\frac{1+\delta_W^{(1)}+\delta_W^{(2)}}{1+\delta_Z^{(1)}+\delta_Z^{(2)}}\right) \\ &= \left(1 - \frac{M_W^2}{M_Z^2}\right) - \frac{M_W^2}{M_Z^2}\left[(\delta_W^{(1)} - \delta_Z^{(1)})(1-\delta_Z^{(1)}) + \delta_W^{(2)} - \delta_Z^{(2)}\right] \qquad (5.8)\end{aligned}$$

where we adopted the notation, $m_V^2/M_V^2 = 1 + \delta_V^{(1)} + \delta_V^{(2)}$. It turns out that in (5.8) all M_H^4 terms cancel. The corresponding results are depicted in Figs. 9 and 10. Again for a "light" Higgs the two-loop corrections are small, while for a heavier Higgs particle the corrections become large. Again, below about 120 GeV we cannot trust the series–expansion any longer.

We would like to mention that the presence of m_H^4 corrections in the relation (5.4) does not contradict Veltman's screening theorem [43] which states, that the L-loop Higgs dependence of a physical observable is at most of the form $(m_H^2)^{L-1}\ln^L m_H^2$ for large Higgs masses. This theorem applies to physical observables like cross sections and asymmetries, whereas our formula is nothing but a relation between parameters of two different schemes.

In conclusion, the main results of our paper are the following: (i) we have presented an independent proof of gauge invariance and infrared stability of the 2-loop electroweak bosonic corrections to the pole of the gauge boson propagators; (ii) analytical results for a number of coefficients of the expansion in $\sin^2\theta_W = 1 - m_W^2/m_Z^2$ and m_V^2/m_H^2 for the 2-loop electroweak bosonic corrections are given for the on-shell mass counter-terms (5.7) and the relationship between $\overline{\rm MS}$ and on-shell masses of the gauge bosons W and Z. All calculations have been performed in the electroweak Standard Model.

Note added: After completion of our paper we received the preprint [52], which presents a complete SM calculation of the Higgs mass dependent terms to the observable Δr which

determines the $M_W - M_Z$ relationship given α, G_μ and M_Z as input parameters.

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A The one-loop master integral and its ε-expansion

For the two-loop calculation we have to take into account the part proportional to ε of the one-loop propagator type integral[16]

$$J = \int \frac{\mathrm{d}^d q}{(q^2 - m_1^2 + \mathrm{i}0)((k-q)^2 - m_2^2 + \mathrm{i}0)} \,, \tag{A.1}$$

where $d = 4 - 2\varepsilon$ and "+i0" is the causal prescription for the propagator. Its finite part has been presented in [44], the $O(\varepsilon)$-part in [45], the terms up to order ε^3 can be extracted by means of Eq. (A.3) of [33] and an all order ε-expansion was obtained in [46]. We write the part of J linear in ε in a form suitable for the implementation in **FORM** [17]

$$\begin{aligned} J &= \mathrm{i}\pi^{2-\varepsilon} \frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)} \Bigg\{ \frac{m_1^{-2\varepsilon} + m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_1^2 - m_2^2}{\varepsilon\, k^2} \left(m_1^{-2\varepsilon} - m_2^{-2\varepsilon} \right) \\ &\quad - 2 \frac{\sqrt{-\lambda(m_1^2, m_2^2, k^2)}}{k^2} \Bigg[\arccos\left(\frac{m_1^2 + m_2^2 - k^2}{2 m_1 m_2} \right) \left(1 - \varepsilon \ln\left(\frac{-\lambda(m_1^2, m_2^2, k^2)}{k^2} \right) \right) \\ &\quad + 2\varepsilon \Big(\mathrm{Cl}_2(\tau_1) - \mathrm{Cl}_2(\pi - \tau_1) + \mathrm{Cl}_2(\tau_2) - \mathrm{Cl}_2(\pi - \tau_2) \Big) \Bigg] + \mathcal{O}(\varepsilon^2) \Bigg\}, \end{aligned} \tag{A.2}$$

where $\lambda(m_1^2, m_2^2, k^2) = (m_1^4 + m_2^4 + k^4 - 2m_1^2 k^2 - 2m_2^2 k^2 - 2m_1^2 m_2^2)$ and the angles τ_i are defined (see [46]) via

$$\cos\tau_1 = \frac{k^2 - m_1^2 + m_2^2}{2 m_2 \sqrt{k^2}} \,, \quad \cos\tau_2 = \frac{k^2 + m_1^2 - m_2^2}{2 m_1 \sqrt{k^2}} \,. \tag{A.3}$$

$\mathrm{Cl}_2(\theta)$ is the Clausen function $\mathrm{Cl}_2(\theta) = \frac{1}{2\mathrm{i}} \left[\mathrm{Li}_2\left(e^{\mathrm{i}\theta}\right) - \mathrm{Li}_2\left(e^{-\mathrm{i}\theta}\right) \right]$. The expansion (A.2) is directly applicable in the region where $\lambda \le 0$, i.e. when $(m_1 - m_2)^2 \le k^2 \le (m_1 + m_2)^2$. For the region $\lambda > 0$ we need the proper analytic continuation which has been given in [47]. Let

[16] In [45] it is denoted as $J^{(2)}(4 - 2\varepsilon; 1, 1)$.

[17] The higher order ε terms can be extracted from [46, 47].

us briefly describe it here. First of all, we rewrite (A.2) in the form (see section (2.2) of [47] for details)

$$J = \mathrm{i}\pi^{2-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left(\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon}+\frac{m_1^2-m_2^2}{\varepsilon k^2}\left(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}\right)\right.$$
$$-\mathrm{i}\frac{[-\lambda(m_1^2,m_2^2,k^2)]^{1/2-\varepsilon}}{(k^2)^{1-\varepsilon}}\left\{\frac{2\Gamma^2(1-\varepsilon)}{\varepsilon\Gamma(1-2\varepsilon)}\,(1-\rho_1-\rho_2)+\frac{1}{\varepsilon}\sum_{i=1}^{2}\left(\rho_i(-z_i)^{-\varepsilon}-(1-\rho_i)(-z_i)^{\varepsilon}\right)\right.$$
$$\left.\left.+2\varepsilon\sum_{i=1}^{2}\left(\rho_i(-z_i)^{-\varepsilon}\mathrm{Li}_2(z_i)-(1-\rho_i)(-z_i)^{\varepsilon}\mathrm{Li}_2(1/z_i)+\mathcal{O}(\varepsilon)\right)\right\}\right), \tag{A.4}$$

where ρ_1 and ρ_2 are some numbers, which we will define later and

$$z_1=\frac{\left[\sqrt{\lambda(m_1^2,m_2^2,k^2)}+m_1^2-m_2^2-k^2\right]^2}{4m_2^2k^2},\quad z_2=\frac{\left[\sqrt{\lambda(m_1^2,m_2^2,k^2)}-m_1^2+m_2^2-k^2\right]^2}{4m_1^2k^2}. \tag{A.5}$$

Firstly, we note that the causal prescription amounts to the following rule for λ ($\lambda>0$)

$$\begin{aligned}\ln(-\lambda(m_1^2,m_2^2,k^2)) &= \ln(\lambda(m_1^2,m_2^2,k^2))-\mathrm{i}\pi,\\ \sqrt{-\lambda(m_1^2,m_2^2,k^2)} &= -\mathrm{i}\sqrt{\lambda(m_1^2,m_2^2,k^2)}.\end{aligned}$$

The function $\mathrm{Li}_2(z)$ is real for real z and $|z|\le 1$. For real z and $|z|>1$ we change argument $z\to 1/z$ using the relation [18]

$$\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{1}{z}\right)=-\frac{1}{2}\ln^2(-z)-\zeta_2,$$

by which an imaginary part shows up. This change of variables can be done from the very beginning in (A.4) by an appropriate choice of the values of the coefficients ρ_j:

$$\begin{aligned}0<z_j<1 &\Rightarrow \rho_j=1;\quad \ln(-z_j)=\ln(z_j)+\mathrm{i}\pi,\\ z_j>1 &\Rightarrow \rho_j=0;\quad \ln(-z_j)=\ln(z_j)-\mathrm{i}\pi.\end{aligned}$$

Assuming $m_1<m_2$ in the following we have

- for $k^2<(m_1-m_2)^2\Rightarrow z_1<1, z_2>1$

$$J = \mathrm{i}\pi^{2-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left(\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon}+\frac{m_1^2-m_2^2}{\varepsilon k^2}\left(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}\right)\right.$$

[18] For the higher order poly-logarithm the relation is [48]

$$\mathrm{Li}_n(z)+(-1)^n\mathrm{Li}_n\left(\frac{1}{z}\right)=-\frac{1}{n!}\ln^n(-z)-\sum_{j=1}^{[n/2]}\frac{\ln^{n-2r}(-z)}{(n-2r)!}\zeta_{2r}.$$

$$+\frac{\sqrt{\lambda(m_1^2,m_2^2,k^2)}}{k^2}\Biggl\{\ln(z_1z_2)+\varepsilon\Biggl[2\mathrm{Li}_2\Bigl(\frac{1}{z_2}\Bigr)-2\mathrm{Li}_2(z_1)-\frac{1}{2}\ln^2 z_1+\frac{1}{2}\ln^2 z_2$$
$$-\ln(z_1z_2)\ln\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)+\mathcal{O}(\varepsilon^2)\Biggr]\Biggr\}\Biggr) \tag{A.6}$$

- for $k^2>(m_1+m_2)^2 \Rightarrow z_1<1, z_2<1$

$$\begin{aligned} J &= \mathrm{i}\pi^{2-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left(\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon}+\frac{m_1^2-m_2^2}{\varepsilon k^2}\left(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}\right)\right.\\ &+\frac{\sqrt{\lambda(m_1^2,m_2^2,k^2)}}{k^2}\Biggl\{\ln(z_1z_2)+\varepsilon\Biggl[-8\zeta_2-2\mathrm{Li}_2(z_1)-2\mathrm{Li}_2(z_2)\\ &-\frac{1}{2}\ln^2 z_1-\frac{1}{2}\ln^2 z_2-\ln(z_1z_2)\ln\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)\Biggr]\\ &+\mathrm{i}\pi\left[2-2\varepsilon\ln\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)\right]+\mathcal{O}(\varepsilon^2)\Biggr\}\Biggr). \end{aligned} \tag{A.7}$$

In a similar manner, starting from Eq. (2.17) of [47] and performing an analytical continuation, it is possible to obtain the higher order terms of the ε expansion [19]. In particular, the imaginary part of J in each order of ε coincides with that obtained from the exact result [21]

$$\mathrm{Im}J=\mathrm{i}\pi\theta\left(k^2-(m_1+m_2)^2\right)\frac{\sqrt{\lambda(m_1^2,m_2^2,k^2)}}{k^2}\left(\frac{\lambda(m_1^2,m_2^2,k^2)}{k^2}\right)^{-\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}.$$

In the limit, when one of the masses vanishes, the result is [47]

$$J|_{m_1=0,\,m_2\equiv m}=\mathrm{i}\pi^{2-\varepsilon}m^{-2\varepsilon}\frac{\Gamma(1+\varepsilon)}{(1-2\varepsilon)}\left\{\frac{1}{\varepsilon}-\frac{1-u}{2u\varepsilon}\left[(1-u)^{-2\varepsilon}-1\right]-\frac{(1-u)^{1-2\varepsilon}}{u}\varepsilon\mathrm{Li}_2(u)+\mathcal{O}(\varepsilon^2)\right\}, \tag{A.8}$$

with $u=k^2/m^2$.

