Hadronic currents and correlators

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Thanks to T. Izubuchi for pointing out an error in $N_f \geq 3$ EC correlator code (resonance conributions taken at wrong energy scale)

We study the possibility of evaluation hadronic correlators in terms of e^+e^--annihilation data. Correlators different from the one of two electromagnetic currents $\langle\gamma\gamma\rangle$, like $\langle\gamma 3\rangle$ or $\langle 33\rangle$ with 3 referring to the neutral 3rd component of the weak isospin current, require a separation of the flavor components, which is particularly problematic for the light flavors u, d and s. Above the heavy quark threshold for charm and beauty the lighter flavors (u, d, s above $M_{J/Psi}$ and (u, d, s, c above M_Υ) can be safely separated using perturbative QCD. While hadronic contributions to the running fine structure constant α can be evaluated directly in terms of experimental data, the calculation of the weak $SU(2)$ coupling α_2 relevant in neutrino scattering or Compton scattering, requires appropriate recombination of the flavor contributions. On of the problems one encounters is that the OZI-rule is violated such that mixed quark correlators like Π_{ud} are non-negligible. In case one is interested in the running of $\alpha_2(s)$ at higher energies i.e. $\alpha_2(M_Z^2)$ it seems reasonable to assume flavor $SU(3)$ for the region below the charm mass, where only 3 flavors are active. Surprisingly, for the mixed correlator $\langle\gamma 3\rangle$, which is relevant for calculating $\alpha_2(s)$, in the $SU(3)$ symmetry limit no assumption concerning the OZI suppressed terms is needed [1, 2]. Here, we consider the calculation of Euclidean correlators, which can be calculated in lattice QCD [3, 4]. The aim is to compare lattice results with evaluations obtainable from the data. As we know, in the low energy region assuming $SU(3)$ flavor symmetry is not a good approximation. What one can use is the less severely broken $SU(2)$ flavor symmetry, to which also corrections like $\rho - \omega$ and $\gamma - \rho$ mixing are well known. Also, the isovector part can be obtained by even-odd number of pions separation. However, separation into u, d and s components is not any longer possible without assuming the OZI-rule violating effects are sufficiently suppressed one would (wrongly) expect. In the following we describe what can be done and the present the results obtained.

Electromagnetic current:

$$j^\mu_{\rm em} = \frac{2}{3}\,\bar{u}\gamma^\mu u - \frac{1}{3}\,\bar{d}\gamma^\mu d - \frac{1}{3}\,\bar{s}\gamma^\mu s + \cdots$$

Weak isovector current:

$$j^\mu_3 = \frac{1}{2}\,\bar{u}\gamma^\mu u - \frac{1}{2}\,\bar{d}\gamma^\mu d - \frac{1}{2}\,\bar{s}\gamma^\mu s + \cdots$$

Correlators: in $SU(3)$ limit

$$\begin{aligned}
\langle\gamma\gamma\rangle &\sim \frac{6}{9}\langle uu\rangle - \frac{6}{9}\langle ud\rangle = \frac{2}{3}\,(\langle uu\rangle - \langle ud\rangle)\\
\langle\gamma 3\rangle &\sim \frac{4}{6}\langle uu\rangle - \frac{4}{6}\langle ud\rangle = \frac{2}{3}\,(\langle uu\rangle - \langle ud\rangle)\\
\langle 33\rangle &\sim \frac{3}{4}\langle uu\rangle - \frac{2}{4}\langle ud\rangle = \frac{3}{4}\,(\langle uu\rangle - \langle ud\rangle) + \frac{1}{4}\,\langle ud\rangle
\end{aligned}$$

In this case

$$\langle\gamma 3\rangle_{uds} = \langle\gamma\gamma\rangle_{uds}\ ;\ \ \langle 33\rangle_{uds} \simeq \frac{9}{8}\,\langle\gamma\gamma\rangle_{uds} + O(\frac{\langle ud\rangle}{\langle uu\rangle})\ .$$

