# ③ Vacuum Polarization in High Energy Physics: $\alpha(M_Z)$ and $\alpha$ at ILC scale

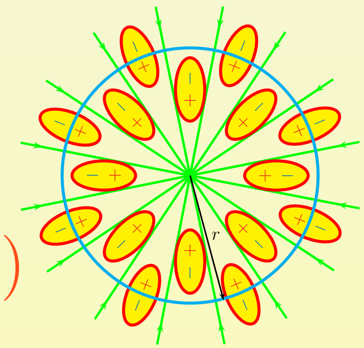
1. Introduction
2. $\alpha(M_Z)$ in precision physics (precision physics limitations)
3. Evaluation of $\alpha(M_Z)$
4. $\Delta\alpha^{\mathrm{had}}$ via the Adler function
5. The running electric charge at high energies

# 1. Introduction

Non-perturbative hadronic effects in electroweak precision observables, main effect via

**effective fine-structure "constant" $\alpha(E)$**

**(charge screening by vacuum polarization)**



**Of particular interest:**

$$\alpha(M_Z) \text{ and } a_\mu \equiv (g-2)_\mu/2 \Leftrightarrow \alpha(m_\mu)$$

- **electroweak effects (leptons etc.) calculable in perturbation theory**
- **strong interaction effects (hadrons/quarks etc.) perturbation theory fails**

$\Longrightarrow$ **Dispersion integrals over $e^+e^-$–data**

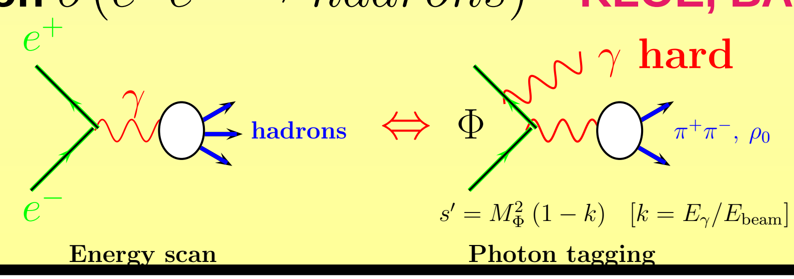
**encoded in**

$$R_\gamma(s) \equiv \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

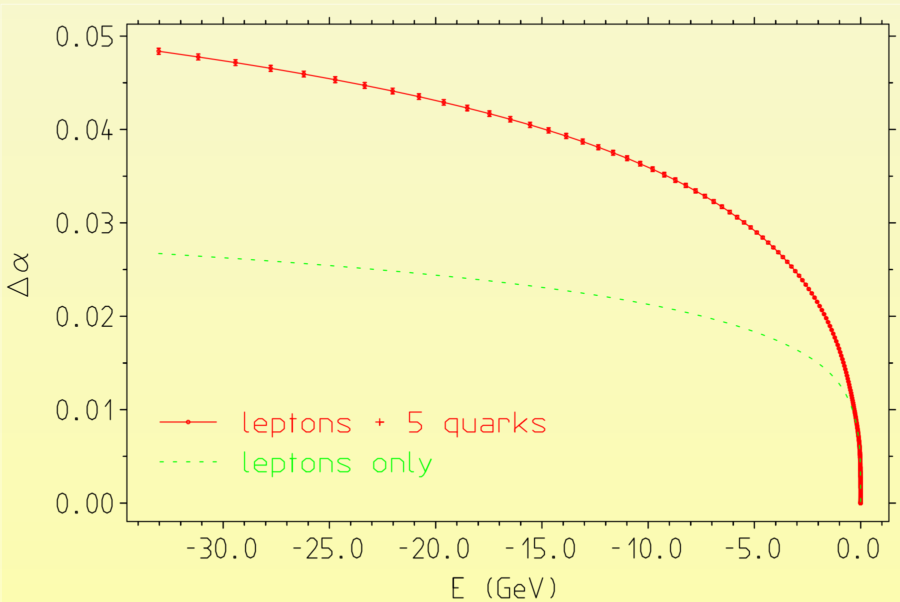
**Errors of data $\Longrightarrow$ theoretical uncertainties !!!**

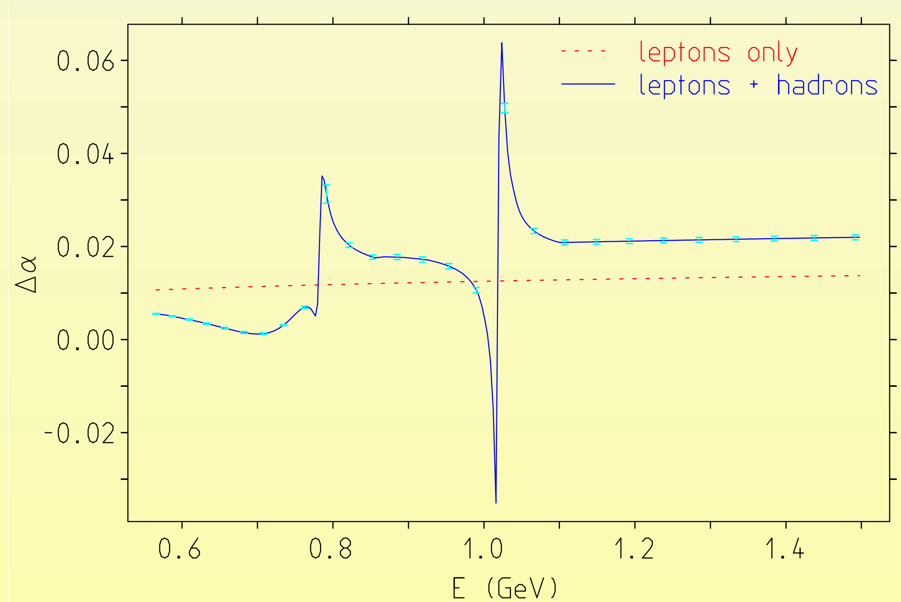
**The art of getting precise results from non-precision measurements !**

**New challenge for precision experiments on $\sigma(e^+e^- \to hadrons)$ KLOE, BABAR, .... $\sigma_{\mathrm{hadronic}}$ via radiative return:**



- **Need to know running of $\alpha_{\mathrm{QED}}$ very precisely.**
  **Large corrections, steeply increasing at low $E$**

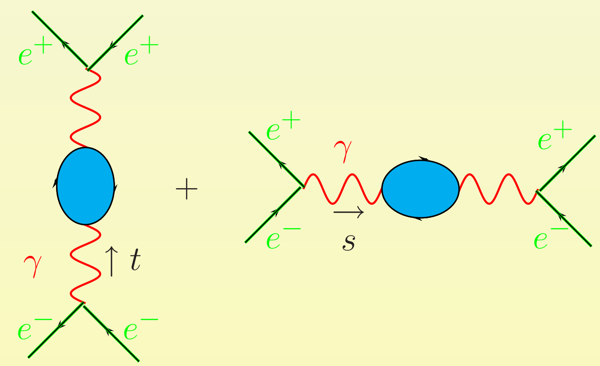



**The running of $\alpha$. The "negative" $E$ axis is chosen to indicate space-like momentum transfer. T vertical bars at selected points indicate the uncertainty. In the time-like region the resonances l to pronounced variations of the effective charge (shown in the $\rho-\omega$ and $\phi$ region).**

**Questions:** why not measure $\alpha_{\rm eff}(E)$ directly, like QCD running coupling $\alpha_s(s)$?

**Problem:** any measurement requires normalizing process like Bhabha,




depends itself on $\alpha_{\rm eff}(t)$ and $\alpha_{\rm eff}(s)$, always measure something like