The transition from the bare parameters to the renormalized ones requires differentiations of the one-loop propagators with respect to all parameters, couplings, masses and external momentum. The integrals obtained thereby can be reduces again to integrals of type (A.4) plus simpler bubble integrals. The expansion of the propagators with respect to small parameters (ratios of the masses or momenta and masses) can be extracted from the exact analytical results written in terms of hyper-geometric functions (see [50]).

[19] A collection of useful expressions for the one-loop two-point function is given also in Appendix A of [49].

B $\overline{\rm MS}$ vs. pole masses at one-loop

In this Appendix we present, for completeness, the well know [16] one-loop relations between pole and $\overline{\rm MS}$ masses of gauge bosons. Using the following notation

$$\frac{M_V^2}{m_V^2} = 1 + \left(\frac{e^2}{16\pi^2 \sin^2\theta_W}\right) X_V^{(1)} \tag{B.1}$$

we have

$$\begin{aligned}
X_H^{(1)} &= \frac{1}{2} - \frac{1}{2}\ln\frac{M_W^2}{\mu^2} - B(m_W^2, m_W^2; m_H^2) \\
&+ \frac{m_H^2}{m_W^2}\left(-\frac{3}{2} + \frac{9}{8}\frac{\pi}{\sqrt{3}} + \frac{3}{8}\ln\frac{M_H^2}{\mu^2} + \frac{1}{4}B(m_W^2, m_W^2; m_H^2) + \frac{1}{8}B(m_Z^2, m_Z^2; m_H^2)\right) \\
&+ \frac{m_Z^2}{m_W^2}\left(\frac{1}{4} - \frac{1}{4}\ln\frac{M_Z^2}{\mu^2} - \frac{1}{2}B(m_Z^2, m_Z; m_H^2)\right) \\
&+ \frac{m_W^2}{m_H^2}\left(3 - 3\ln\frac{M_W^2}{\mu^2} + 3B(m_W^2, m_W; m_H^2)\right) \\
&+ \frac{m_Z^4}{m_W^2 m_H^2}\left(\frac{3}{2} - \frac{3}{2}\ln\frac{M_Z^2}{\mu^2} + \frac{3}{2}B(m_Z^2, m_Z; m_H^2)\right)
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
X_W^{(1)} &= \frac{73}{9} - 3\ln\frac{M_W^2}{\mu^2} + 2\ln\frac{M_Z^2}{\mu^2} - \frac{17}{3}B(m_Z^2, m_W^2; m_W^2) + B(m_H^2, m_W^2; m_W^2) \\
&+ \frac{m_H^4}{m_W^4}\left(\frac{1}{12} - \frac{1}{12}\ln\frac{M_H^2}{\mu^2} + \frac{1}{12}B(m_H^2, m_W^2; m_W^2)\right) \\
&+ \frac{m_H^2}{m_W^2}\left(\frac{7}{12} + \frac{1}{12}\ln\frac{M_W^2}{\mu^2} - \frac{1}{2}\ln\left(\frac{M_H^2}{\mu^2}\right) - \frac{1}{3}B(m_H^2, m_W^2; m_W^2)\right) \\
&+ \frac{m_Z^4}{m_W^4}\left(\frac{1}{12} - \frac{1}{12}\ln\frac{M_Z^2}{\mu^2} + \frac{1}{12}B(m_Z^2, m_W^2; m_W^2)\right) \\
&+ \frac{m_Z^2}{m_W^2}\left(\frac{3}{4} + \frac{1}{12}\ln\frac{M_W^2}{\mu^2} - \frac{2}{3}\ln\frac{M_Z^2}{\mu^2} + \frac{4}{3}B(m_Z^2, m_W^2; m_W^2)\right) \\
&+ \frac{m_W^2}{m_Z^2}\left(-8 + 4\ln\frac{M_W^2}{\mu^2} - 4B(m_Z^2, m_W^2, m_W^2)\right) \\
&+ \frac{m_Z^4}{m_W^2 m_H^2}\left(\frac{1}{2} - \frac{3}{2}\ln\frac{M_Z^2}{\mu^2}\right) + \frac{m_W^2}{m_H^2}\left(1 - 3\ln\frac{M_W^2}{\mu^2}\right)
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
X_Z^{(1)} &= \frac{13}{18} - \frac{1}{6}\ln\frac{M_W^2}{\mu^2} + \frac{4}{3}B(m_W^2, m_W; m_Z^2) \\
&+ \frac{m_H^4}{m_W^2 m_Z^2}\left(\frac{1}{12} - \frac{1}{12}\ln\frac{M_H^2}{\mu^2} + \frac{1}{12}B(m_H^2, m_Z^2; m_Z^2)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{m_H^2}{m_W^2}\left(\frac{7}{12}+\frac{1}{12}\ln\frac{M_Z^2}{\mu^2}-\frac{1}{2}\ln\frac{M_H^2}{\mu^2}-\frac{1}{3}B(m_H^2,m_Z^2;m_Z^2)\right)\\
&+ \frac{m_Z^2}{m_W^2}\left(\frac{2}{9}-\frac{1}{6}\ln\frac{M_Z^2}{\mu^2}+\frac{1}{12}B(m_W^2,m_W^2;m_Z^2)+B(m_H^2,m_Z^2;m_Z^2)\right)\\
&+ \frac{m_W^2}{m_Z^2}\left(-\frac{4}{3}\ln\frac{M_W^2}{\mu^2}-\frac{17}{3}B(m_W^2,m_W^2;m_Z^2)\right)+\frac{m_W^4}{m_Z^4}\left(4\ln\frac{M_W^2}{\mu^2}-4B(m_W^2,m_W^2;m_Z^2)\right)\\
&+ \frac{m_W^2}{m_H^2}\left(1-3\ln\frac{M_W^2}{\mu^2}\right)+\frac{m_Z^4}{m_W^2m_H^2}\left(\frac{1}{2}-\frac{3}{2}\ln\frac{M_Z^2}{\mu^2}\right). \qquad \text{(B.4)}
\end{aligned}
$$

where we have used the following function:

$$
B(m_1^2,m_2^2;p^2)=\int_0^1 dx\log\left(\frac{m_1^2}{\mu^2}x+\frac{m_2^2}{\mu^2}(1-x)-\frac{p^2}{\mu^2}x(1-x)-\mathrm{i}0\right)\ .
$$

C Unrenormalized one-loop expressions in d dimension

The computation of higher loop corrections requires a deeper expansion in ε of lower order terms. In this Appendix we present for completeness the results for unrenormalized one-loop corrections to the relation between pole and $\overline{\rm MS}$ masses of the gauge bosons in arbitrary dimension d without expanding it in ε. Using the following notation

$$
\frac{M_V^2}{m_{0,V}^2}=1+\left(\frac{g_0^2}{16\pi^2}\right)\left[X_{0,V}^{(1),\rm boson}+X_{0,V}^{(1),\rm fermion}\right]\ , \qquad \text{(C.1)}
$$

we have

$$
\begin{aligned}
X_{0,W}^{(1),\rm boson} &= \frac{1}{4(d-1)}\Bigg[\frac{m_H^4}{m_W^4}\Big(-B_0(m_H^2,m_W^2,m_W^2)-A_0(m_H^2)\Big)\\
&\qquad +\frac{m_H^2}{m_W^2}\Big(A_0(m_W^2)+4B_0(m_H^2,m_W^2,m_W^2)\Big)\\
&\qquad +\frac{m_Z^4}{m_W^4}\Big(-B_0(m_Z^2,m_W^2,m_W^2)-A_0(m_Z^2)\Big)-16B_0(m_Z^2,m_W^2,m_W^2)\\
&\qquad +\frac{m_Z^2}{m_W^2}\Big(A_0(m_W^2)+4A_0(m_Z^2)+8B_0(m_Z^2,m_W^2,m_W^2)\Big)\Bigg]\\
&\quad -\frac{m_Z^2}{m_W^2}\Big(2B_0(m_Z^2,m_W^2,m_W^2)+A_0(m_Z^2)\Big)+7B_0(m_Z^2,m_W^2,m_W^2)\\
&\quad +2\frac{m_W^2}{m_Z^2}\Big(2B_0(m_Z^2,m_W^2,m_W^2)+A_0(m_W^2)\Big)-B_0(m_H^2,m_W^2,m_W^2)\\
&\quad +(d-5)A_0(m_W^2)+(d-2)A_0(m_Z^2)-\frac{d-1}{2}\frac{m_Z^4}{m_H^2m_W^2}A_0(m_Z^2)\\
&\quad -\frac{1}{2}\frac{m_H^2}{m_W^2}A_0(m_H^2)-\frac{m_W^2}{m_H^2}(d-1)A_0(m_W^2)-2\frac{1}{d-3}A_0(m_W^2)\left(1-\frac{m_W^2}{m_Z^2}\right)\ , \qquad \text{(C.2)}
\end{aligned}
$$