Correlators: in $SU(2)$ limit, neglecting OZI violating contributions

$$\begin{aligned}
\langle\gamma\gamma\rangle &\sim \frac{5}{9}\langle uu\rangle - \frac{4}{9}\langle ud\rangle + \frac{1}{9}\langle ss\rangle - \frac{2}{9}\langle us\rangle \sim \frac{5}{9}\langle uu\rangle + \frac{1}{9}\langle ss\rangle + O(\frac{\langle ud\rangle}{\langle uu\rangle},\frac{\langle us\rangle}{\langle ss\rangle})\\
\langle\gamma 3\rangle &\sim \frac{1}{2}\langle uu\rangle - \frac{1}{2}\langle ud\rangle + \frac{1}{6}\langle ss\rangle - \frac{1}{6}\langle us\rangle \sim \frac{1}{2}\langle uu\rangle + \frac{1}{6}\langle ss\rangle + O(\frac{\langle ud\rangle}{\langle uu\rangle},\frac{\langle us\rangle}{\langle ss\rangle})\\
\langle 33\rangle &\sim \frac{1}{2}\langle uu\rangle - \frac{1}{2}\langle ud\rangle + \frac{1}{4}\langle ss\rangle \sim \frac{1}{2}\langle uu\rangle + \frac{1}{4}\langle ss\rangle + O(\frac{\langle ud\rangle}{\langle uu\rangle})
\end{aligned}$$

As indicated, here there are no simple relations between (in the symmetry limit) known combinations. The only way is to assume that the off-diagonal elements $|\langle ud\rangle| \ll |\langle uu\rangle| = |\langle dd\rangle|$ as well as $|\langle us\rangle| = |\langle ds\rangle| \ll |\langle ss\rangle|$. This yields the re-weightings:

$$\langle uu\rangle \ \simeq\ \frac{9}{5}\,\langle\gamma\gamma\rangle_{u,d}\ ;\ \ \langle ss\rangle \simeq 9\langle\gamma\gamma\rangle_s$$

$$\begin{aligned}\langle\gamma 3\rangle_{ud} &\simeq \frac{9}{10}\langle\gamma\gamma\rangle_{ud}\ ;\ \ \langle\gamma 3\rangle_s \simeq \frac{9}{6}\langle\gamma\gamma\rangle_s \\ \langle 33\rangle_{ud} &\simeq \frac{9}{10}\langle\gamma\gamma\rangle_{ud}\ ;\ \ \langle 33\rangle_s \simeq \frac{9}{4}\langle\gamma\gamma\rangle_s\end{aligned}$$

This $SU(2)$ version assuming OZI violating effects to be negligible corresponds to a **perturbative reweighting**! This has been implemented in the 2012 version of the `alphaQED` package. Later, lattice evaluations [6] revealed this to be wrong! while the "old" [1] agreed much better, see [7]. Nevertheless, the $SU(3)$ flavor symmetry argument also looks the be rather crude when looking at correlator in the low energy regime. In place of the untenable assumption that OZI violating terms are small, we may argue by isovector ρ meson dominance (VMD isovector) which suggests an isospin factor 1/2 in place of 9/20 suggested by perturbative reweighting.

For the resonance contributions in the spirit of the large–N_c vector meson dominance picture we proceed as follows: in terms of single quark currents j^q, where $j^q_\mu = \bar{q}\gamma_\mu q$, we may define currents associated with the resonances $j^\rho = \frac{1}{2}\left(j^u - j^d\right)$, $j^\omega = \frac{1}{6}\left(j^u + j^d\right)$ and $j^\phi = -\frac{1}{3}\,j^s$, which corresponds to the ideally mixed $J^{PC} = 1^{--}$ states ρ_0, ω_0 and ϕ_0, we may write

$$\begin{aligned} j^\gamma &= j^\rho + j^\omega + j^\phi + j^{J/\psi} + j^\Upsilon \\ j^3 &= \frac{1}{2}\,j^\rho + \frac{3}{4}\,j^\phi + \frac{3}{8}\,j^{J/\psi} + \frac{3}{4}\,j^\Upsilon \end{aligned}$$