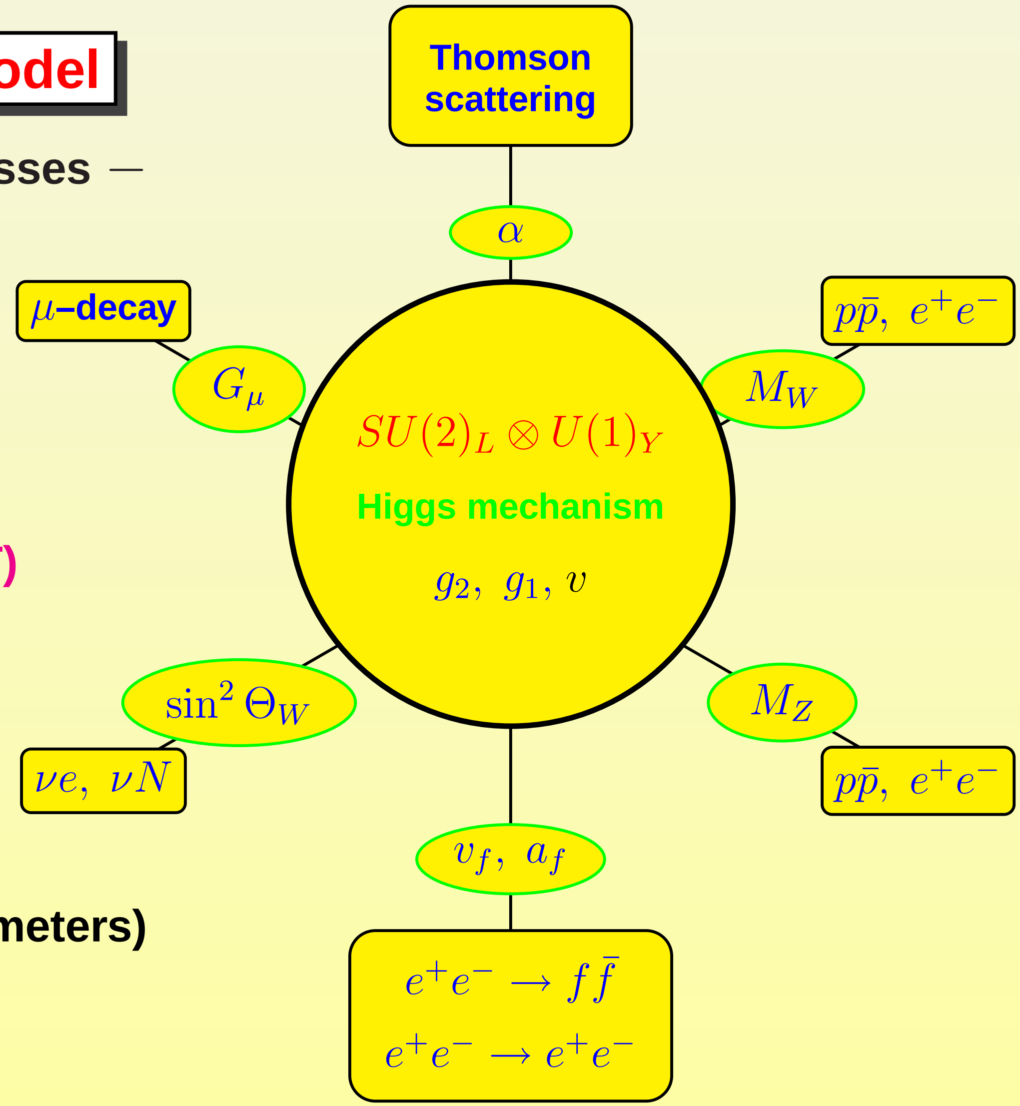
$$r(E) \propto \left(\alpha_{\rm eff}(s)/\alpha_{\rm eff}(t)\right)^2 \quad , \quad t = -\frac{1}{2}\left(s-4m_e^2\right)(1-\cos\theta)$$

where large part of the effect drops out, especially the strongly raising low energy piece, which includes substantial non-perturbative effects.

**Higher energies:** for all processes which are not dominated by a single one photon exchange, $\alpha_{\rm eff}(E)$ enters in complicated way in observables and cannot by measured in any direct way. Extraction gets model-dependent e.g. via $\sin^2\Theta_{\rm eff}$ from LEP $\Rightarrow$ depends on unknown Higgs ma ! (see below).

## The Parameters of the Standard Model

— in four fermion and vector boson processes —




**unlike in QED and QCD in SM (SBGT)**

**parameter interdependence**

☞

**only 3 independent quantities**

**(besides fermion masses and mixing parameters)**

$\alpha$, $G_\mu$, $M_Z$

$\Downarrow$

**parameter relationships between very precisely measurable quantities**

**precision tests, possible sign of new physics**

## 2. $\alpha(M_Z)$ in precision physics (precision physics limitations)

**Uncertainties of hadronic contributions to effective $\alpha$ are a problem for electroweak precision physics:**

$\alpha\ ,\ G_\mu, M_Z$ **most precise input parameters**

$\Updownarrow$

50% non-perturbative relationship

**precision predictions** $\sin^2\Theta_f, v_f, a_f, M_W, \Gamma_Z, \Gamma_W, \cdots$

$\alpha(M_Z), G_\mu, M_Z$ **best effective input parameters for VB physics (Z,W) etc.**

$$\frac{\delta\alpha}{\alpha} \sim 3.6 \times 10^{-9}$$
$$\frac{\delta G_\mu}{G_\mu} \sim 8.6 \times 10^{-6}$$
$$\frac{\delta M_Z}{M_Z} \sim 2.4 \times 10^{-5}$$
$$\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} \sim 1.6 \div 6.8 \times 10^{-4} \quad (\text{present : lost } 10^5 \text{ in precision!})$$
$$\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} \sim 5.3 \times 10^{-5} \quad (\texttt{ILC requirement})$$

**LEP/SLD:** $\sin^2\Theta_{\rm eff} = (1-g_{Vl}/g_{Al})/4 = 0.23148 \pm \boxed{0.00017}$

$$\delta\Delta\alpha(M_Z) = 0.00036 \quad \Rightarrow \quad \delta\sin^2\Theta_{\rm eff} = \boxed{0.00013}$$

**affects Higgs mass bounds, precision tests and new physics searches!!!**

**For pQCD contributions very crucial: precise QCD parameters $\alpha_s,\ m_c,\ m_b,\ m_t \Rightarrow$ Lattice-QCD**

**Indirect**

**Higgs boson mass "measurement"**

$m_H = 87^{+35}_{-26}$ **GeV**

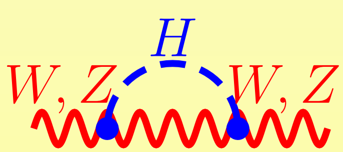
**CDF/D0 exclude 160-170 GeV 95% C.L.**

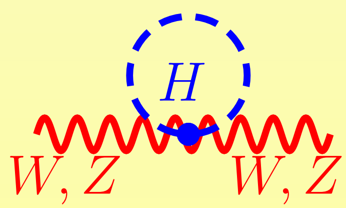
**Direct lower bound:**

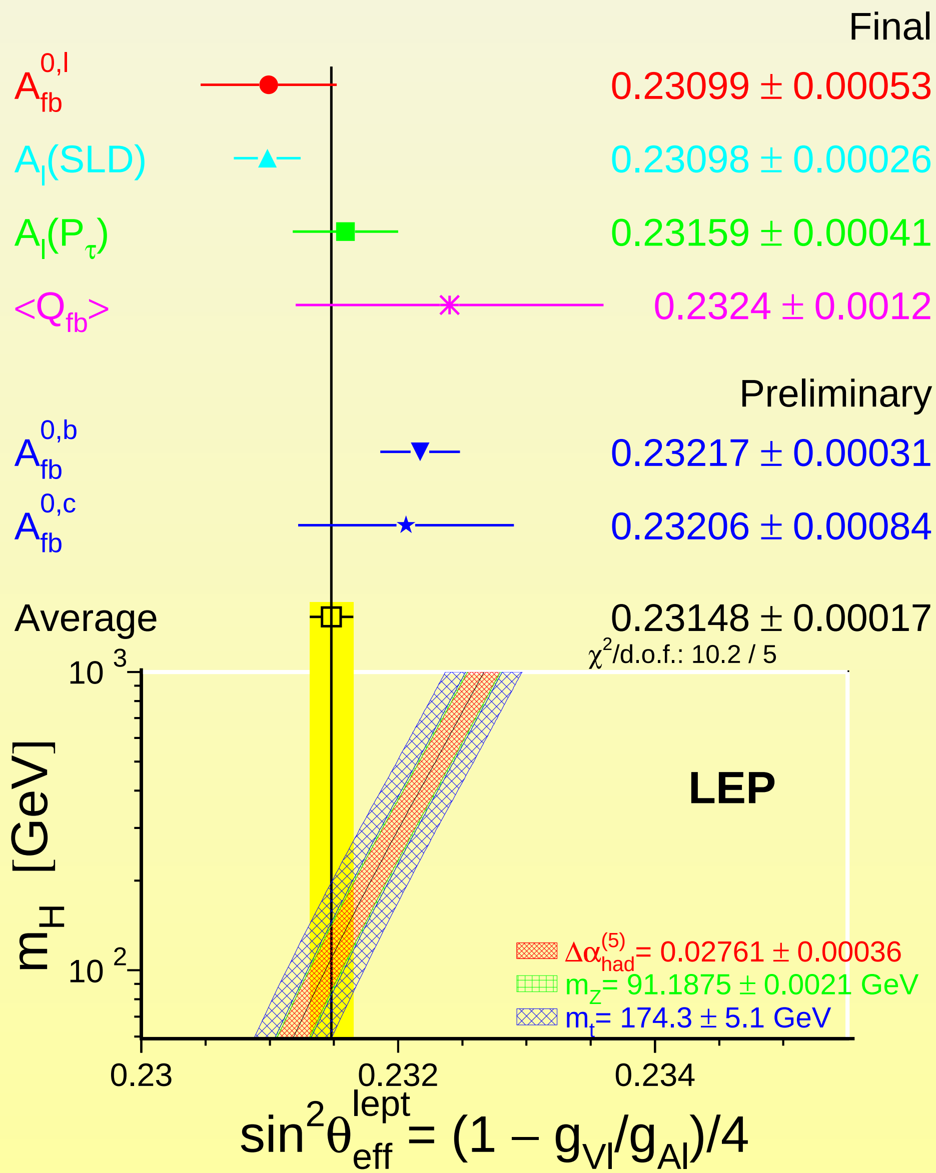
$\boxed{m_H > 114}$ **GeV at 95% CL**

**Indirect upper bound:**

$\boxed{m_H < 186}$ **GeV at 95% CL**




**What is the point once $m_H$ has been measured by the LHC? $\Rightarrow$ $\sin^2\Theta_{\rm eff}$ turns into an excellent monitor for new physics!**