$$
\begin{aligned}
X_{0,W}^{(1),\text{fermion}} &= -\frac{1}{2}\frac{d-2}{d-1}\sum_{\text{lepton}}\left(B_0(0,m_l^2,m_W^2)+\frac{m_l^2}{m_W^2}A_0(m_l^2)\right)+\sum_{\text{lepton}}2\frac{m_l^4}{m_W^2m_H^2}A_0(m_l^2)\\
&+\frac{1}{2}\frac{d-3}{d-1}\sum_{\text{lepton}}\frac{m_l^2}{m_W^2}B_0(0,m_l^2,m_W^2)\\
&+\frac{1}{2}\frac{1}{d-1}\sum_{\text{lepton}}\frac{m_l^4}{m_W^4}\left(B_0(0,m_l^2,m_W^2)+A_0(m_l^2)\right)\\
&-\frac{N_c}{2(d-1)}\sum_{i,j=1}^{3}\Bigg[\left(2\frac{m_{u_i}^2m_{d_j}^2}{m_W^4}-\frac{m_{u_i}^4}{m_W^4}-\frac{m_{d_j}^4}{m_W^4}\right)-(d-3)\left(\frac{m_{u_i}^2}{m_W^2}+\frac{m_{d_j}^2}{m_W^2}\right)\\
&\qquad\qquad+(d-2)\Bigg]\times K_{ij}K_{ji}^*B_0(m_{u_i}^2,m_{d_j}^2,m_W^2)\\
&-\frac{N_c}{2(d-1)}\sum_{i=1}^{3}\frac{m_{u_i}^2}{m_W^2}A_0(m_{u_i}^2)\left(\sum_{j=1}^{3}\frac{m_{d_j}^2}{m_W^2}K_{ij}K_{ij}^*-\frac{m_{u_i}^2}{m_W^2}+(d-2)-4(d-1)\frac{m_{u_i}^2}{m_H^2}\right)\\
&-\frac{N_c}{2(d-1)}\sum_{i=1}^{3}\frac{m_{d_i}^2}{m_W^2}A_0(m_{d_i}^2)\left(\sum_{j=1}^{3}\frac{m_{u_j}^2}{m_W^2}K_{ji}K_{ij}^*-\frac{m_{d_i}^2}{m_W^2}+(d-2)-4(d-1)\frac{m_{d_i}^2}{m_H^2}\right)
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
X_{0,Z}^{(1),\text{boson}} &= \frac{1}{4(d-1)}\Bigg[\frac{m_H^4}{m_W^2m_Z^2}\left(-B_0(m_H^2,m_Z^2,m_Z^2)-A_0(m_H^2)\right)\\
&\qquad+\frac{m_H^2}{m_W^2}\left(A_0(m_Z^2)+4B_0(m_H^2,m_Z^2,m_Z^2)\right)\\
&\qquad-\frac{m_Z^2}{m_W^2}\left(B_0(m_W^2,m_W^2,m_Z^2)+2A_0(m_Z^2)\right)\\
&\qquad+8\frac{m_W^2}{m_Z^2}\left(-2B_0(m_W^2,m_W^2,m_Z^2)+A_0(m_W^2)\right)\\
&\qquad-2\left(A_0(m_W^2)-4B_0(m_W^2,m_W^2,m_Z^2)\right)\Bigg]\\
&-\frac{m_W^2}{m_Z^2}\left(2A_0(m_W^2)-7B_0(m_W^2,m_W^2,m_Z^2)\right)-\frac{1}{2}\frac{m_H^2}{m_W^2}A_0(m_H^2)\\
&-\frac{m_Z^2}{m_W^2}B_0(m_H^2,m_Z^2,m_Z^2)-2B_0(m_W^2,m_W^2,m_Z^2)-\frac{d-1}{2}\frac{m_Z^4}{m_W^2m_H^2}A_0(m_Z^2)\\
&+2\frac{m_W^4}{m_Z^4}\left((d-2)A_0(m_W^2)+2B_0(m_W^2,m_W^2,m_Z^2)\right)-(d-1)\frac{m_W^2}{m_H^2}A_0(m_W^2)
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
X_{0,Z}^{(1),\text{fermion}} &= -\frac{1}{4}\frac{d-2}{d-1}\sum_{\text{lepton}}\Bigg[\frac{m_Z^2}{m_W^2}B_0(0,0,m_Z^2)-\left(12-8\frac{m_W^2}{m_Z^2}-5\frac{m_Z^2}{m_W^2}\right)B_0(m_l^2,m_l^2,m_Z^2)\\
&\qquad+\left(10\frac{m_l^2}{m_W^2}+16\frac{m_l^2m_W^2}{m_Z^4}-24\frac{m_l^2}{m_Z^2}\right)\left(A_0(m_l^2)+\frac{2}{d-2}B_0(m_l^2,m_l^2,m_Z^2)\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{N_c}{36}\frac{d-2}{d-1}\sum_{\rm up}\Bigg[-\left(40-17\frac{m_Z^2}{m_W^2}-32\frac{m_W^2}{m_Z^2}\right)B_0(m_u^2,m_u^2,m_Z^2)\\
&\qquad+\left(34\frac{m_u^2}{m_W^2}+64\frac{m_u^2m_W^2}{m_Z^4}-80\frac{m_u^2}{m_Z^2}\right)\left(A_0(m_u^2)+\frac{2}{d-2}B_0(m_u^2,m_u^2,m_Z^2)\right)\Bigg]\\
&-\frac{N_c}{36}\frac{d-2}{d-1}\sum_{\rm down}\Bigg[-\left(4-5\frac{m_Z^2}{m_W^2}-8\frac{m_W^2}{m_Z^2}\right)B_0(m_d^2,m_d^2,m_Z^2)\\
&\qquad+\left(10\frac{m_d^2}{m_W^2}+16\frac{m_d^2m_W^2}{m_Z^4}-8\frac{m_d^2}{m_Z^2}\right)\left(A_0(m_d^2)+\frac{2}{d-2}B_0(m_d^2,m_d^2,m_Z^2)\right)\Bigg]\\
&+\sum_{\rm lepton}\left\{2\frac{m_l^4}{m_W^2m_H^2}A_0(m_l^2)+\frac{1}{2}\frac{m_l^2}{m_W^2}B_0(m_l^2,m_l^2,m_Z^2)\right\}\\
&+N_c\sum_{\rm quark}\left\{2\frac{m_q^4}{m_W^2m_H^2}A_0(m_q^2)+\frac{1}{2}\frac{m_q^2}{m_W^2}B_0(m_q^2,m_q^2,m_Z^2)\right\}
\end{aligned}
\tag{C.5}
$$

In above formulae we use the following functions

$$
\begin{aligned}
A_0(m_1^2) &= \frac{1}{m_1^2}\left(\frac{\mu^2e^\gamma}{4\pi}\right)^\varepsilon\int\frac{{\rm d}^dq}{i\pi^{d/2}}\frac{1}{(q^2-m_1^2)}\,,\\
B_0(m_1^2,m_2^2;p^2) &= \left(\frac{\mu^2e^\gamma}{4\pi}\right)^\varepsilon\int\frac{{\rm d}^dq}{i\pi^{d/2}}\frac{1}{(q^2-m_1^2)((p-q)^2-m_2^2)}\,.
\end{aligned}
\tag{C.6}
$$

m_{u_i} and m_{d_i} denote the masses of corresponding up- and down-quarks, N_c is a number of color ($N_c=3$) and K_{ij} is the element of the Kobayashi-Maskawa matrix.

D $\overline{\rm MS}$ vs. pole masses at two-loop

After expansion of the diagrams with respect to $\sin^2\theta_W$ we get rid of one of the boson masses and write the functions $X_V^{(2)}$ introduced in (5.4) in the form

$$
X_V^{(2)}=\frac{m_H^4}{m_V^4}\sum_{k=0}^{5}\sin^{2k}\theta_W\,A_k^V\,.
\tag{D.7}
$$

In particular for the Z boson propagator we eliminate m_W and vice versa. Consequently, the coefficients A_i^V in the above formula are functions of the Higgs mass and one of the boson masses. We expand this function with respect to m_V^2/m_H^2

$$
A_i^V=\sum_{j=0}^{5}A_{i,j}^V\left(\frac{m_V^2}{m_H^2}\right)^j
$$

and calculate analytically the first six coefficients. This is not a naive Taylor expansion. The general rules for asymptotic expansions [12] allow us to extract also logarithmic dependences, or in other words, to preserve all analytical properties of the original diagrams. In the result

of the asymptotic expansion all propagator diagrams are reduced to single scale massive diagrams (including the two-loop bubbles). As a consequence, the finite as well as the ε-part of the corresponding diagrams, are characterized by a restricted set of transcendental numbers [51] which may appear in the coefficients $A_{i,j}$. We find the following constants:

$$\begin{aligned} S_0 &= \frac{\pi}{\sqrt{3}} \sim 1.813799365..., & S_1 &= \frac{\pi}{\sqrt{3}} \ln 3 \sim 1.992662272..., \\ S_2 &= \frac{4}{9} \frac{\mathrm{Cl}_2\left(\frac{\pi}{3}\right)}{\sqrt{3}} \sim 0.260434137632162..., & S_3 &= \pi \mathrm{Cl}_2\left(\frac{\pi}{3}\right) \sim 3.188533097... \end{aligned} \quad \text{(D.8)}$$

Furthermore, $\ln(m_H^2)$ denotes $\ln\left(m_H^2/\mu^2\right)$ where μ is the 't Hooft scale. We also introduce the notation $w_H = m_W^2/m_H^2$ and $z_H = m_Z^2/m_H^2$.

D.1 Analytical results for the two-loop finite part of W-boson

$$\begin{aligned} A_0^W &= \left[-\tfrac{359}{128} + \tfrac{243}{32} S_2 - \tfrac{1}{24}\pi^2 + \tfrac{33}{16} \ln(m_H^2) - \tfrac{9}{32} \left(\ln(m_H^2)\right)^2 \right] \\ &+ w_H \left[\tfrac{2483}{192} - \tfrac{231}{32} S_0 + \tfrac{243}{32} S_2 + \tfrac{433}{864}\pi^2 - \tfrac{161}{32} \ln(w_H) + \tfrac{69}{16} \ln(w_H)\ln(m_H^2) \right. \\ &\quad \left. - \tfrac{519}{32} \ln(m_H^2) + \tfrac{99}{16} \ln(m_H^2) S_0 + \tfrac{43}{8} \left(\ln(m_H^2)\right)^2 \right] \\ &+ w_H^2 \left[\tfrac{8311573}{20736} - \tfrac{153}{8} S_3 - \tfrac{243}{4} S_0 - \tfrac{87651}{128} S_2 - \tfrac{47893}{3456}\pi^2 + \tfrac{459}{16}\zeta(3) - \tfrac{399943}{1728} \ln(w_H) \right. \\ &\quad + \tfrac{1705}{16} \ln(w_H) S_0 + \tfrac{1805}{48} \left(\ln(w_H)\right)^2 + \tfrac{1951}{24} \ln(w_H)\ln(m_H^2) - \tfrac{356585}{1728} \ln(m_H^2) \\ &\quad \left. + \tfrac{1595}{16} \ln(m_H^2) S_0 + \tfrac{10013}{288} \left(\ln(m_H^2)\right)^2 \right] \\ &+ w_H^3 \left[\tfrac{10730119}{86400} - \tfrac{3451}{96} S_0 - \tfrac{35739}{80} S_2 + \tfrac{3167}{1080}\pi^2 - \tfrac{1173881}{4320} \ln(w_H) \right. \\ &\quad + \tfrac{517}{8} \ln(w_H) S_0 + \tfrac{6767}{96} \left(\ln(w_H)\right)^2 + \tfrac{1743}{16} \ln(w_H)\ln(m_H^2) - \tfrac{30383}{192} \ln(m_H^2) \\ &\quad \left. + \tfrac{297}{8} \ln(m_H^2) S_0 + \tfrac{177}{4} \left(\ln(m_H^2)\right)^2 \right] \\ &+ w_H^4 \left[-\tfrac{276774409}{1296000} + \tfrac{86473}{3200} S_0 - \tfrac{3807}{20} S_2 + \tfrac{859}{80}\pi^2 - \tfrac{168691}{1152} \ln(w_H) + \tfrac{10323}{160} \ln(w_H) S_0 \right. \\ &\quad \left. + \tfrac{90121}{960} \left(\ln(w_H)\right)^2 + \tfrac{187}{2} \ln(w_H)\ln(m_H^2) + \tfrac{15733}{240} \ln(m_H^2) + \tfrac{81}{4} \left(\ln(m_H^2)\right)^2 \right] \\ &+ w_H^5 \left[-\tfrac{13424129921}{14112000} + \tfrac{416057}{2400} S_0 - \tfrac{452727}{1120} S_2 + \tfrac{1395311}{30240}\pi^2 - \tfrac{3016477}{10080} \ln(w_H) \right. \\ &\quad \left. + \tfrac{7739}{40} \ln(w_H) S_0 + \tfrac{16007}{60} \left(\ln(w_H)\right)^2 + \tfrac{12347}{72} \ln(w_H)\ln(m_H^2) + \tfrac{6333137}{30240} \ln(m_H^2) \right] \end{aligned}$$