Denoting the diagonal amplitudes by $\Pi^{(V)}$ we obtain

$$\begin{aligned} \hat{\Pi}^{\gamma\gamma} &\simeq \Pi^{(\rho)} + \Pi^{(\omega)} + \Pi^{(\phi)} + \Pi^{(J/\psi)} + \Pi^{(\Upsilon)} \\ \hat{\Pi}^{3\gamma} &\simeq \frac{1}{2}\,\Pi^{(\rho)} + \frac{3}{4}\,\Pi^{(\phi)} + \frac{3}{8}\,\Pi^{(J/\psi)} + \frac{3}{4}\,\Pi^{(\Upsilon)} \\ \hat{\Pi}^{33} &\simeq \frac{1}{4}\,\Pi^{(\rho)} + \frac{9}{16}\,\Pi^{(\phi)} + \frac{9}{64}\,\Pi^{(J/\psi)} + \frac{9}{16}\,\Pi^{(\Upsilon)} \end{aligned}$$

provided mixing is small. For the combination $3\,\hat{\Pi}^{33} - \hat{\Pi}^{3\gamma} = -\frac{1}{2}\Pi^{\rho\omega} + \frac{3}{8}\Pi^{(\phi)} - \frac{3}{32}\Pi(J/\psi) + \cdots$ the (ud) contribution is solely due to $\rho - \omega$ mixing, as an example. In any case, we apply the resonance reweighting for corresponding contributions.

Besides the flavor $SU(3)$ inspired weighting

$$\Pi^{3\gamma}_{uds} = \frac{1}{2}\,\Pi^{\gamma\gamma}_{uds}$$

the ρ dominance (exact in the isospin limit) assignment reads

$$\Pi^{3\gamma}_{ud} = \frac{1}{2}\,\Pi^{\gamma\gamma}_{ud}\ ;\ \ \Pi^{3\gamma}_s = \frac{3}{4}\,\Pi^{\gamma\gamma}_s$$

Table 1: Variants of flavor recombination of $\langle 3\gamma\rangle$ in terms of $\langle\gamma\gamma\rangle$. LQCD tests strongly disfavor "$SU(2)$" [6, 7], i.e. perturbative reweighting and/or neglecting OZI suppressed terms is obsolete.

variant		weights			"model"	
$SU(3)$	=	$\frac{1}{2}\,[ud]^{I=1}$	+	$\frac{1}{2}[s]$	assuming $SU(3)$ symmetry	
"$SU(2)$"	=	$\frac{9}{20}\,[ud]^{I=1}$	+	$\frac{3}{4}[s]$	perturbative reweighting	✘
VMD [iso]	=	$\frac{1}{2}\,[ud]^{I=1}$	+	$\frac{3}{4}[s]$	VDM isovector	✔

which agrees well with lattice data.

This has been implemented in the 2016/17 versions of the `alphaQED` package. The changes affect the $\alpha_2(s)$ routines `alpha2SMr17.f, alpha2SMc17.f` and the $\sin^2\theta_{\rm eff}$ routine `ACWMsin2theta.f`. The different trials are compared in Tab. 1 and Fig. 1.

Flavor separation by hand: (in particular for extracting the isovector part)

we skip all final states involving photons like: $\pi^0\gamma$, $\eta\gamma$ channels, including η' has not been reported so far

as ud, $I = 0$ we include states with odd number of pions

as ud, $I = 1$ we include states with even number of pions

as $\bar{s}s$ we count all states with Kaons

States ηX with X some other hadrons are collected separately, and then split into $q = u, d$ and s components by appropriate mixing.

Flavor separation is possible only in regions where exclusive channel cross sections are available. We perform this in the region 0.61 GeV to 2.1 GeV. Above this energy only inclusive $R(s)$ measurements are available, and a pQCD reweighting is applied.

Results for Euclidean correlators:

$$I(t) = t^3 \int_a^\infty \mathrm{d}\,\omega\,\omega^2\,\rho(\omega^2)\,\mathrm{e}^{-\omega t}\ ;\ \ \rho(s) = \frac{R(s)}{12\pi^2}$$