**Input parameter for ILC physics:**

$$
\begin{array}{cccccccccc}
\frac{\delta\alpha}{\alpha} & \sim & 3.6 & \times & 10^{-9} & \qquad & \frac{\delta\alpha(M_Z)}{\alpha(M_Z)} & \sim & 1.6 \div 6.8 & \times & 10^{-4} \\
\frac{\delta G_\mu}{G_\mu} & \sim & 8.6 & \times & 10^{-6} & \qquad & \frac{\delta M_Z}{M_Z} & \sim & 2.4 & \times & 10^{-5}
\end{array}
$$

**accuracy in $\delta\alpha(M_Z)$ roughly one order of magnitude worse than $M_Z$ !**

$$
\sin^2\Theta_i\ \cos^2\Theta_i\ = \frac{\pi\alpha}{\sqrt{2}\,G_\mu\,M_Z^2}\,\frac{1}{1-\Delta r_i}
$$

**where**

$$
\Delta r_i \;=\; \Delta r_i(\alpha,\,G_\mu,\,M_Z,m_H,\,m_{f\neq t},\,m_t)
$$

**quantum corrections from gauge boson self-energies, vertex– and box–corrections.**

**Propagation of uncertainty:** $\delta\Delta\alpha \Rightarrow \delta M_W$**,** $\delta\sin^2\Theta_f$**:**

$$
\begin{aligned}
\frac{\delta M_W}{M_W} &\sim \frac{1}{2}\,\frac{\sin^2\Theta_W}{\cos^2\Theta_W-\sin^2\Theta_W}\,\delta\Delta\alpha \sim 0.23\,\delta\Delta\alpha \\
\frac{\delta\sin^2\Theta_f}{\sin^2\Theta_f} &\sim \frac{\cos^2\Theta_f}{\cos^2\Theta_f-\sin^2\Theta_f}\,\delta\Delta\alpha \sim 1.54\,\delta\Delta\alpha
\end{aligned}
$$

**e.g., obscure in particular the indirect bounds on the Higgs mass obtained from electroweak precision measureme**

**Precision predictions:**

$$
\begin{array}{lrcl}
M_W: & \sin^2\Theta_W & = & 1-\frac{M_W^2}{M_Z^2} \\
g_2: & \sin^2\Theta_g & = & e^2/g_2^2 = \frac{\pi\alpha}{\sqrt{2}\,G_\mu\,M_W^2} \\
a_f: & \sin^2\Theta_f & = & \frac{1}{4|Q_f|}\left(1-\frac{v_f}{a_f}\right)\;,\quad f \neq \nu \\
a_f: & \rho_f & = & \frac{1}{1-\Delta\rho}\;,\quad \text{independent on}\quad \alpha
\end{array}
$$

**for the most important cases and the general form of $\Delta r_i$ reads**

$$\Delta r_i \quad = \quad \Delta\alpha - f_i(\sin^2\Theta_i)\,\Delta\rho + \Delta r_{i\,\mathrm{remainder}}$$

**with a universal term $\Delta\alpha$ which affects the predictions for $M_W$, $A_{LR}$, $A^f_{FB}$, $\Gamma_f$, etc.**

**Equally important in:**

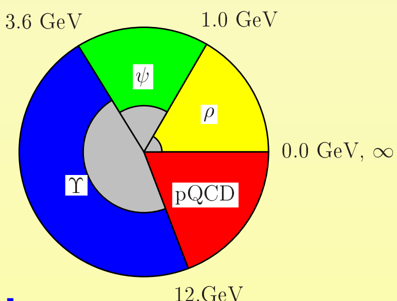
- **Bhahba scattering ($\alpha(t)$)**
- $V_{ud}$ **superallowed $\beta$–decay ($\alpha(m_p)$)**
- $\cdots$

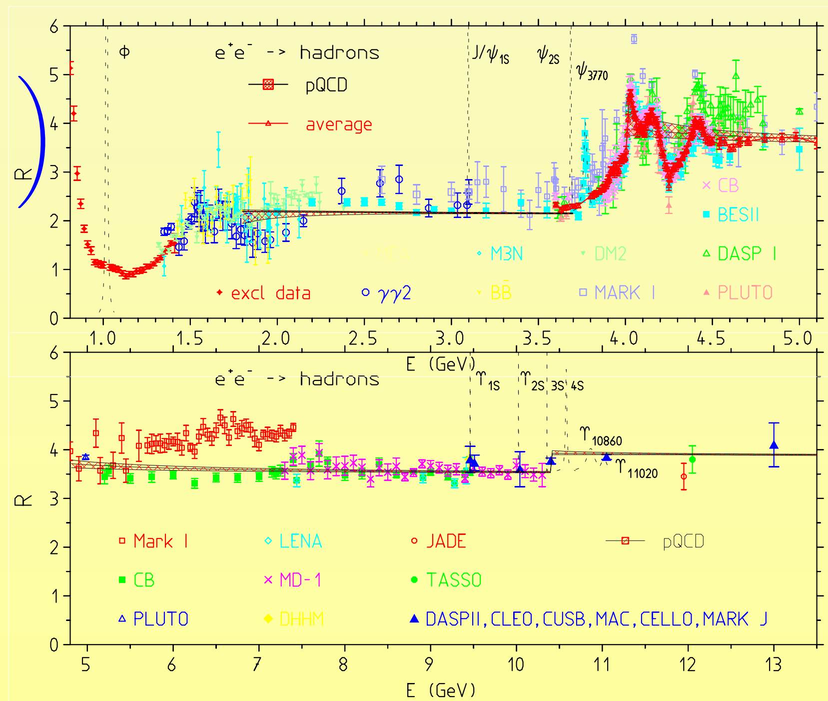
# 3. Evaluation of $\alpha(M_Z)$

**Non-perturbative hadronic contributions $\Delta\alpha^{(5)}_{\rm had}(s)$ can be evaluated in terms of $\sigma(e^+e^- \to \text{hadrons})$ data via dispersion integral:**

$$\Delta\alpha^{(5)}_{\rm had}(s) = -\frac{\alpha s}{3\pi}\left( \quad \mathop{\int\!\!\!\!\!\!-}_{4m_\pi^2}^{E^2_{\rm cut}} ds' \frac{R^{\rm data}_\gamma(s')}{s'(s'-s)} + \mathop{\int\!\!\!\!\!\!-}_{E^2_{\rm cut}}^{\infty} ds' \frac{R^{\rm pQCD}_\gamma(s')}{s'(s'-s)} \right)$$

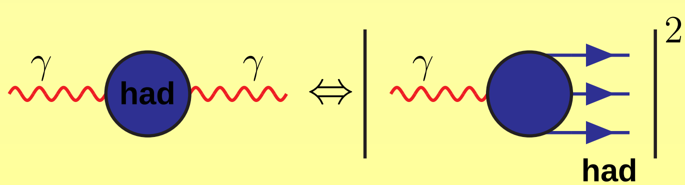
**Compilation:** FJ 10
**Theory = pQCD:** Gorishny et al. 91, Chetyrkin et al. 97...09




**where**

$$R_\gamma(s) \equiv \frac{\sigma^{(0)}(e^+e^-\to\gamma^*\to\text{hadrons})}{\frac{4\pi\alpha^2}{3s}}$$




$\Pi'_\gamma(q^2)$ $\qquad \sim \sigma^{\rm had}_{\rm tot}(q^2)$

**Evaluation FJ 2006 update:** at $M_Z = 91.19$ GeV

- $R(s)$ data up to $\sqrt{s} = E_{cut} = 5$ GeV
  and for $\Upsilon$ resonances region between 9.6 and 13 GeV

- perturbative QCD from 5.0 to 9.6 GeV
  and for the high energy tail above 13 GeV

$$\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M_Z^2) = 0.027773 \pm 0.000354$$

$$0.027664 \pm 0.000173 \quad \textbf{Adler}$$

$$\alpha^{-1}(M_Z^2) = 128.922 \pm 0.049$$

$$128.937 \pm 0.024 \quad \textbf{Adler}$$

Contributions and uncertainties $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)^{\mathrm{data}} \cdot 10^4$. Direct integration method. In red the results relevant fo

VEPP-2000/DAFNE-II.