$$
\begin{aligned}
A_1^W &= w_H \left[\tfrac{79}{48} + \tfrac{77}{32}\,S_0 + \tfrac{3}{16}\,\pi^2 + \tfrac{21}{32}\,\ln(w_H) - \tfrac{9}{16}\,\ln(w_H)\ln(m_H^2) - \tfrac{25}{32}\,\ln(m_H^2) - \tfrac{33}{16}\,\ln(m_H^2)S_0\right] \\
&+ w_H^2\Bigg[-\tfrac{15491}{576} + \tfrac{65}{6}\,S_3 - \tfrac{11183}{48}\,S_0 + 66\,S_1 - \tfrac{4239}{16}\,S_2 + \tfrac{8837}{216}\,\pi^2 - \tfrac{22}{3}\,\pi^2\ln(3) + \tfrac{67}{4}\,\zeta_3 \\
&\quad -\tfrac{149}{216}\,\ln(w_H) - \tfrac{209}{12}\,\ln(w_H)S_0 - \tfrac{277}{72}\,(\ln(w_H))^2 - \tfrac{5}{4}\,\ln(w_H)\ln(m_H^2) + \tfrac{559}{72}\,\ln(m_H^2) \\
&\quad -\tfrac{473}{24}\,\ln(m_H^2)S_0 - \tfrac{5}{16}\,\left(\ln(m_H^2)\right)^2\Bigg] \\
&+ w_H^3\Bigg[-\tfrac{14221}{216} + \tfrac{977}{72}\,S_0 + \tfrac{8181}{64}\,S_2 + \tfrac{5705}{1728}\,\pi^2 - \tfrac{5923}{216}\,\ln(w_H) + \tfrac{77}{3}\,\ln(w_H)S_0 \\
&\quad +\tfrac{2801}{144}\,(\ln(w_H))^2 + \tfrac{397}{16}\,\ln(w_H)\ln(m_H^2) - \tfrac{1361}{192}\,\ln(m_H^2) + \tfrac{99}{8}\,\ln(m_H^2)S_0 + 12\,\left(\ln(m_H^2)\right)^2\Bigg] \\
&+ w_H^4\Bigg[-\tfrac{43842631}{288000} + \tfrac{259993}{4800}\,S_0 + \tfrac{4329}{160}\,S_2 + \tfrac{21149}{1440}\,\pi^2 + \tfrac{26903}{1200}\,\ln(w_H) + \tfrac{4431}{80}\,\ln(w_H)S_0 \\
&\quad +\tfrac{40579}{720}\,(\ln(w_H))^2 + \tfrac{423}{8}\,\ln(w_H)\ln(m_H^2) + \tfrac{635}{32}\,\ln(m_H^2) + 27\,\left(\ln(m_H^2)\right)^2\Bigg] \\
&+ w_H^5\Bigg[-\tfrac{367144853}{518400} + \tfrac{2094431}{7200}\,S_0 + \tfrac{29151}{320}\,S_2 + \tfrac{210541}{2880}\,\pi^2 + \tfrac{8105093}{43200}\,\ln(w_H) \\
&\quad +\tfrac{28277}{120}\,\ln(w_H)S_0 + \tfrac{120943}{720}\,(\ln(w_H))^2 - \tfrac{53}{8}\,\ln(w_H)\ln(m_H^2) - \tfrac{4829}{480}\,\ln(m_H^2)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
A_2^W &= w_H\left[-\tfrac{29}{32} + \tfrac{35}{16}\,S_0 + \tfrac{3}{16}\,\pi^2 + \tfrac{21}{32}\,\ln(w_H) - \tfrac{9}{16}\,\ln(w_H)\ln(m_H^2) + \tfrac{45}{32}\,\ln(m_H^2) - \tfrac{15}{8}\,\ln(m_H^2)S_0\right] \\
&+ w_H^2\Bigg[\tfrac{439321}{3456} + \tfrac{7}{36}\,S_3 + \tfrac{18119}{144}\,S_0 - 44\,S_1 + \tfrac{14469}{64}\,S_2 - \tfrac{19417}{576}\,\pi^2 - 16\,\pi^2\ln(2) + \tfrac{110}{9}\,\pi^2\ln(3) \\
&\quad -\tfrac{751}{24}\,\zeta_3 - \tfrac{973}{432}\,\ln(w_H) - \tfrac{233}{12}\,\ln(w_H)S_0 - \tfrac{581}{144}\,(\ln(w_H))^2 \\
&\quad +\tfrac{45}{16}\,\ln(w_H)\ln(m_H^2) + \tfrac{281}{96}\,\ln(m_H^2) - \tfrac{71}{4}\,\ln(m_H^2)S_0 + \tfrac{55}{32}\,\left(\ln(m_H^2)\right)^2\Bigg] \\
&+ w_H^3\Bigg[-\tfrac{304691}{3456} + \tfrac{15601}{576}\,S_0 + \tfrac{10659}{64}\,S_2 + \tfrac{9167}{1728}\,\pi^2 + \tfrac{4267}{864}\,\ln(w_H) + \tfrac{1073}{48}\,\ln(w_H)S_0 \\
&\quad +\tfrac{3509}{144}\,(\ln(w_H))^2 + \tfrac{393}{16}\,\ln(w_H)\ln(m_H^2) + \tfrac{945}{64}\,\ln(m_H^2) + \tfrac{141}{8}\,\ln(m_H^2)S_0 + \tfrac{45}{4}\,\left(\ln(m_H^2)\right)^2\Bigg] \\
&+ w_H^4\Bigg[-\tfrac{107428459}{432000} + \tfrac{990193}{9600}\,S_0 + \tfrac{5103}{40}\,S_2 + \tfrac{111}{5}\,\pi^2 + \tfrac{10042841}{86400}\,\ln(w_H) + \tfrac{10771}{160}\,\ln(w_H)S_0 \\
&\quad +\tfrac{32969}{320}\,(\ln(w_H))^2 + \tfrac{801}{8}\,\ln(w_H)\ln(m_H^2) + \tfrac{1881}{32}\,\ln(m_H^2) + \tfrac{99}{2}\,\left(\ln(m_H^2)\right)^2\Bigg] \\
&+ w_H^5\Bigg[-\tfrac{3726216829}{2592000} + \tfrac{2622653}{3600}\,S_0 + \tfrac{98229}{320}\,S_2 + \tfrac{1160293}{8640}\,\pi^2 + \tfrac{21966593}{43200}\,\ln(w_H)
\end{aligned}
$$

$$+\frac{27221}{60}\,\ln(w_H)S_0+\frac{253819}{720}\,\left(\ln(w_H)\right)^2+\frac{1}{8}\,\ln(w_H)\ln(m_H^2)-\frac{199}{96}\,\ln(m_H^2)\Bigg]$$

$$\begin{aligned}
A_3^W &= w_H\left[-2+\frac{49}{18}\,S_0+\frac{3}{16}\,\pi^2+\frac{21}{32}\,\ln(w_H)-\frac{9}{16}\,\ln(w_H)\ln(m_H^2)+\frac{75}{32}\,\ln(m_H^2)-\frac{7}{3}\,\ln(m_H^2)S_0\right]\\
&+ w_H^2\Bigg[-\frac{799}{12}-\frac{199}{54}\,S_3-\frac{26975}{144}\,S_0-\frac{5}{3}\,S_1+\frac{4745}{24}\,S_2+\frac{1669}{54}\,\pi^2+\frac{104}{27}\,\pi^2\ln(3)-\frac{425}{36}\,\zeta(3)\\
&\quad+\frac{284}{27}\,\ln(w_H)-\frac{1709}{108}\,\ln(w_H)S_0-\frac{38}{9}\,\left(\ln(w_H)\right)^2+\frac{55}{8}\,\ln(w_H)\ln(m_H^2)+\frac{47}{4}\,\ln(m_H^2)\\
&\quad-\frac{859}{108}\,\ln(m_H^2)S_0+\frac{15}{4}\,\left(\ln(m_H^2)\right)^2\Bigg]\\
&+ w_H^3\Bigg[-\frac{857575}{6912}+\frac{196669}{5184}\,S_0+\frac{49193}{192}\,S_2+\frac{13715}{1728}\,\pi^2+\frac{26065}{576}\,\ln(w_H)+\frac{2429}{432}\,\ln(w_H)S_0\\
&\quad+\frac{7795}{288}\,\left(\ln(w_H)\right)^2+\frac{309}{16}\,\ln(w_H)\ln(m_H^2)+\frac{2169}{64}\,\ln(m_H^2)+\frac{125}{8}\,\ln(m_H^2)S_0+8\,\left(\ln(m_H^2)\right)^2\Bigg]\\
&+ w_H^4\Bigg[-\frac{3314383}{9000}+\frac{1531019}{10800}\,S_0+\frac{42717}{160}\,S_2+\frac{53773}{1440}\,\pi^2+\frac{6187879}{21600}\,\ln(w_H)+\frac{7507}{120}\,\ln(w_H)S_0\\
&\quad+\frac{19279}{120}\,\left(\ln(w_H)\right)^2+\frac{1323}{8}\,\ln(w_H)\ln(m_H^2)+\frac{3943}{32}\,\ln(m_H^2)+81\,\left(\ln(m_H^2)\right)^2\Bigg]\\
&+ w_H^5\Bigg[-\frac{3404318779}{1296000}+\frac{1761697}{1296}\,S_0+\frac{288607}{480}\,S_2+\frac{1098061}{4320}\,\pi^2+\frac{4020149}{3600}\,\ln(w_H)\\
&\quad+\frac{38929}{54}\,\ln(w_H)S_0+\frac{43325}{72}\,\left(\ln(w_H)\right)^2+\frac{55}{8}\,\ln(w_H)\ln(m_H^2)+\frac{2839}{480}\,\ln(m_H^2)\Bigg]
\end{aligned}$$