Normalization (as in [1] i.e. as weak currents in SM):

$$D_{\gamma\gamma}(t) = \langle\gamma\gamma\rangle$$

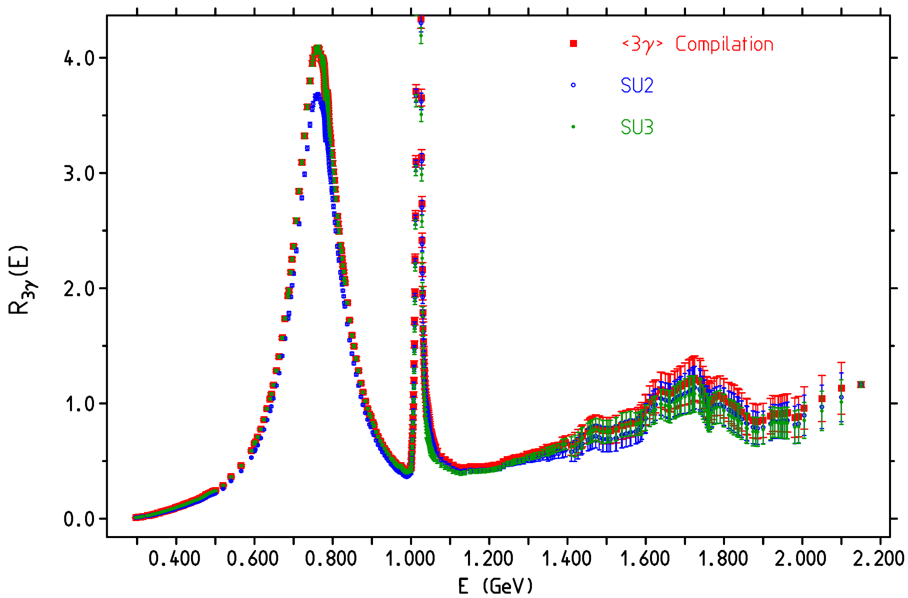

Figure 1: $R_{3\gamma}$ compilations for the VMD [iso] inspired flavor separation which fits the lattice data best versus the $SU(2)$ [i.e. perturbative] (which had been implemented in `alphQED12`) and badly fails to match lattice data and the old $SU(3)$ (which had been implemented in `alphQED09`) and also is in fairly good agreement with lattice results [7]. What is compared to lattice QCD is the corresponding dispersion integrals either for the Euclidean time correlator or the space-like hadronic VP function.

$$D_{\gamma 3}(t) = \frac{1}{2}\,\langle \gamma 3 \rangle$$
$$D_{33}(t) \leftrightarrow \frac{1}{4}\,\langle 33 \rangle$$

The Euclidean time variable t is in units of 1 fermi fm = 0.1973269631 in GeV^{-1}, i.e. $t = \mathrm{fm}/E[\mathrm{GeV}]$.

For $R(s) = 1$ the integral is given by

$$\begin{aligned} I(t,a,L)[R=1] &= \frac{1}{12\pi^2}\, t^3 \int_a^L \mathrm{d}\,\omega\,\omega^2\, \mathrm{e}^{-\omega t} \\ &= \frac{1}{12\pi^2} \left\{ \left(a^2 t^2 + 2at + 2\right) \mathrm{e}^{-at} - \left(L^2 t^2 + 2Lt + 2\right) \mathrm{e}^{-Lt} \right\} \end{aligned}$$

$$\begin{aligned} J(t,a,L)[R=1] &= \frac{1}{12\pi^2}\, t^4 \int_a^L \mathrm{d}\,\omega\,\omega^3\, \mathrm{e}^{-\omega t} \\ &= \frac{1}{12\pi^2} \left\{ \left(a^3\, t^3 + 3\, a^2\, t^2 + 6\, a\, t + 6\right) \mathrm{e}^{-a\,t} \right. \\ &\qquad \left. - \left(t^3\, L^3 + 3\, t^2\, L^2 + 6\, t\, L + 6\right) \mathrm{e}^{-t\,L} \right\} \end{aligned}$$

Results obtainable with the program `intRdatx.f`[1] with input from `intRdatx.inp` and driven by `intRdatx.sh` are shown in Figs. 2 and 3 for the $\langle \gamma\gamma \rangle$ correlator as a function of the Euclidean imaginary time t.

[1] I thank Taku Izubuchi for communicating a bug in that program, which concerned the inclusion of the narrow resonance contributions in the Euclidean Correlator for $N_f > 2$ [8].