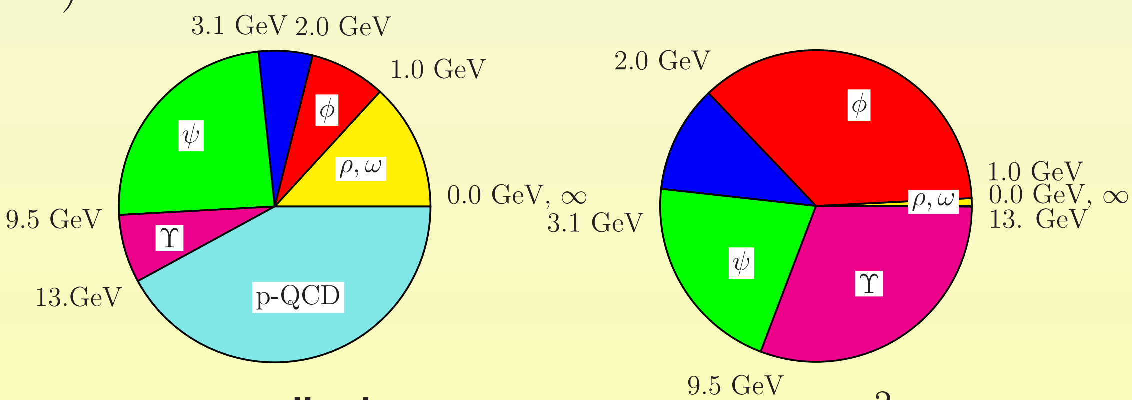
| | $\Delta\alpha^{(5)}_{\mathrm{had}} \times 10^4$ | rel. err. | abs. err. |
|---|---|---|---|
| $\rho, \omega$ ($E < 2M_K$) | 36.38 [ 13.1](0.96) | 2.6 % | 5.8 % |
| $2M_K < E < 2$ GeV | 22.20 [ 8.0](1.51) | 6.8 % | 14.3 % |
| $2$ GeV $< E < M_{J/\psi}$ | 15.77 [ 5.7](0.97) | 6.2 % | 6.0 % |
| $M_{J/\psi} < E < M_\Upsilon$ | 68.53 [ 24.6](3.13) | 4.6 % | 61.7 % |
| $M_\Upsilon < E < E_{\mathrm{cut}}$ | 19.85 [ 7.1](1.39) | 7.0 % | 12.1 % |
| $E_{\mathrm{cut}} < E$ pQCD | 115.57 [ 41.5](0.12) | 0.1 % | 0.1 % |
| $E < E_{\mathrm{cut}}$ data | 162.72 [ 58.5](3.98) | 2.4 % | 99.9 % |
| total | 278.29 [100.0](3.98) | 1.4 % | 100.0 % |

**Contributions and uncertainties $\Delta\alpha^{(5)}_{\rm had}(-M_0^2)^{\rm data} \cdot 10^4$ ($M_0 = 2.5$ GeV). Adler function method. In red the resu relevant for VEPP-2000/DAFNE.**

| | $\Delta\alpha^{(5)}_{\rm had} \times 10^4$ | rel. err. | abs. err. |
|---|---|---|---|
| $\rho,\omega$ ($E < 2M_K$) | 33.43 [ 44.8](0.95) | 2.8 % | 34.5 % |
| $2M_K < E < 2$ GeV | 16.81 [ 22.5](1.09) | 6.5 % | 45.6 % |
| $2$ GeV $< E < M_{J/\psi}$ | 7.93 [ 10.6](0.49) | 6.2 % | 9.1 % |
| $M_{J/\psi} < E < M_\Upsilon$ | 14.47 [ 19.4](0.52) | 3.6 % | 10.6 % |
| $M_\Upsilon < E < E_{\rm cut}$ | 0.97 [ 1.3](0.07) | 7.0 % | 0.2 % |
| $E_{\rm cut} < E$ pQCD | 1.09 [ 1.5](0.00) | 0.1 % | 0.0 % |
| $E < E_{\rm cut}$ data | 73.61 [ 98.5](1.61) | 2.2 % | 100.0 % |
| total | 74.69 [100.0](1.61) | 2.2 % | 100.0 % |

## present distribution of contributions and errors

a)

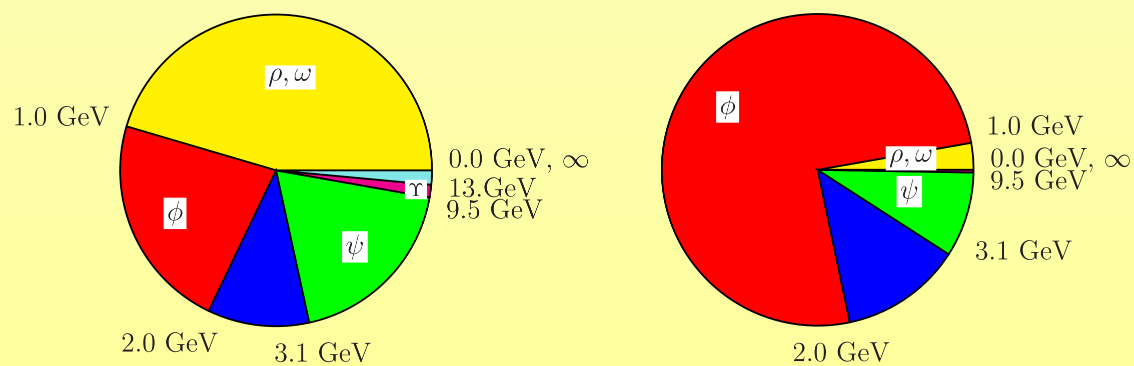



**contributions** **error$^2$**

**direct integration of data**

$$\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M_Z^2)$$

b)




**contributions** **error$^2$**

**"Adler function controlled"**

$\Delta\alpha^{(5)}_{\mathrm{had}}(-M_0^2)^{\mathrm{data}}$ **(**$M_0 = 2.5$ **GeV)**

## 4. $\Delta\alpha^{\rm had}$ via the Adler function

✗ use old idea: Adler function: Monitor for comparing theory and data

$$D(-s) \doteq \frac{3\pi}{\alpha}\, s\, \frac{d}{ds}\Delta\alpha_{\rm had}(s) = -\left(12\pi^2\right)\; s\, \frac{d\Pi'_\gamma(s)}{ds}$$

$$\Rightarrow \qquad D(Q^2) = Q^2 \int\limits_{4m_\pi^2}^{\infty} ds\, \frac{R(s)}{(s+Q^2)^2}$$

| pQCD $\leftrightarrow R(s)$ | pQCD $\leftrightarrow D(Q^2)$ |
|---|---|
| very difficult to obtain in theory | smooth simple function in Euclidean region |

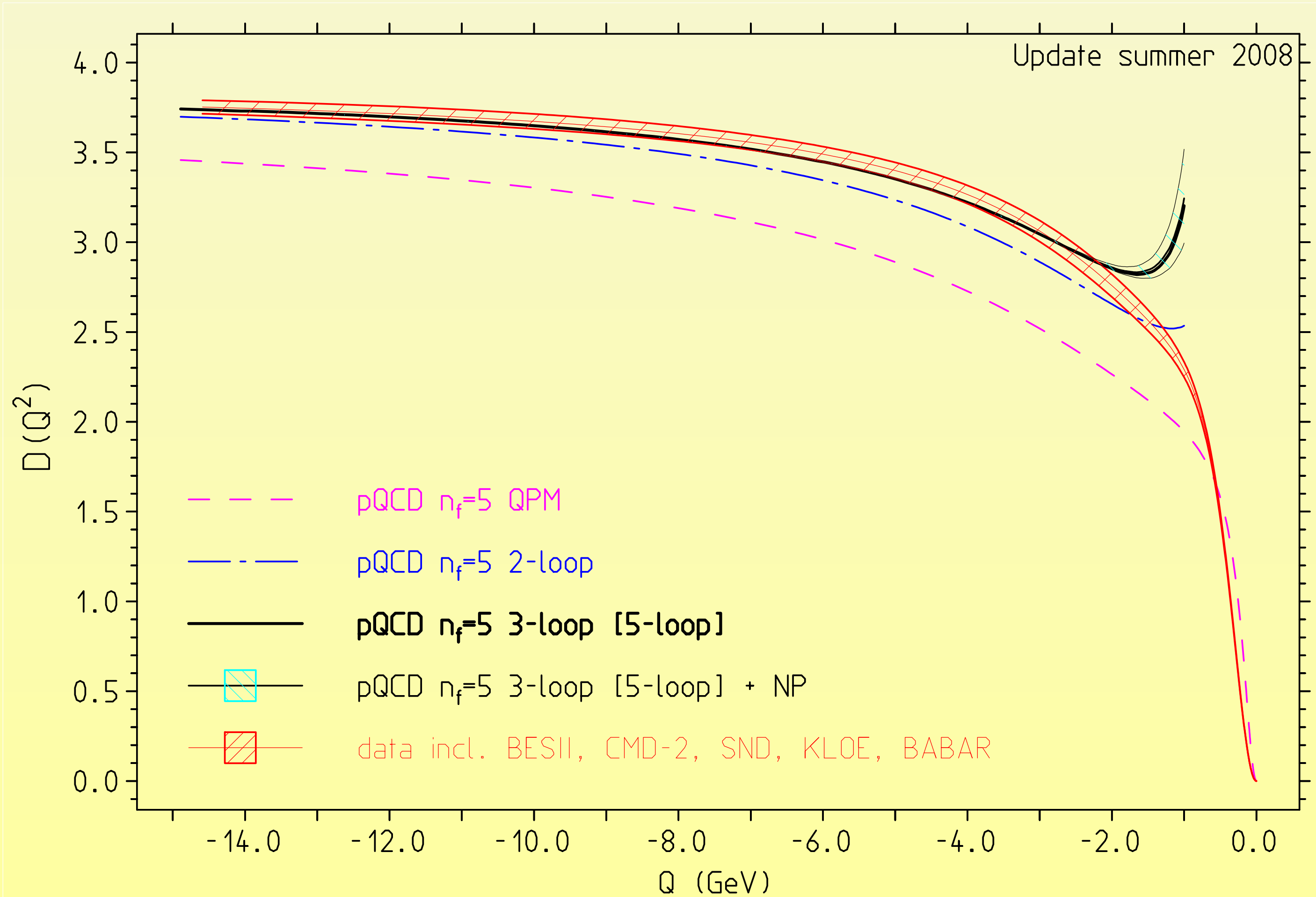
**Conservative conclusion:**

❖ time-like approach: pQCD works well in "perturbative windows" 3.00 - 3.73 GeV, 5.00 - 10.52 GeV and 11.50 - $\infty$ (Kühn,Steinhauser)