$$\begin{aligned}
A_4^W &= w_H\Bigg[-\frac{3095}{1152}+\frac{2723}{864}\,S_0+\frac{3}{16}\,\pi^2+\frac{21}{32}\,\ln(w_H)-\frac{9}{16}\,\ln(w_H)\ln(m_H^2)+\frac{563}{192}\,\ln(m_H^2)\\
&\quad-\frac{389}{144}\,\ln(m_H^2)S_0\Bigg]\\
&+ w_H^2\Bigg[-\frac{40045}{432}-\frac{841}{216}\,S_3-\frac{747209}{5184}\,S_0+\frac{1}{9}\,S_1+\frac{125399}{384}\,S_2+\frac{660755}{31104}\,\pi^2+\frac{68}{27}\,\pi^2\ln(3)-\frac{791}{144}\,\zeta_3\\
&\quad+\frac{4151}{216}\,\ln(w_H)-\frac{26723}{1296}\,\ln(w_H)S_0-\frac{635}{144}\,\left(\ln(w_H)\right)^2+\frac{175}{16}\,\ln(w_H)\ln(m_H^2)+\frac{27841}{1728}\,\ln(m_H^2)\\
&\quad-\frac{7153}{1296}\,\ln(m_H^2)S_0+\frac{185}{32}\,\left(\ln(m_H^2)\right)^2\Bigg]\\
&+ w_H^3\Bigg[-\frac{803113}{5184}+\frac{323699}{7776}\,S_0+\frac{33121}{96}\,S_2+\frac{28757}{2592}\,\pi^2+\frac{85597}{864}\,\ln(w_H)-\frac{30247}{1296}\,\ln(w_H)S_0\\
&\quad+\frac{7933}{288}\,\left(\ln(w_H)\right)^2+\frac{145}{16}\,\ln(w_H)\ln(m_H^2)+\frac{10643}{192}\,\ln(m_H^2)+\frac{103}{12}\,\ln(m_H^2)S_0+\frac{9}{4}\,\left(\ln(m_H^2)\right)^2\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad w_H^4 \Bigg[-\tfrac{1919013}{4000} + \tfrac{40790531}{259200}\, S_0 + \tfrac{869239}{1920}\, S_2 + \tfrac{1015361}{17280}\, \pi^2 + \tfrac{16268087}{28800}\, \ln(w_H) + \tfrac{7991}{360}\, \ln(w_H) S_0 \\
&\quad + \tfrac{660151}{2880}\, (\ln(w_H))^2 + \tfrac{2025}{8}\, \ln(w_H) \ln(m_H^2) + \tfrac{7121}{32}\, \ln(m_H^2) + \tfrac{495}{4}\, \left(\ln(m_H^2)\right)^2 \Bigg] \\
&+ \quad w_H^5 \Bigg[-\tfrac{305951057}{72000} + \tfrac{414264923}{194400}\, S_0 + \tfrac{265819}{240}\, S_2 + \tfrac{955459}{2160}\, \pi^2 + \tfrac{11862391}{5400}\, \ln(w_H) \\
&\quad + \tfrac{3022481}{3240}\, \ln(w_H) S_0 + \tfrac{40741}{45}\, (\ln(w_H))^2 + \tfrac{109}{8}\, \ln(w_H) \ln(m_H^2) + \tfrac{6673}{480}\, \ln(m_H^2) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
A_5^W \quad = \quad & w_H \Bigg[-\tfrac{5947}{1920} + \tfrac{2975}{864}\, S_0 + \tfrac{3}{16}\, \pi^2 + \tfrac{21}{32}\, \ln(w_H) - \tfrac{9}{16}\, \ln(w_H) \ln(m_H^2) \\
& \quad + \tfrac{1051}{320}\, \ln(m_H^2) - \tfrac{425}{144}\, \ln(m_H^2) S_0 \Bigg] \\
& + \quad w_H^2 \Bigg[-\tfrac{942115}{5184} - \tfrac{37}{9}\, S_3 - \tfrac{3403793}{19440}\, S_0 - \tfrac{1}{54}\, S_1 + \tfrac{643159}{1440}\, S_2 + \tfrac{3758107}{116640}\, \pi^2 + \tfrac{10}{9}\, \pi^2 \ln(3) + \tfrac{7}{6}\, \zeta_3 \\
& \quad + \tfrac{61481}{2160}\, \ln(w_H) - \tfrac{4435}{162}\, \ln(w_H) S_0 - \tfrac{331}{72}\, (\ln(w_H))^2 + 15\, \ln(w_H) \ln(m_H^2) + \tfrac{3829}{180}\, \ln(m_H^2) \\
& \quad - \tfrac{995}{216}\, \ln(m_H^2) S_0 + \tfrac{125}{16}\, \left(\ln(m_H^2)\right)^2 \Bigg] \\
& + \quad w_H^3 \Bigg[-\tfrac{4522879}{25920} + \tfrac{1329751}{38880}\, S_0 + \tfrac{120049}{288}\, S_2 + \tfrac{39089}{2592}\, \pi^2 + \tfrac{16039}{96}\, \ln(w_H) - \tfrac{84613}{1296}\, \ln(w_H) S_0 \\
& \quad + \tfrac{929}{36}\, (\ln(w_H))^2 - \tfrac{99}{16}\, \ln(w_H) \ln(m_H^2) + \tfrac{77011}{960}\, \ln(m_H^2) - \tfrac{35}{9}\, \ln(m_H^2) S_0 - 6\, \left(\ln(m_H^2)\right)^2 \Bigg] \\
& + \quad w_H^4 \Bigg[-\tfrac{239678009}{432000} + \tfrac{15484933}{129600}\, S_0 + \tfrac{635063}{960}\, S_2 + \tfrac{771181}{8640}\, \pi^2 + \tfrac{42518293}{43200}\, \ln(w_H) - \tfrac{9928}{135}\, \ln(w_H) S_0 \\
& \quad + \tfrac{74083}{240}\, (\ln(w_H))^2 + \tfrac{2943}{8}\, \ln(w_H) \ln(m_H^2) + \tfrac{58767}{160}\, \ln(m_H^2) + 180\, \left(\ln(m_H^2)\right)^2 \Bigg] \\
& + \quad w_H^5 \Bigg[-\tfrac{16217189681}{2592000} + \tfrac{138228439}{48600}\, S_0 + \tfrac{1072891}{576}\, S_2 + \tfrac{6308171}{8640}\, \pi^2 + \tfrac{171948913}{43200}\, \ln(w_H) \\
& \quad + \tfrac{604271}{648}\, \ln(w_H) S_0 + \tfrac{895867}{720}\, (\ln(w_H))^2 + \tfrac{163}{8}\, \ln(w_H) \ln(m_H^2) + \tfrac{10507}{480}\, \ln(m_H^2) \Bigg]
\end{aligned}
$$

D.2 Analytical results for the two-loop finite part of Z-boson

$$
\begin{aligned}
A_0^Z \quad = \quad & \Bigg[-\tfrac{359}{128} + \tfrac{243}{32}\, S_2 - \tfrac{1}{24}\, \pi^2 + \tfrac{33}{16}\, \ln(m_H^2) - \tfrac{9}{32}\, \left(\ln(m_H^2)\right)^2 \Bigg] \frac{m_Z^4}{m_W^4} \\
& + \quad z_H \Bigg[\tfrac{2483}{192} - \tfrac{231}{32}\, S_0 + \tfrac{243}{32}\, S_2 + \tfrac{433}{864}\, \pi^2 - \tfrac{161}{32}\, \ln(z_H) + \tfrac{69}{16}\, \ln(z_H) \ln(m_H^2)
\end{aligned}
$$

$$
\begin{aligned}
& -\tfrac{519}{32}\,\ln(m_H^2) + \tfrac{99}{16}\,\ln(m_H^2) S_0 + \tfrac{43}{8}\,\left(\ln(m_H^2)\right)^2 \Bigg] \\
+ \;& z_H^2 \Bigg[\tfrac{8311573}{20736} - \tfrac{153}{8}\,S_3 - \tfrac{243}{4}\,S_0 - \tfrac{87651}{128}\,S_2 - \tfrac{47893}{3456}\,\pi^2 + \tfrac{459}{16}\,\zeta(3) - \tfrac{399943}{1728}\,\ln(z_H) \\
& + \tfrac{1705}{16}\,\ln(z_H) S_0 + \tfrac{1805}{48}\,\left(\ln(z_H)\right)^2 + \tfrac{1951}{24}\,\ln(z_H)\ln(m_H^2) - \tfrac{356585}{1728}\,\ln(m_H^2) \\
& + \tfrac{1595}{16}\,\ln(m_H^2) S_0 + \tfrac{10013}{288}\,\left(\ln(m_H^2)\right)^2 \Bigg] \\
+ \;& z_H^3 \Bigg[\tfrac{10730119}{86400} - \tfrac{3451}{96}\,S_0 - \tfrac{35739}{80}\,S_2 + \tfrac{3167}{1080}\,\pi^2 - \tfrac{1173881}{4320}\,\ln(z_H) + \tfrac{517}{8}\,\ln(z_H) S_0 \\
& + \tfrac{6767}{96}\,\left(\ln(z_H)\right)^2 + \tfrac{1743}{16}\,\ln(z_H)\ln(m_H^2) - \tfrac{30383}{192}\,\ln(m_H^2) + \tfrac{297}{8}\,\ln(m_H^2) S_0 + \tfrac{177}{4}\,\left(\ln(m_H^2)\right)^2 \Bigg] \\
+ \;& z_H^4 \Bigg[-\tfrac{276774409}{1296000} + \tfrac{86473}{3200}\,S_0 - \tfrac{3807}{20}\,S_2 + \tfrac{859}{80}\,\pi^2 - \tfrac{168691}{1152}\,\ln(z_H) + \tfrac{10323}{160}\,\ln(z_H) S_0 \\
& + \tfrac{90121}{960}\,\left(\ln(z_H)\right)^2 + \tfrac{187}{2}\,\ln(z_H)\ln(m_H^2) + \tfrac{15733}{240}\,\ln(m_H^2) + \tfrac{81}{4}\,\left(\ln(m_H^2)\right)^2 \Bigg] \\
+ \;& z_H^5 \Bigg[-\tfrac{13424129921}{14112000} + \tfrac{416057}{2400}\,S_0 - \tfrac{452727}{1120}\,S_2 + \tfrac{1395311}{30240}\,\pi^2 - \tfrac{3016477}{10080}\,\ln(z_H) \\
& + \tfrac{7739}{40}\,\ln(z_H) S_0 + \tfrac{16007}{60}\,\left(\ln(z_H)\right)^2 + \tfrac{12347}{72}\,\ln(z_H)\ln(m_H^2) + \tfrac{6333137}{30240}\,\ln(m_H^2) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
A_1^Z \;=\; & z_H \Bigg[-\tfrac{2243}{288} + \tfrac{77}{8}\,S_0 + \tfrac{243}{16}\,S_2 + \tfrac{271}{432}\,\pi^2 + \tfrac{7}{6}\,\ln(z_H) - \ln(z_H)\ln(m_H^2) \\
& + \tfrac{91}{12}\,\ln(m_H^2) - \tfrac{33}{4}\,\ln(m_H^2) S_0 \Bigg] \\
+ \;& z_H^2 \Bigg[-\tfrac{4637683}{10368} + \tfrac{589}{12}\,S_3 - \tfrac{12295}{24}\,S_0 + 66\,S_1 + \tfrac{157725}{64}\,S_2 + \tfrac{99227}{1728}\,\pi^2 + \tfrac{11}{3}\,\pi^2\ln(3) \\
& - \tfrac{853}{8}\,\zeta(3) + \tfrac{225601}{864}\,\ln(z_H) - \tfrac{4279}{24}\,\ln(z_H) S_0 - \tfrac{733}{18}\,\left(\ln(z_H)\right)^2 - \tfrac{632}{9}\,\ln(z_H)\ln(m_H^2) \\
& + \tfrac{203531}{864}\,\ln(m_H^2) - \tfrac{3839}{24}\,\ln(m_H^2) S_0 - \tfrac{5119}{144}\,\left(\ln(m_H^2)\right)^2 \Bigg] \\
+ \;& z_H^3 \Bigg[-\tfrac{4480681}{43200} + \tfrac{6445}{144}\,S_0 + \tfrac{126999}{160}\,S_2 - \tfrac{10643}{1440}\,\pi^2 + \tfrac{694519}{2160}\,\ln(z_H) - \tfrac{2167}{12}\,\ln(z_H) S_0 \\
& - \tfrac{11395}{144}\,\left(\ln(z_H)\right)^2 - 122\,\ln(z_H)\ln(m_H^2) + \tfrac{5381}{24}\,\ln(m_H^2) - 99\,\ln(m_H^2) S_0 - \tfrac{129}{2}\,\left(\ln(m_H^2)\right)^2 \Bigg] \\
+ \;& z_H^4 \Bigg[\tfrac{284392801}{1296000} - \tfrac{155881}{960}\,S_0 + \tfrac{37971}{40}\,S_2 - \tfrac{661}{18}\,\pi^2 + \tfrac{6123413}{43200}\,\ln(z_H) - \tfrac{3837}{16}\,\ln(z_H) S_0 \\
& - \tfrac{11129}{96}\,\left(\ln(z_H)\right)^2 - \tfrac{19}{4}\,\ln(z_H)\ln(m_H^2) - \tfrac{763}{240}\,\ln(m_H^2) - \tfrac{27}{2}\,\left(\ln(m_H^2)\right)^2 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad z_H^5 \left[\tfrac{6259218463}{3969000} - \tfrac{1671301}{1800}\, S_0 + \tfrac{685701}{224}\, S_2 - \tfrac{5834099}{30240}\, \pi^2 - \tfrac{270677}{6048}\, \ln(z_H) - \tfrac{25747}{30}\, \ln(z_H) S_0 \right. \\
&\left. - \tfrac{204503}{360}\, (\ln(z_H))^2 + \tfrac{163}{2}\, \ln(z_H) \ln(m_H^2) + \tfrac{89983}{840}\, \ln(m_H^2) \right]
\end{aligned}
$$