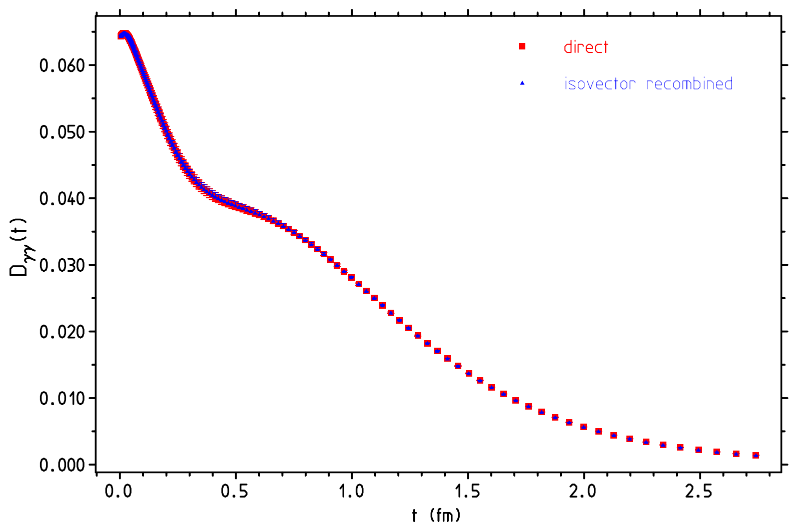

Figure 2: $D_{\gamma\gamma}(E)$ including 5 flavors ($N_F = 5$): a) direct evaluation based on e^+e^--annihilation data compared with b) flavor separation in the isovector limit recombined. This is a test that channels dropped because they violate isospin symmetry are close to the full answer.

For $N_f = 5$ results for the different correlators (1) are displayed in Fig. 4, where $D_{3\gamma}(t)$ is rescaled by 2 and $D_{33}(t)$ is rescaled by a factor 4.

In Fig. 6 we compare phenomenological evaluations with lattice data. The results show substantial differences in the intermediate time range. Above 0.6 fm the result is independent of the number of flavors, while at short times the flavor dependence is strong (of course). Only above about 2.5 fm the result does not depend substantially on the two flavor separation approaches. The $SU(3)$ inspired approach is definitely too crude to get reasonably good approximation in the intermediate distance regime. How good the improved flavor separation works is a matter to be tested by comparison with the lattice results. We have not yet taken out $\rho - \gamma$ mixing effects [5], which are included in the data but not automatically in the lattice simulation.

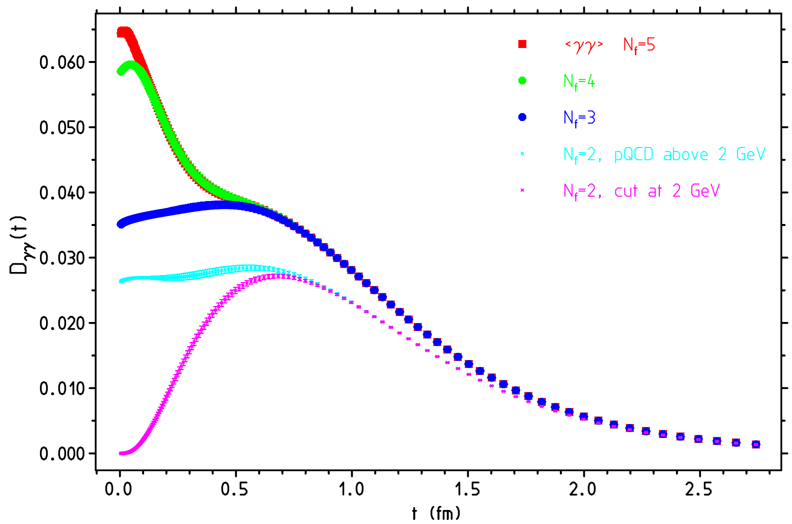

Figure 3: Euclidean correlator $\langle\gamma\gamma\rangle$ for $N_f = 3, 4$ and 5 flavors and for $N_f = 2$ isovector