❖ space-like approach: pQCD works well for $\sqrt{Q^2=-q^2} > 2.5$ GeV (see plot)

## "Experimental" Adler–function versus theory (pQCD + NP)

**Error includes statistical + systematic here (in contrast to most $R$-plots showing statistical errors only)!**




**(Eidelman, F.J., Kataev, Veretin 98, FJ 08 update) theory based on results by Chetyrkin, Kühn e**

$\Rightarrow$ **pQCD works well to predict $D(Q^2)$ down to $s_0 = (2.5\,\text{GeV})^2$; use this to calculate**

$$\Delta\alpha_{\rm had}(-Q^2) \sim \frac{\alpha}{3\pi}\int dQ'^2\,\frac{D(Q'^2)}{Q'^2}$$

$$\boxed{\begin{aligned}\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) \;=\; & \left[\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) - \Delta\alpha^{(5)}_{\rm had}(-s_0)\right]^{\rm pQCD} \\ & + \Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data}\end{aligned}}$$

**and obtain, for $s_0 = (2.5\,\text{GeV})^2$:** **(FJ 98/1**

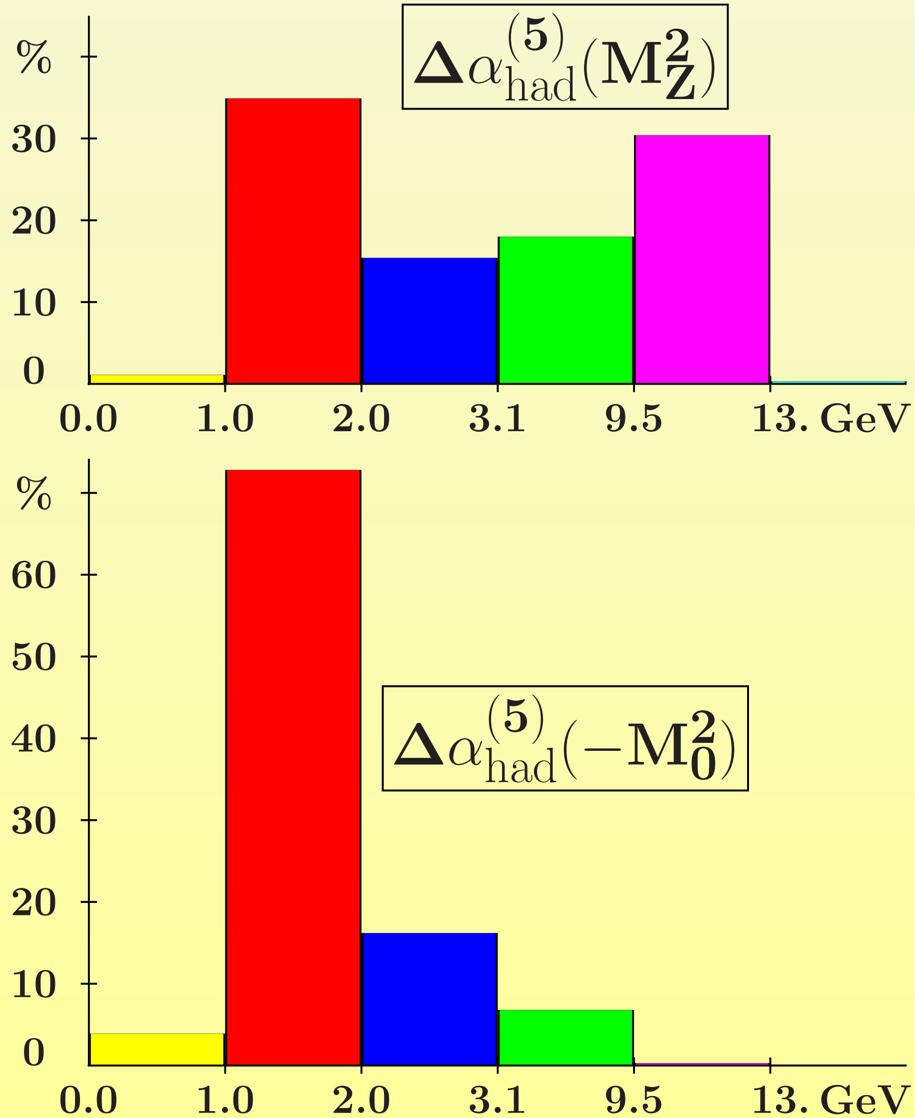
$$\begin{aligned}
\Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data} &= 0.007337 \pm 0.000090 \\
\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) &= 0.027460 \pm 0.000134 \\
\Delta\alpha^{(5)}_{\rm had}(M_Z^2) &= 0.027498 \pm 0.000135
\end{aligned}$$

- **shift $+0.000008$ from the 5-loop contribution**
- **error $\pm 0.000103$ added in quadrature form perturbative part**

**QCD parameters:** $\alpha_s(M_Z) = 0.1189(20)$, $m_c(m_c) = 1.286(13)\ [M_c = 1.666(17)]$ **GeV**, $m_b(m_c) = 4.164(25)\ [M_b = 4.800(29)]$ **GeV**

**based on a complete 3-loop massive QCD analysis (Kühn et al 2007)**

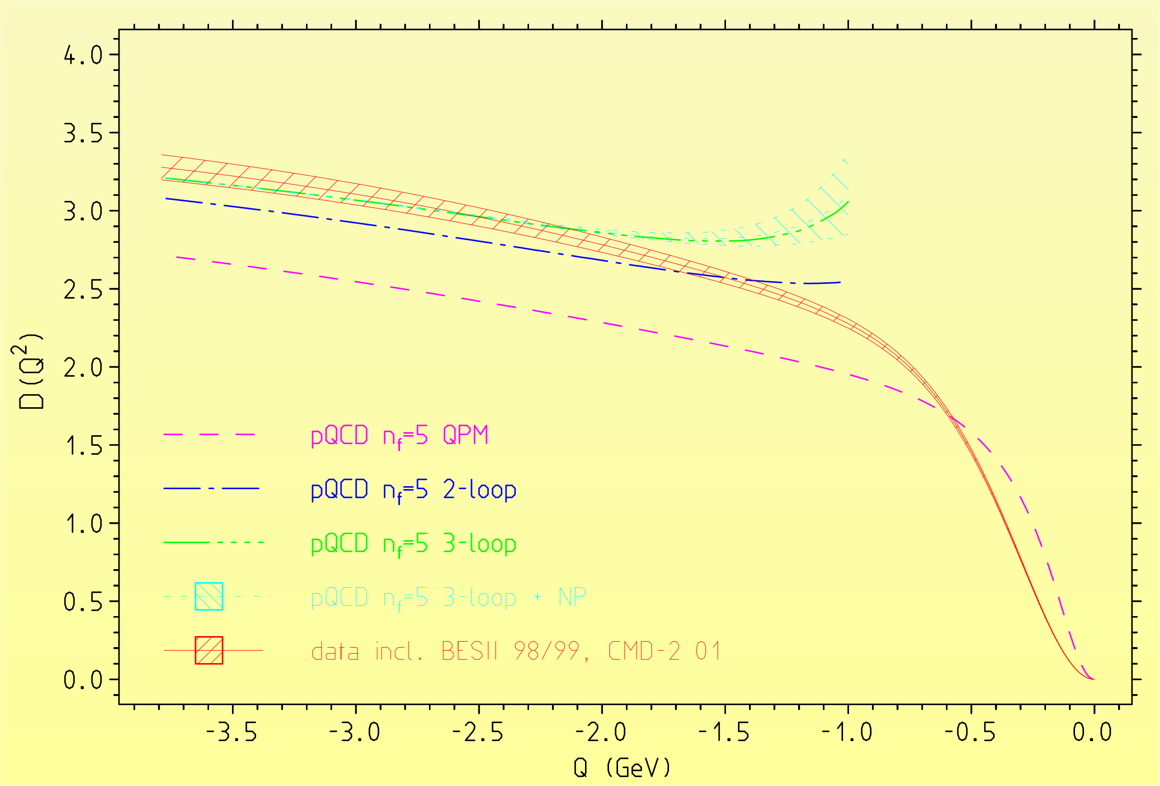
**Comparison of error profiles between** $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$**,** $\Delta\alpha^{(5)}_{\mathrm{had}}(-s_0)$ **and** $a_\mu$**:**



**Mandatory pQCD improvements required are:**

- **4–loop massive pQCD calculation of Adler function; required are a number of terms in the low and high momentum series expansions which allow for the appropriate Padé improvements [essentially equivalent to a massive 4–loop calculation of $R(s)$];**
- **$m_c$ improvement by sum rule and/or lattice QCD evaluations;**
- **improved $\alpha_s$ in low $Q^2$ region.**