$$
\begin{aligned}
A_2^Z \;=\;& z_H \left[\tfrac{3307}{576} - \tfrac{7}{32}\, S_0 + \tfrac{729}{32}\, S_2 + \tfrac{217}{288}\, \pi^2 + \tfrac{119}{96}\, \ln(z_H) - \tfrac{17}{16}\, \ln(z_H) \ln(m_H^2) - \tfrac{383}{96}\, \ln(m_H^2) \right. \\
&\left. + \tfrac{3}{16}\, \ln(m_H^2) S_0 - \tfrac{1}{8}\, \left(\ln(m_H^2)\right)^2 \right] \\
&+ \quad z_H^2 \left[- \tfrac{14576777}{20736} - \tfrac{2923}{72}\, S_3 + \tfrac{110455}{72}\, S_0 - 176\, S_1 - \tfrac{349773}{128}\, S_2 - \tfrac{111577}{1152}\, \pi^2 - \tfrac{121}{9}\, \pi^2 \ln(3) \right. \\
&\left. + \tfrac{8731}{48}\, \zeta(3) + \tfrac{25009}{576}\, \ln(z_H) - \tfrac{87}{16}\, \ln(z_H) S_0 + \tfrac{1}{3}\, (\ln(z_H))^2 + \tfrac{1195}{72}\, \ln(z_H) \ln(m_H^2) \right. \\
&\left. + \tfrac{112861}{1728}\, \ln(m_H^2) - \tfrac{731}{48}\, \ln(m_H^2) S_0 + \tfrac{239}{32}\, \left(\ln(m_H^2)\right)^2 \right] \\
&+ \quad z_H^3 \left[- \tfrac{452509}{3200} + \tfrac{6145}{144}\, S_0 + \tfrac{2913}{80}\, S_2 + \tfrac{2279}{270}\, \pi^2 - \tfrac{251521}{1440}\, \ln(z_H) + \tfrac{1943}{12}\, \ln(z_H) S_0 \right. \\
&\left. + \tfrac{5623}{288}\, (\ln(z_H))^2 + \tfrac{485}{16}\, \ln(z_H) \ln(m_H^2) - \tfrac{2951}{64}\, \ln(m_H^2) + \tfrac{735}{8}\, \ln(m_H^2) S_0 + \tfrac{37}{4}\, \left(\ln(m_H^2)\right)^2 \right] \\
&+ \quad z_H^4 \left[- \tfrac{106557211}{144000} + \tfrac{3265133}{9600}\, S_0 + \tfrac{41949}{160}\, S_2 + \tfrac{65291}{1440}\, \pi^2 - \tfrac{16156451}{86400}\, \ln(z_H) + \tfrac{51121}{160}\, \ln(z_H) S_0 \right. \\
&\left. + \tfrac{109207}{960}\, (\ln(z_H))^2 + \tfrac{139}{2}\, \ln(z_H) \ln(m_H^2) + \tfrac{6093}{80}\, \ln(m_H^2) + \tfrac{63}{4}\, \left(\ln(m_H^2)\right)^2 \right] \\
&+ \quad z_H^5 \left[- \tfrac{109475704943}{25401600} + \tfrac{15528551}{7200}\, S_0 + \tfrac{1335741}{1120}\, S_2 + \tfrac{9380087}{30240}\, \pi^2 + \tfrac{34762379}{151200}\, \ln(z_H) \right. \\
&\left. + \tfrac{188237}{120}\, \ln(z_H) S_0 + \tfrac{36539}{45}\, (\ln(z_H))^2 + \tfrac{10021}{72}\, \ln(z_H) \ln(m_H^2) + \tfrac{5529703}{30240}\, \ln(m_H^2) \right]
\end{aligned}
$$

$$
\begin{aligned}
A_3^Z \;=\;& z_H \left[\tfrac{53}{24} + \tfrac{77}{144}\, S_0 + \tfrac{243}{8}\, S_2 + \tfrac{95}{108}\, \pi^2 + \tfrac{21}{16}\, \ln(z_H) - \tfrac{9}{8}\, \ln(z_H) \ln(m_H^2) - \tfrac{15}{16}\, \ln(m_H^2) \right. \\
&\left. - \tfrac{11}{24}\, \ln(m_H^2) S_0 - \frac{1}{4}\, \left(\ln(m_H^2)\right)^2 \right] \\
&+ \quad z_H^2 \left[\tfrac{6479761}{5184} + \tfrac{365}{54}\, S_3 - \tfrac{11525}{8}\, S_0 + \tfrac{457}{3}\, S_1 + \tfrac{102293}{96}\, S_2 + \tfrac{21419}{288}\, \pi^2 + \tfrac{377}{27}\, \pi^2 \ln(3) \right. \\
&\left. - \tfrac{4889}{36}\, \zeta(3) - \tfrac{65147}{432}\, \ln(z_H) + \tfrac{9091}{108}\, \ln(z_H) S_0 + \tfrac{5}{9}\, (\ln(z_H))^2 + \tfrac{87}{4}\, \ln(z_H) \ln(m_H^2) \right. \\
&\left. - \tfrac{65201}{432}\, \ln(m_H^2) + \tfrac{2339}{27}\, \ln(m_H^2) S_0 + \tfrac{695}{72}\, \left(\ln(m_H^2)\right)^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad z_H^3 \left[\tfrac{315263}{43200} - \tfrac{160771}{2592}\, S_0 + \tfrac{92279}{480}\, S_2 - \tfrac{3983}{4320}\, \pi^2 + \tfrac{187433}{2160}\, \ln(z_H) - \tfrac{12461}{216}\, \ln(z_H) S_0 \right. \\
&\quad \left. - \tfrac{437}{72}\, (\ln(z_H))^2 - \tfrac{99}{8}\, \ln(z_H) \ln(m_H^2) + \tfrac{9839}{96}\, \ln(m_H^2) - \tfrac{149}{4}\, \ln(m_H^2) S_0 - \tfrac{29}{2}\, \left(\ln(m_H^2)\right)^2 \right] \\
&+ \quad z_H^4 \left[\tfrac{275866331}{1296000} - \tfrac{3508117}{10800}\, S_0 + \tfrac{13353}{20}\, S_2 - \tfrac{1403}{90}\, \pi^2 + \tfrac{1202389}{10800}\, \ln(z_H) - \tfrac{2932}{15}\, \ln(z_H) S_0 \right. \\
&\quad \left. - \tfrac{179}{40}\, (\ln(z_H))^2 + \tfrac{287}{4}\, \ln(z_H) \ln(m_H^2) + \tfrac{15001}{240}\, \ln(m_H^2) + 9\, \left(\ln(m_H^2)\right)^2 \right] \\
&+ \quad z_H^5 \left[\tfrac{178942516547}{63504000} - \tfrac{871139}{324}\, S_0 + \tfrac{8825401}{3360}\, S_2 - \tfrac{6422629}{30240}\, \pi^2 - \tfrac{5656397}{50400}\, \ln(z_H) \right. \\
&\quad \left. - \tfrac{162991}{108}\, \ln(z_H) S_0 - \tfrac{11701}{24}\, (\ln(z_H))^2 + \tfrac{7087}{36}\, \ln(z_H) \ln(m_H^2) + \tfrac{3910009}{15120}\, \ln(m_H^2) \right]
\end{aligned}
$$