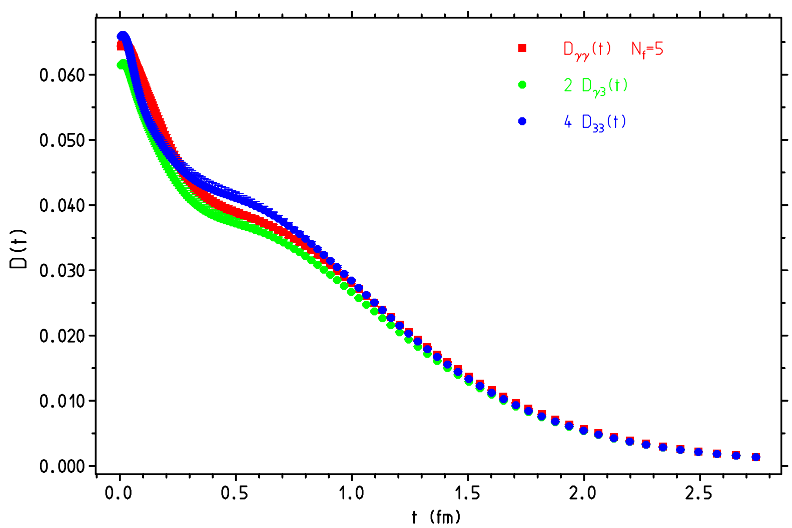

Figure 4: Comparison of the different Euclidean correlators: $D_{\gamma\gamma}$, $2\,D_{3\gamma}$ and $4\,D_{33}$.

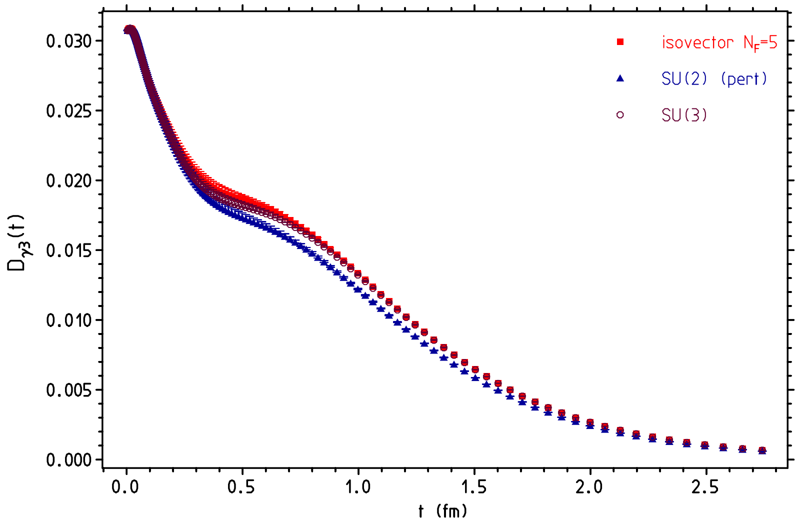

Figure 5: $D_{\gamma 3}(t)$ versions of flavor separation a) VMD isovector, b) in the $SU(2)$ and neglecting OZI suppressed terms = perturbative reweighting, with c) flavor separation in the $SU(3)$ limit including OZI suppressed contributions. Version a) fits best to lattice data, c) shows also reasonable agreement, while b) is significantly off, i.e. perturbative reweighting and/or neglecting OZI suppressed effects is inadequate.

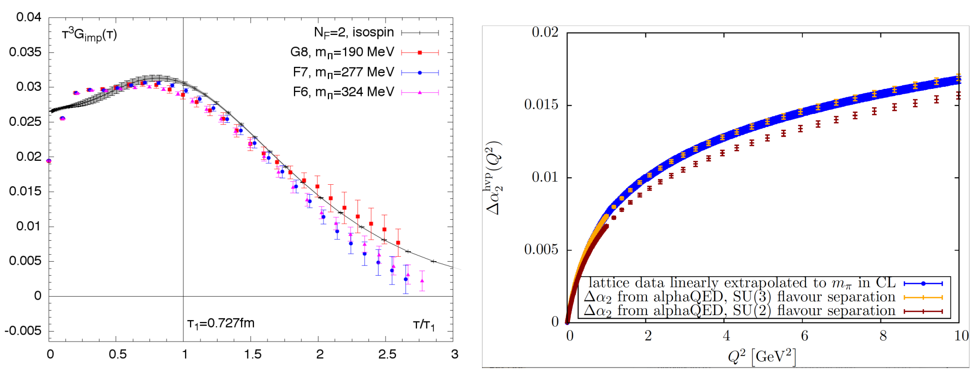

Figure 6: Testing flavor separation H. Meyer et al. [l], arXiv:1312.0035, K. Jansen et al. arXiv:1505.03283[r]

1 The $N_f = 2$ component in the isospin symmetry limit (I=1)

To be more precise what we compare in the left Fig. 6. Data are separated in arrays `RN2DAT=1/2 R3GDAT` below 3π and "strangeness threshold" i.e. where no K's etc.