**"History": (EJKV 98)**



## ❑ Testing non–perturbative hadronic effects via the Adler function

**Addendum: Adler function in theory**

❖ **non-perturbative Adler function in terms of $R(s))$ (experimental $e^+e^-$-annihilation data)**

$$D(Q^2) \quad = \quad Q^2 \left( \int_{4m_\pi^2}^{E^2_{\mathrm{cut}}} \frac{R^{\mathrm{data}}(s)}{(s+Q^2)^2}\,\mathrm{d}s + \int_{E^2_{\mathrm{cut}}}^{\infty} \frac{R^{\mathrm{pQCD}}(s)}{(s+Q^2)^2}\,\mathrm{d}s \right) \; .$$

❖ **formal definition: in terms of $\hat{\Pi}'_\gamma\,(q^2)$ (photon vacuum polarization amplitude)**

$$\frac{D(Q^2)}{Q^2} = (12\pi^2)\,\frac{\mathrm{d}\hat{\Pi}'_\gamma\,(q^2)}{\mathrm{d}q^2} = -\frac{3\pi}{\alpha}\,\frac{\mathrm{d}}{\mathrm{d}q^2}\Delta\alpha_{\mathrm{had}}(q^2)\;,$$

**evaluated in the Euclidean at $Q^2 = -q^2$**

**In contrast to $R(s)$ away from thresholds, $D(Q^2)$ must be evaluated in full massive QCD !!!**

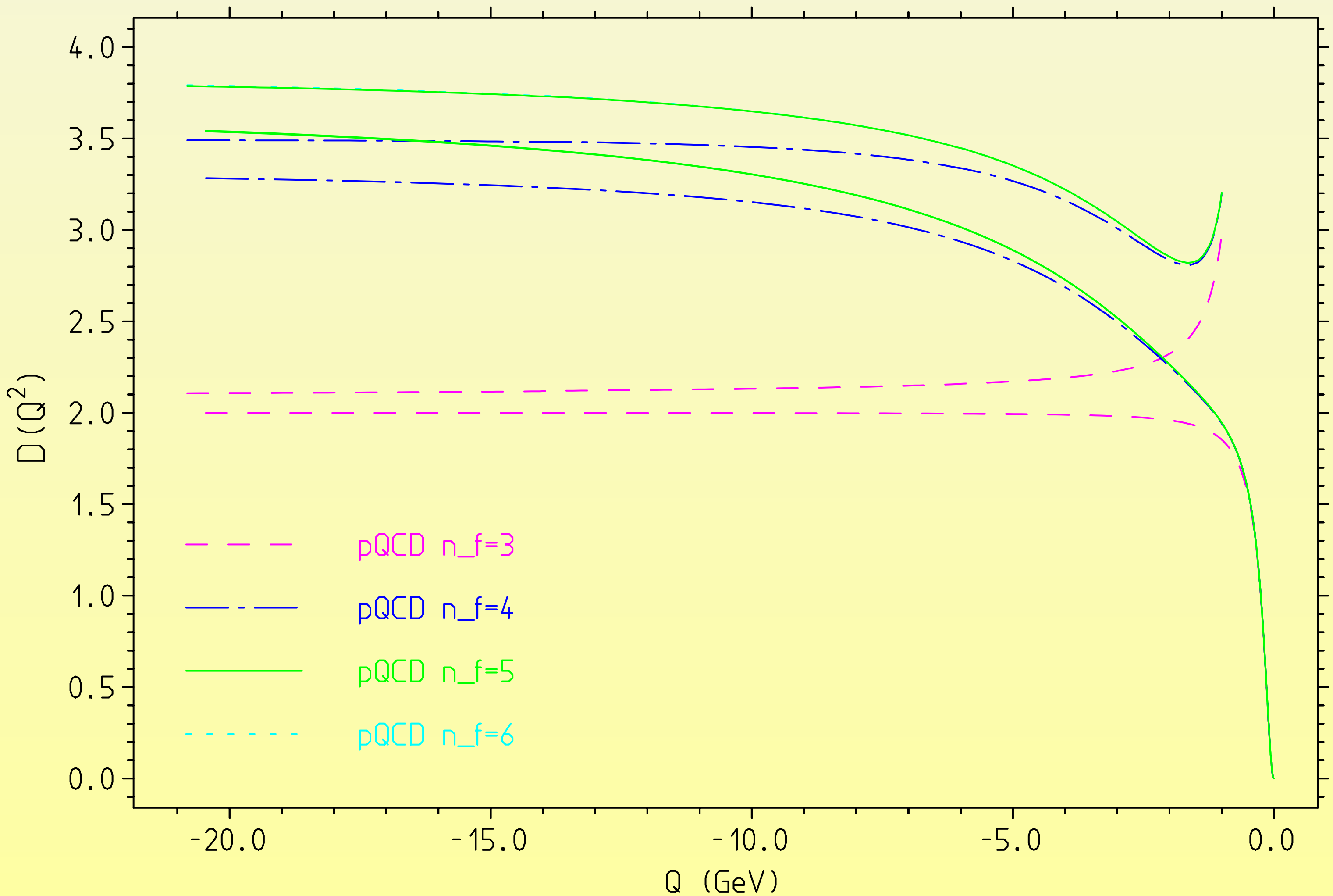
❑ **main $Q^2$ dependence of $D(Q^2)$ is due to the quark masses $m_c$ and $m_b$**

❑ **without mass effects, up to small effects from the running of $\alpha_s$,**

$$D(Q^2) = 3\sum_f Q_f^2\,(1 + O(\alpha_s))$$

**is a constant depending on the number of active flavors**

**Adler function in $\overline{\mathrm{MS}}$**



We write the contributions as a loop expansion

$$D(Q^2) = D^{(0)}(Q^2) + D^{(1)}(Q^2) + D^{(2)}(Q^2) + \cdots + D^{\rm NP}(Q^2)$$

At one loop, taking the derivative of (??) with appropriate coefficient, we obtain

$$D^{(0)}(Q^2) = \sum_f Q_f^2 N_{cf} H^{(0)}$$

with $H \equiv (12\pi^2)\,\dot{\hat{\Pi}}'_V$ , $\dot{\hat{\Pi}}'_V \equiv -s\, d\hat{\Pi}'_V/ds$ , in terms of the vector current amplitude $\hat{\Pi}'_V$. Explicitly,

$$H^{(0)} = 1 + \frac{3y}{2} - \frac{3y^2}{4}\frac{1}{\sqrt{1-y}}\ln\xi \quad y = 4m_f^2/s \ , \quad \xi = \frac{\sqrt{1-y}-1}{\sqrt{1-y}+1}$$

where $\xi$ is taking values $0 \leq \xi \leq 1$ for $s \leq 0$ .

Asymptotically we have the expansion

$$H^{(0)} \to \begin{cases} \frac{1}{5}\frac{Q^2}{m_f^2} - \frac{3}{70}\left(\frac{Q^2}{m_f^2}\right)^2 + \frac{1}{105}\left(\frac{Q^2}{m_f^2}\right)^3 + \cdots & Q^2 \ll m_f^2 \\ 1 - 6\frac{m_f^2}{Q^2} - 12\left(\frac{m_f^2}{Q^2}\right)^2 \ln\frac{m_f^2}{Q^2} + 24\left(\frac{m_f^2}{Q^2}\right)^3\left(\ln\frac{m_f^2}{Q^2} + 1\right) + \cdots & Q^2 \gg m_f^2 \end{cases}$$

and this behavior determines the quark parton model (QPM) (leading order QCD) property of the Adler function: he quarks ($m_f^2 \gg Q^2$) decouple like $Q^2/m_f^2$ while light modes ($m_f^2 \ll Q^2$) contribute $Q_f^2 N_{cf}$ to $D^{(0)}$.

At two loops we have the known analytic result (Broadhurst 85 and others)

$$D^{(1)}(Q^2) = \frac{\alpha_s(Q^2)}{\pi} \sum_f Q_f^2 N_{cf} \, H^{(1)}$$

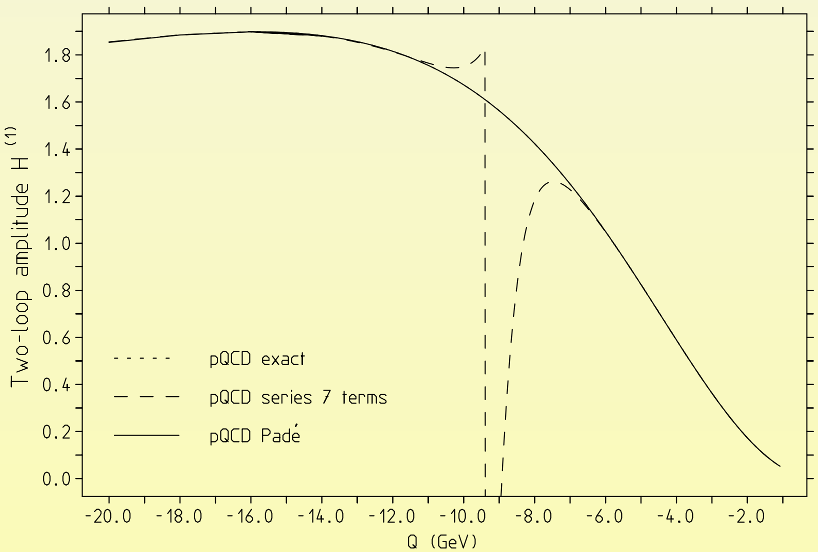
where $H^{(1)} = (12\pi^2)\, \hat{\dot{\Pi}}{}_V^{\prime (2)}(-Q^2, m_f^2)$. At three loops the result is known as low and large momentum expansion (Chetyrkin, Harlander, Kühn, Steinhauser 96/97)