$$
\begin{aligned}
A_4^Z \quad = \quad & z_H \left[\tfrac{7}{9} + \tfrac{371}{864}\, S_0 + \tfrac{1215}{32}\, S_2 + \tfrac{869}{864}\, \pi^2 + \tfrac{133}{96}\, \ln(z_H) - \tfrac{19}{16}\, \ln(z_H) \ln(m_H^2) + \tfrac{5}{16}\, \ln(m_H^2) \right. \\
& \quad \left. - \tfrac{53}{144}\, \ln(m_H^2) S_0 - \tfrac{3}{8}\, \left(\ln(m_H^2)\right)^2 \right] \\
& + \quad z_H^2 \left[- \tfrac{12699875}{20736} + \tfrac{793}{216}\, S_3 + \tfrac{1537817}{2592}\, S_0 - \tfrac{365}{9}\, S_1 - \tfrac{22477}{128}\, S_2 - \tfrac{1229275}{31104}\, \pi^2 - \tfrac{95}{27}\, \pi^2 \ln(3) \right. \\
& \quad + \tfrac{3767}{144}\, \zeta(3) + \tfrac{28561}{1728}\, \ln(z_H) - \tfrac{18155}{1296}\, \ln(z_H) S_0 + \tfrac{7}{9}\, (\ln(z_H))^2 + \tfrac{1937}{72}\, \ln(z_H) \ln(m_H^2) \\
& \quad \left. + \tfrac{34757}{1728}\, \ln(m_H^2) - \tfrac{17485}{1296}\, \ln(m_H^2) S_0 + \tfrac{3409}{288}\, \left(\ln(m_H^2)\right)^2 \right] \\
& + \quad z_H^3 \left[- \tfrac{3647159}{25920} + \tfrac{142763}{7776}\, S_0 + \tfrac{23303}{96}\, S_2 + \tfrac{4529}{2592}\, \pi^2 - \tfrac{1921}{144}\, \ln(z_H) + \tfrac{743}{324}\, \ln(z_H) S_0 \right. \\
& \quad \left. - \tfrac{1775}{288}\, (\ln(z_H))^2 - \tfrac{209}{16}\, \ln(z_H) \ln(m_H^2) + \tfrac{9841}{192}\, \ln(m_H^2) + \tfrac{53}{24}\, \ln(m_H^2) S_0 - \tfrac{69}{4}\, \left(\ln(m_H^2)\right)^2 \right] \\
& + \quad z_H^4 \left[- \tfrac{696762899}{1296000} + \tfrac{36167453}{259200}\, S_0 + \tfrac{832139}{960}\, S_2 + \tfrac{38173}{8640}\, \pi^2 - \tfrac{1862941}{17280}\, \ln(z_H) + \tfrac{56477}{1440}\, \ln(z_H) S_0 \right. \\
& \quad \left. + \tfrac{86023}{2880}\, (\ln(z_H))^2 + 92\, \ln(z_H) \ln(m_H^2) + \tfrac{21263}{240}\, \ln(m_H^2) + \tfrac{45}{4}\, \left(\ln(m_H^2)\right)^2 \right] \\
& + \quad z_H^5 \left[- \tfrac{430433439517}{127008000} + \tfrac{343121543}{194400}\, S_0 + \tfrac{3744143}{1120}\, S_2 + \tfrac{49949}{672}\, \pi^2 - \tfrac{54474521}{151200}\, \ln(z_H) \right. \\
& \quad \left. + \tfrac{2240561}{3240}\, \ln(z_H) S_0 + \tfrac{11753}{60}\, (\ln(z_H))^2 + \tfrac{6109}{24}\, \ln(z_H) \ln(m_H^2) + \tfrac{3370111}{10080}\, \ln(m_H^2) \right]
\end{aligned}
$$

$$
\begin{aligned}
A_5^Z \quad = \quad & z_H \left[- \tfrac{281}{720} + \tfrac{7}{24}\, S_0 + \tfrac{729}{16}\, S_2 + \tfrac{163}{144}\, \pi^2 + \tfrac{35}{24}\, \ln(z_H) - \tfrac{5}{4}\, \ln(z_H) \ln(m_H^2) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{107}{80}\,\ln(m_H^2) - \tfrac{1}{4}\,\ln(m_H^2)S_0 - \tfrac{1}{2}\,\left(\ln(m_H^2)\right)^2\Bigg] \\
+\;& z_H^2\Bigg[\tfrac{33773}{384} - \tfrac{1}{108}\,S_3 - \tfrac{2421059}{19440}\,S_0 - \tfrac{103}{54}\,S_1 + \tfrac{219449}{2880}\,S_2 + \tfrac{1585397}{233280}\,\pi^2 + \tfrac{1}{27}\,\pi^2\ln(3) \\
&-\tfrac{23}{72}\,\zeta(3) - \tfrac{20023}{1440}\,\ln(z_H) - \tfrac{19}{24}\,\ln(z_H)S_0 + \left(\ln(z_H)\right)^2 + \tfrac{577}{18}\,\ln(z_H)\ln(m_H^2) \\
&-\tfrac{13429}{1440}\,\ln(m_H^2) - \tfrac{493}{648}\,\ln(m_H^2)S_0 + \tfrac{673}{48}\,\left(\ln(m_H^2)\right)^2\Bigg] \\
+\;& z_H^3\Bigg[-\tfrac{32327203}{259200} + \tfrac{90853}{38880}\,S_0 + \tfrac{202459}{720}\,S_2 + \tfrac{17237}{6480}\,\pi^2 - \tfrac{33671}{4320}\,\ln(z_H) - \tfrac{311}{648}\,\ln(z_H)S_0 \\
&-\tfrac{901}{144}\,\left(\ln(z_H)\right)^2 - \tfrac{55}{4}\,\ln(z_H)\ln(m_H^2) + \tfrac{16597}{240}\,\ln(m_H^2) - \tfrac{7}{18}\,\ln(m_H^2)S_0 - 20\,\left(\ln(m_H^2)\right)^2\Bigg] \\
+\;& z_H^4\Bigg[-\tfrac{81023989}{216000} - \tfrac{727891}{43200}\,S_0 + \tfrac{164217}{160}\,S_2 + \tfrac{10243}{1440}\,\pi^2 - \tfrac{78513}{1600}\,\ln(z_H) - \tfrac{3853}{2160}\,\ln(z_H)S_0 \\
&+\tfrac{53587}{1440}\,\left(\ln(z_H)\right)^2 + \tfrac{449}{4}\,\ln(z_H)\ln(m_H^2) + \tfrac{8863}{80}\,\ln(m_H^2) + \tfrac{27}{2}\,\left(\ln(m_H^2)\right)^2\Bigg] \\
+\;& z_H^5\Bigg[-\tfrac{23064502157}{31752000} - \tfrac{107194699}{194400}\,S_0 + \tfrac{19739897}{5040}\,S_2 + \tfrac{361861}{15120}\,\pi^2 + \tfrac{4365419}{151200}\,\ln(z_H) \\
&-\tfrac{206327}{1620}\,\ln(z_H)S_0 + \tfrac{25421}{360}\,\left(\ln(z_H)\right)^2 + \tfrac{2810}{9}\,\ln(z_H)\ln(m_H^2) + \tfrac{1550081}{3780}\,\ln(m_H^2)\Bigg]
\end{aligned}
$$

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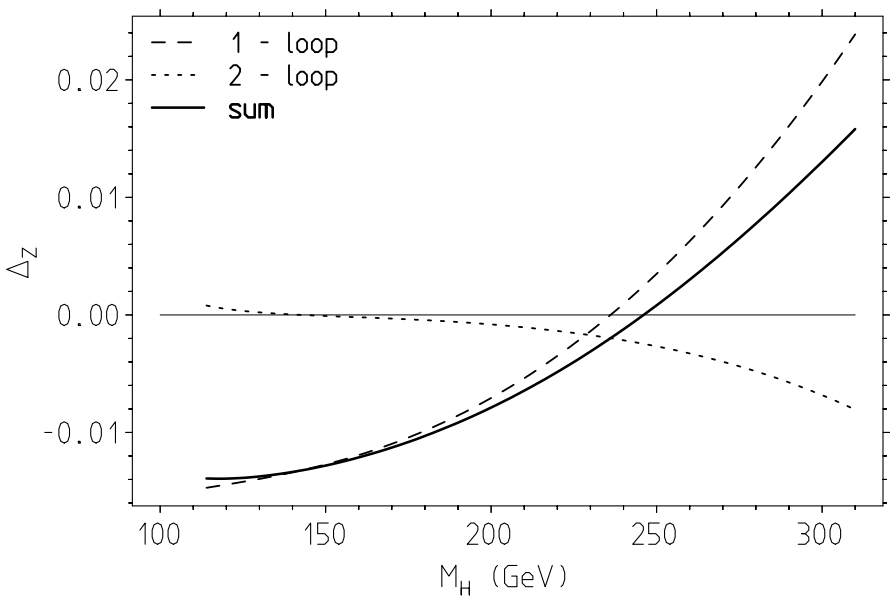

Figure 3: One- and two-loop corrections to the relation $\Delta_Z \equiv m_Z^2(M_Z)/M_Z^2 - 1$ as a function of the Higgs mass M_H for intermediate Higgs masses.

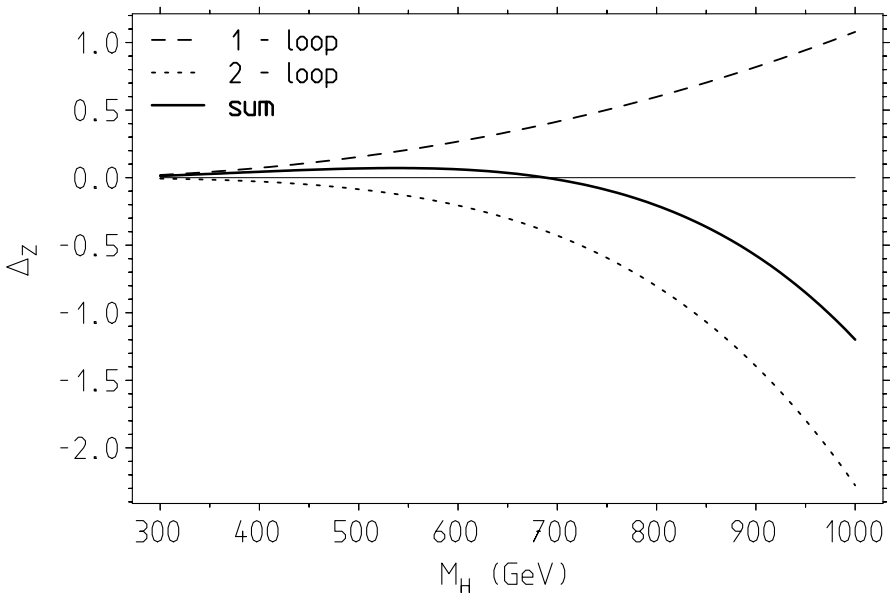

Figure 4: One- and two-loop corrections to the relation $\Delta_Z \equiv m_Z^2(M_Z)/M_Z^2 - 1$ as a function of the Higgs mass M_H for heavy Higgs masses.

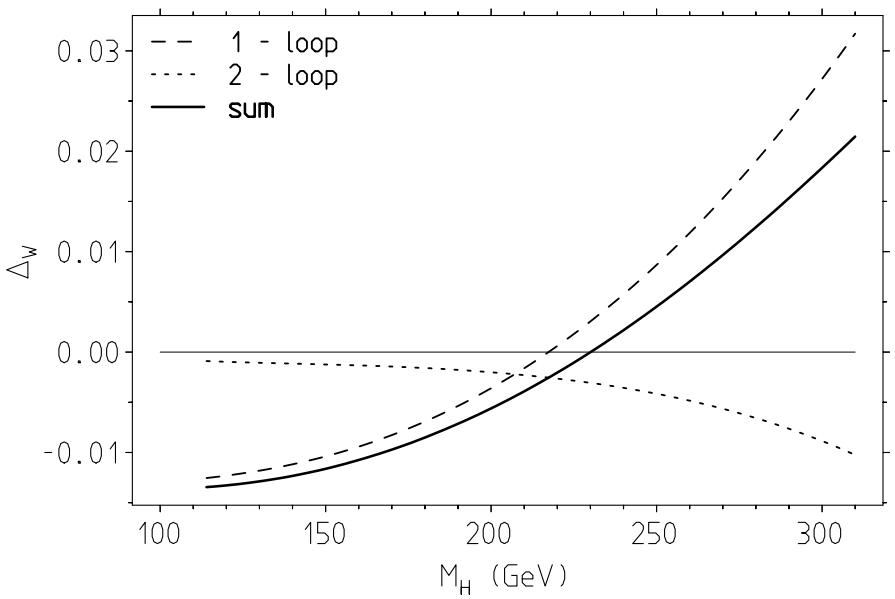

Figure 5: One- and two-loop corrections to the relation $\Delta_W \equiv m_W^2(M_W)/M_W^2 - 1$ as a function of the Higgs mass m_H for intermediate Higgs masses.