We separate out all exclusive channels with an even number of pions, (no K's, no ω, no ϕ). In the $\langle 3\gamma \rangle$ normalization the comparison between $R3G$ ($N_f = 3$) and $RN2$ ($N_f = 2$) are shown in Fig. 7

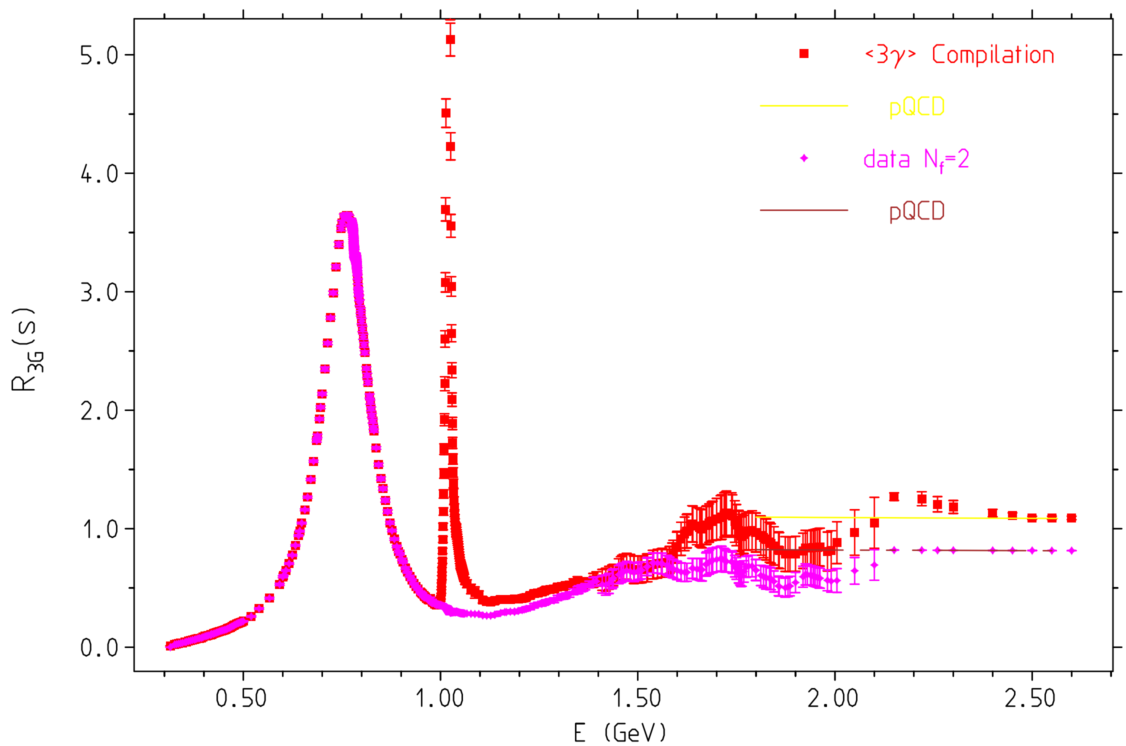

Figure 7: $R(s)$: the $N_F = 2$ isovector part ($I = 1$) vs the full $N_f = 3$ $\langle 3\gamma \rangle$ recombination.