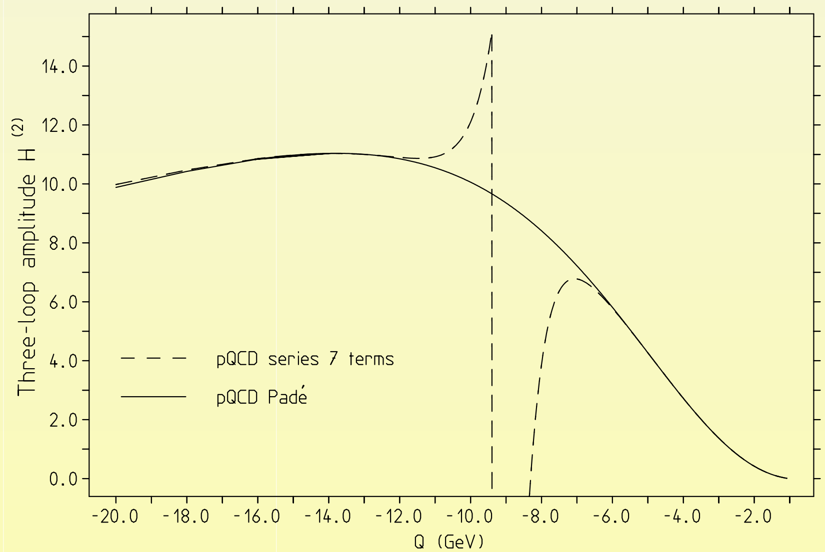
$$D^{(2)}(Q^2) = \left(\frac{\alpha_s(Q^2)}{\pi}\right)^2 \sum_f Q_f^2 N_{cf} \, H^{(2)}$$

where $H^{(2)} = (12\pi^2)\, \hat{\dot{\Pi}}{}_V^{\prime (3)}(-Q^2, m_f^2)$. Both series expansions diverge at the boundary of the circle of converge $Q^2 = 4m^2$, which means that we have a problem in the region where mass effects are of the order of unity in the Euclidean region. Here we apply a conformal mapping (Schwinger 48)

$$y^{-1} = \frac{-Q^2}{4m^2} \to \omega = \frac{1-\sqrt{1-1/y}}{1+\sqrt{1-1/y}}$$

from the complex negative $q^2$ half–plane to the interior of the unit circle $|\omega| < 1$ together with Padé resummation (Fleischer and Tarasov 94). The Padé approximant provides a good estimation to much higher values of $1/y$ up to about $1/y \sim 4$.




Perturbative coefficient functions $H_i$ for the Adler function in full QCD at 2 [left] and 3 [right] loops. The curvature entirely due to the non-zero masses. In the massless limit the $H_i$'s are normalized to unity.

**Padé improvement allows us to obtain reliable results also in the relevant Euclidean "threshold region", around $y$ =**

**We also include the 4–loop and 5–loop contributions in the high energy limit (massless approximation)**

$$D(Q^2) \simeq 3 \sum_f Q_f^2 \; \left(1 + a + d_2 a^2 + d_3 a^3 + d_4 a^4 + \cdots\right)$$

with $a = \alpha_s(Q^2)/\pi$,

$$\begin{aligned} d_2 &= 1.9857 - 0.1153\,N_f\,, \\ d_3 &= 18.2428 - 4.2159\,N_f + 0.0862\,N_f^2 - 1.2395\,(\sum Q_f)^2/(3\sum Q_f^2)\,, \\ d_4 &= -0.010\,N_f^3 \,+\, 1.88\,N_f^2 \,-\, 34.4\,N_f \,+\, 135.8\ \ . \end{aligned}$$

The corresponding formula for $R(s)$ only differs at the 4–loop and 5–loop level due to the effect from the analytic continuation from the Euclidean to the Minkowski region which yields $r_3^R = d_3 - \pi^2\beta_0^2\frac{d_1}{3}$ with $\beta_0 = (11 - 2/3\,N_f)/4$, $d_1 = 1$ and $r_4^R = d_4 - \pi^2\beta_0^2\left(d_2 + \frac{5\beta_1}{6\beta_0}\,d_1\right)$ with $\beta_1 = (102 - 38/3\,N_f)/16$. Numerica the 4–loop term proportional to $d_3$ amounts to $-0.0036$% at 100 GeV and increases to about $0.32$% at 2.5 GeV. The higher order massless results only improve the perturbative high energy tail. Towards low $Q^2$ we also approach th Landau pole of $\alpha_s(Q^2)$, present typically in $\overline{\mathrm{MS}}$ type schemes, and pQCD ceases to "converge".

The Adler–function is a good monitor to compare the pQCD as well as the NP results with experimental data. The Adler–function shows that pQCD in the Euclidean region works very well for $\sqrt{Q^2}\gtrsim 2.5$ GeV. The NP effects just s to be numerically significant where pQCD starts to fail. Thus, no significant NP effects can be established from this plot.

## ❖ Non perturbative effects?

**Parametrize NP effects at sufficiently large energies and away from resonances as prescribed by the OPE. Non–vanishing gluon and light quark condensates (Shifman, Vainshtein, Zakharov 79) imply the leading power corrections**

$$
\begin{aligned}
D^{\mathrm{NP}}(Q^2) \;=\; & \sum_{q=u,d,s} Q_q^2 N_{cq}\,(8\pi^2)\cdot\Bigg[\frac{1}{12}\left(1-\frac{11}{18}\frac{\alpha_s(\mu^2)}{\pi}\right)\frac{<\frac{\alpha_s}{\pi}GG>}{Q^4} \\
& + \; 2\left(1+\frac{\alpha_s(\mu^2)}{3\pi}+\left(\frac{47}{8}-\frac{3}{4}l_{q\mu}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\right)\frac{<m_q\bar{q}q>}{Q^4} \\
& + \left(\frac{4}{27}\frac{\alpha_s(\mu^2)}{\pi}+\left(\frac{4}{3}\zeta_3-\frac{88}{243}-\frac{1}{3}l_{q\mu}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\right)\sum_{q'=u,d,s}\frac{<m_{q'}\bar{q}'q'>}{Q^4}+\cdots\Bigg]
\end{aligned}
$$

**where $a \equiv \alpha_s(\mu^2)/\pi$ and $l_{q\mu} \equiv \ln(Q^2/\mu^2)$.**
**$<\frac{\alpha_s}{\pi}GG>$ and $<m_q\bar{q}q>$ are the scale-invariantly defined condensates. Sum rule estimates of the condensates yield typically (large uncertainties)**

$$
\begin{aligned}
&<\frac{\alpha_s}{\pi}GG>\sim \quad (0.389\ \textbf{GeV})^4 \\
&<m_q\bar{q}q>\sim -(0.098\ \textbf{GeV})^4 \quad \text{for} \quad q=u,d \\
&<m_q\bar{q}q>\sim -(0.218\ \textbf{GeV})^4 \quad \text{for} \quad q=s
\end{aligned}
$$

**Note that the above expansion is just a parameterization of the high energy tail of NP effects associated with the existence of non–vanishing condensates. There are other kind of NP phenomena like bound states, resonances, instantons and what else. The dilemma with is that it works only for $Q^2$ large enough and it has been successfully applied in heavy quark physics. It fails do describe NP physics at lower $Q^2$, once it starts to be numerically relevant.**

**pQCD starts to fail because of the growth of the strong coupling constant.**

**The virtues of Adler function approach are obvious:**

- **no problems with physical threshold and resonances**
- **pQCD is used only where we can check it to work (Euclidean, $Q^2 \gtrsim$ 2.5 GeV).**
- **no manipulation of data, no assumptions about global or local duality.**
- **non–perturbative "remainder" $\Delta\alpha_{\rm had}^{(5)}(-s_0)$ is mainly sensitive to low energy data !!!**

# 5. The running electric charge at high energies

**Prolog:**

- **At high energies, like in QCD, one usually used the $\overline{\mathrm{MS}}$ scheme**
- **The latter is mass independent and hence not very physical (not a form factor, just UV singular poles)**
- **Most problematic lack of decoupling, for each $N_f$ separate effective theory (matching at new thresholds)**

**Above we were considering OS scheme, relationship:**

**$\alpha$ in the on–shell versus $\alpha$ in the $\overline{\mathrm{MS}}$ scheme**

**In our discussion of renormalizing QED we were considering originally the *on–shell renormalization scheme* , whil the RG provides $\alpha$ in the $\overline{\mathrm{MS}}$ scheme. Here we briefly discuss the relationship between the OS and the $\overline{\mathrm{MS}}$ fine structure constants $\alpha_{\mathrm{OS}} = \alpha$ and $\alpha_{\bar{\mathrm{MS}}}$, respectively. Since the bare fine structure constant**

$$\alpha_0 = \alpha_{\bar{\mathrm{MS}}} \left( 1 + \left. \frac{\delta\alpha}{\alpha} \right|_{\bar{\mathrm{MS}}} \right) = \alpha_{\mathrm{OS}} \left( 1 + \left. \frac{\delta\alpha}{\alpha} \right|_{\mathrm{OS}} \right)$$