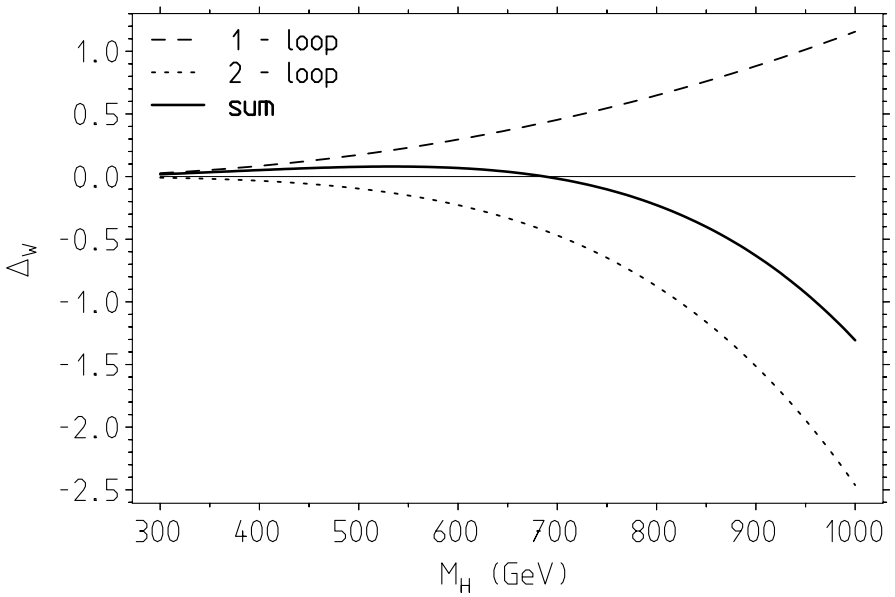

Figure 6: One- and two-loop corrections to the relation $\Delta_W \equiv m_W^2(M_W)/M_W^2 - 1$ as a function of the Higgs mass m_H for heavy Higgs masses.

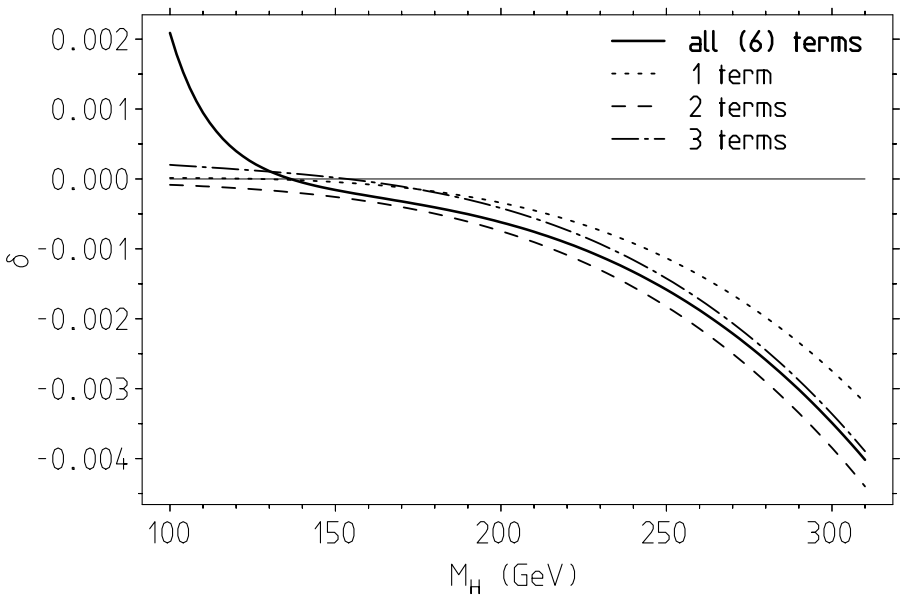

Figure 7: The dependence on the number of coefficients of the expansion (5.4) used for the evaluation of the two-loop corrections. We show $\delta \equiv -\left(\hat{\Pi}_Z^{(2)} + \hat{\Pi}_Z^{(1)}\hat{\Pi}_Z^{(1)\prime}\right)/M_Z^2$ (see 5.3) as a function of the Higgs mass. The dotted, dashed, dot-dashed and full lines show results obtained with the first one, two, three and all calculated (six) coefficients, respectively.

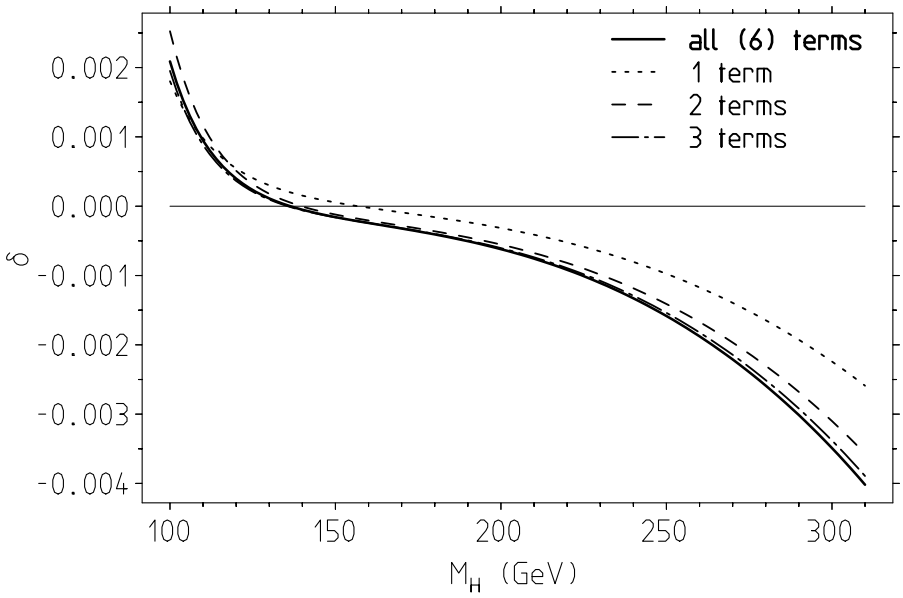

Figure 8: The dependence on the number of coefficients of the expansion with respect to $\sin^2\theta_W$ of the two-loop corrections. We show $\delta \equiv -\left(\hat{\Pi}_Z^{(2)} + \hat{\Pi}_Z^{(1)}\hat{\Pi}_Z^{(1)\prime}\right)/M_Z^2$ (see 5.3) as a function of the Higgs mass. The dotted, dashed, dot-dashed and full lines show results obtained with the first one, two, three and all calculated (six) coefficients, respectively.

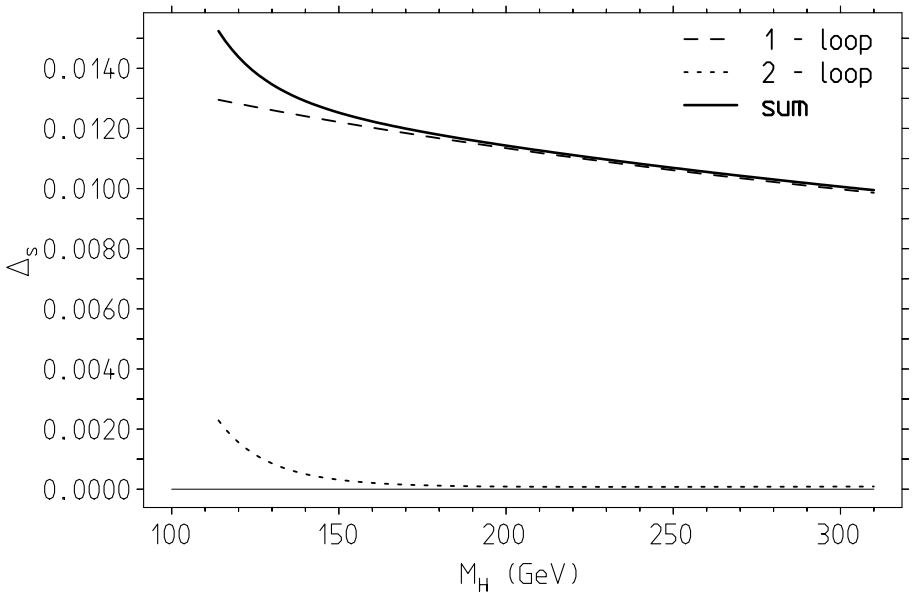

Figure 9: One- and two-loop corrections to $\Delta_s = \sin^2\theta_W^{\overline{\rm MS}}/\sin^2\theta_W^{\rm OS} - 1$ (see (5.8)) as a function of the Higgs mass m_H for intermediate Higgs masses ($\mu = M_Z$).

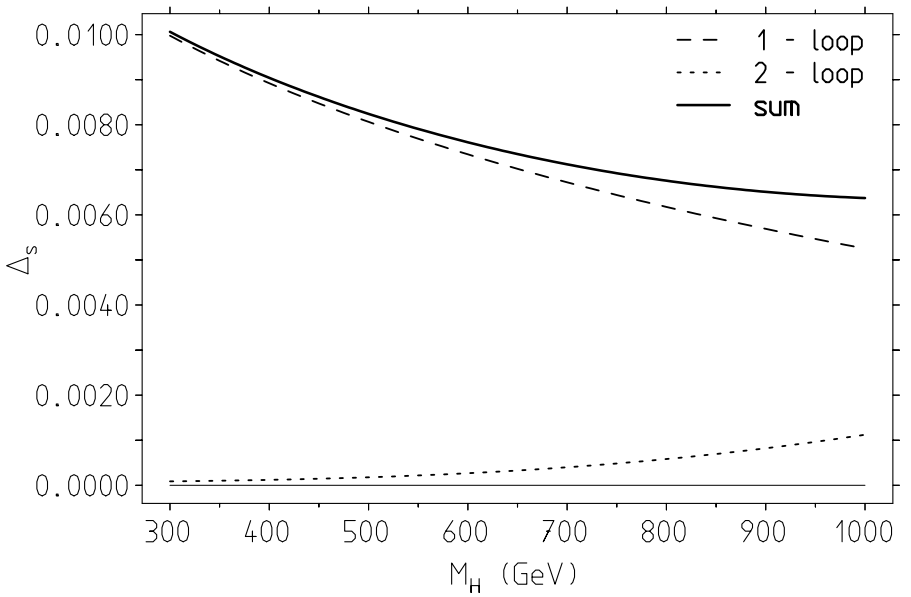

Figure 10: One- and two-loop corrections to $\Delta_s = \sin^2\theta_W^{\overline{\rm MS}}/\sin^2\theta_W^{\rm OS} - 1$ (see (5.8)) as a function of the Higgs mass m_H for heavy Higgs masses ($\mu = M_Z$).