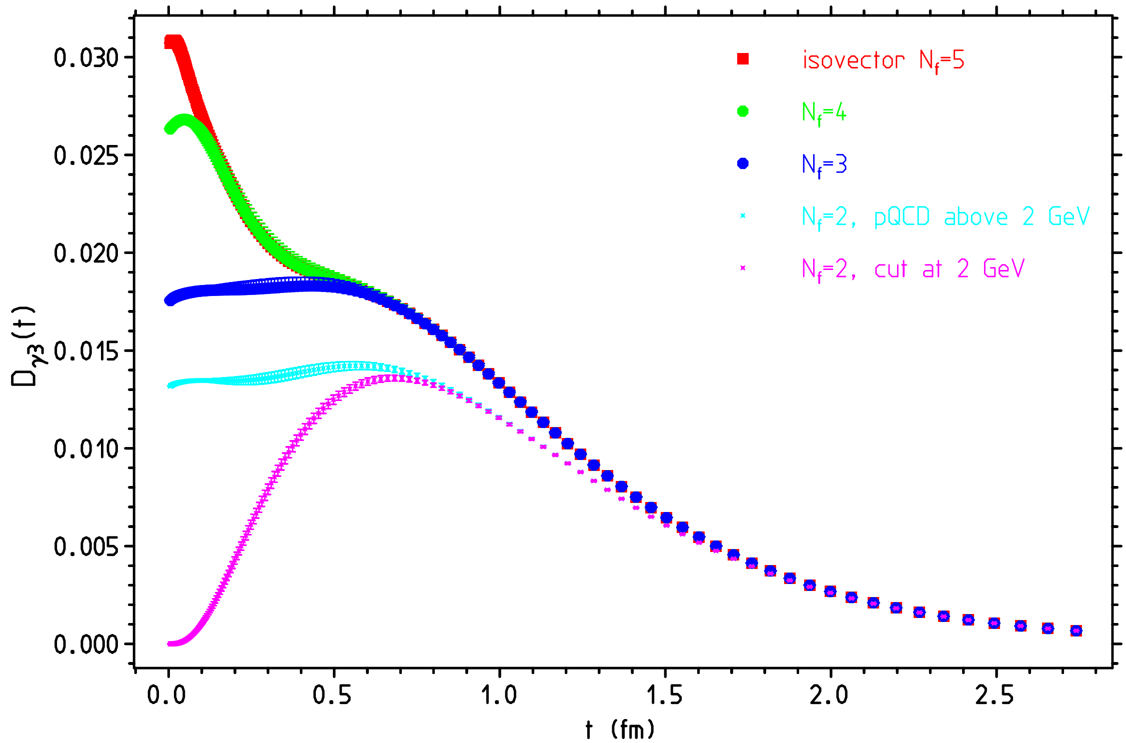

Figure 8: $\langle 3\gamma \rangle$ time correlator for $N_f = 2$ vs. $N_f = 3, 4, 5$ flavor versions. In the (u, d) sector $\langle 3\gamma \rangle$ is $1/2\, \langle \gamma\gamma \rangle$.

In Fig. 8 we compare the $\langle 3\gamma \rangle(t)$ correlator for different flavors. The large t, equivalently, the low Q^2 behavior is universal while the short distance (high energy) behavior is strongly flavor dependent.

Data sets of Euclidean time correlators as a linear [lin] or logarithmic [log] function of Euclidean time are listed in Tab. 2

Table 2: List of plot data sets shown in Figures.

Plot-data	correlator	Fig.
`RS3GISO.dat`	reweighted $R(s)$ data – VMD isovector	Fig. 1
`RS3GSU2.dat`	ibid. – $SU(2)$ OZI neglected	Fig. 1
`RS3GSU3.dat`	ibid. – $SU(3)$ incl. OZI	Fig. 1
`ECgg5log.dap`		Fig. 2
`ECgg5log.dap_rec`		Fig. 2
`ECgg4log.dap`		Fig. 3
`ECgg3log.dap`		Fig. 3
`ECgg5log.dap`	$\langle\gamma\gamma\rangle$	Fig. 4
`EC3g5log.dap`	$\langle 3\gamma\rangle$	Fig. 4
`EC335log.dap`	$\langle 33\rangle$	Fig. 4
`EC3g3log.dap`	$\langle 3\gamma\rangle$ $N_f = 3$ [log/lin]	Fig. 8
`EC3g4log.dap`	$\langle 3\gamma\rangle$ $N_f = 4$ [log/lin]	Fig. 8
`EC3g5log.dap`	$\langle 3\gamma\rangle$ $N_f = 5$ [log/lin]	Fig. 8
`EC3g2log.dap_cut`	$\langle 3\gamma\rangle$ $N_f = 2$ cut at 2 GeV [log/lin]	Fig. 8
`EC3g2log.dap_inf`	$\langle 3\gamma\rangle$ $N_f = 2$ pQCD above 2 GeV [log/lin]	Fig. 8

References

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[7] F. Burger, K. Jansen, M. Petschlies, G. Pientka, JHEP **1511** (2015) 215

[8] Taku Izubuchi, private communication.