**is independent of the renormalization scheme. The one–loop calculation in the SM yields (including the charged $W$**

contribution for completeness)

$$\begin{aligned}
\left.\frac{\delta\alpha}{\alpha}\right|_{\bar{\mathrm{M}}\mathrm{S}} &= \frac{\alpha}{3\pi}\sum Q_f^2 \ln\frac{\mu^2}{m_f^2} - \frac{\alpha}{3\pi}\frac{21}{4}\ln\frac{\mu^2}{M_W^2} \\
\left.\frac{\delta\alpha}{\alpha}\right|_{\mathrm{OS}} &= \Pi'_\gamma(0) + \frac{\alpha}{\pi}\ln\frac{M_W^2}{\mu^2} \\
&= \left.\frac{\delta\alpha}{\alpha}\right|_{\bar{\mathrm{M}}\mathrm{S}} - \frac{\alpha}{6\pi}
\end{aligned}$$

and thus

$$\alpha^{-1}_{\overline{\mathrm{MS}}}(0) = \alpha^{-1} + \frac{1}{6\pi}$$

as a low energy matching condition. The $\alpha$–shift in the $\overline{\mathrm{MS}}$ scheme is very simple, just the UV logs,

$$\Delta\alpha_{\overline{\mathrm{MS}}}(\mu) = \frac{\alpha}{3\pi}\sum Q_f^2 N_{cf}\ln\frac{\mu^2}{m_f^2} - \frac{\alpha}{3\pi}\frac{21}{4}\ln\frac{\mu^2}{M_W^2}$$

such that

$$\Delta\alpha_{\overline{\mathrm{MS}}}(\mu) = \Delta\alpha_{\mathrm{OS}}(\mu) + \frac{\alpha}{\pi}\frac{5}{3}\sum Q_f^2 N_{cf}$$

where the sum goes over all fermions $f$ with $N_{cf} = 1$ for leptons and $N_{cf} = 3$ for quarks.

In perturbation theory, the leading light fermion ($m_f \ll M_W, \sqrt{s}$) contribution in the OS scheme is given by

$$\Delta\alpha(s) \quad = \quad \frac{\alpha}{3\pi} \sum_f Q_f^2 N_{cf} (\ln \frac{s}{m_f^2} - \frac{5}{3}) \quad .$$

**We distinguish the contributions from the leptons, for which the perturbative expression is appropriate, the five lig quarks ($u, d, s, c, b$) and the top**

$$\Delta\alpha = \Delta\alpha_{\rm lep} + \Delta\alpha_{\rm had} + \Delta\alpha_{top} \ .$$

**Since the top quark is heavy we cannot use the light fermion approximation for it. A very heavy top in fact decoupl like**

$$\Delta\alpha_{top} \simeq -\frac{\alpha}{3\pi} \frac{4}{15} \frac{s}{m_t^2} \to 0$$

**when $m_t \gg s$. Since pQCD does not apply at low energies, $\Delta\alpha_{\rm had}$ has to be evaluated via dispersion relations fro $e^+e^-$–annihilation data.**

**Types of VP contributions in SM:**

- **below $M_Z$: light fermion contributions: $e, \mu, \tau, u, d, s, c, b$**
- **beyond $M_Z$: the charged $W$ as well as the top contribute**

**simplest at one loop:**

❑ **fermion loop gauge invariant** ❑ **$W$-boson loop not gauge invariant**

• **Measuring form factor: simplest in "$2 \to 2$ fermions" process**

**However: all types of corrections, self-energy, vertex, boxes**

**form-factor observable only in factorization:**

❑ **at low $q^2$ when one-photon approximation is good**

❑ **at $Z$ resonance, when background is small (residue of pole gauge invariant)**

**Lesson: running coupling is not an observable except in exceptional situations, rather a book–keeping device, convenient for parametrization in context of RG summation of large logs**

**In general form-factors not directly accessible in experiment**

**A new problem shows up in the SM because of $\gamma - Z$ –mixing:**

- **What is the proper definition of the photon field?**
- **In SM mixing Abelian and non-Abelian components, photon should be Abelian!**
- **Correct photon should satisfy Maxwell's equations, which is not automatic!**

**In the SM, adopting the standard lowest order definition, the photon field**

$$A_\mu = (gB_\mu + g'W_{\mu 3})/\sqrt{g'^2+g^2}$$

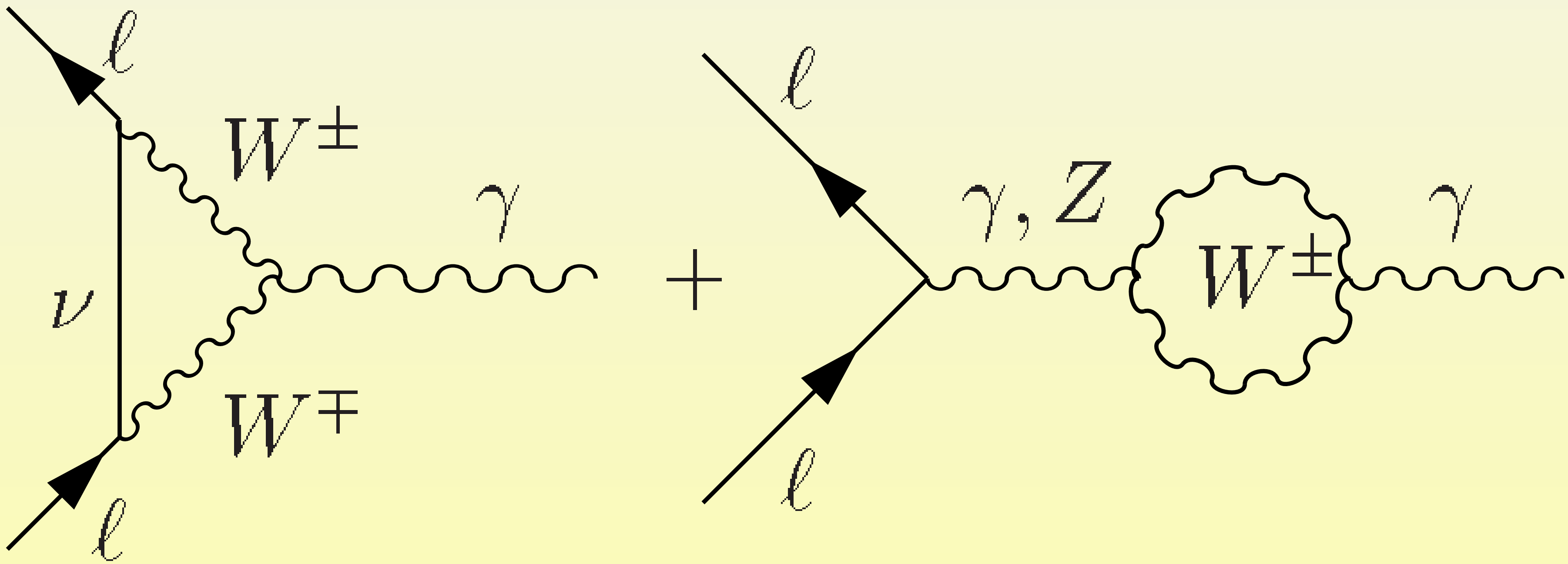
**has a non-Abelian component. This fact at higher orders causes problems which do not appear in pure QED. A manifestly Abelian photon may be defined by (Fleischer, FJ 1985)**

$$\begin{aligned} A^a_\mu &= (gB_\mu + g'W^s_{\mu 3})/\sqrt{g'^2+g^2} \\ &= (\sqrt{g'^2+g^2}/g)\, B_\mu + (g'/g)\, Z^s_\mu \end{aligned}$$

**where $Z^s_\mu$ defined by**

$$\begin{aligned} Z^s_\mu &= (gW^s_{\mu 3} - g'B_\mu)/\sqrt{g'^2+g^2} \\ &= -(i/\sqrt{g'^2+g^2})\, (\Phi^+ D_\mu \Phi - \mathrm{h.c.})/(\Phi^+\Phi) \end{aligned}$$

**obviously is a singlet, with respect to the SM gauge group, and $W^s_{\mu 3}$ is Abelian. $\Phi$ is the Higgs doublet field and $D_\mu\Phi$ its covariant derivative. In the *unitary gauge* $A_\mu$ and $A^a_\mu$ coincide, which means that in the unitary gauge we automatically are dealing with the Abelian photon field, which satisfies the correct Maxwell equations. The gauge dependent part is originating at the one–loop level solely from the $W$–pair excitation, described by the diagrams**




**Because we are mainly interested in the high energy behavior and in order to avoid lengthy expressions, we present the results for $|q^2| \gg M_W^2$ only. The one–loop contributions to the singlet form–factor may be written as**

$$\Delta\alpha = \Delta\alpha_W + \Delta\alpha_Z + \Delta\alpha_\gamma + \Delta\alpha_f$$

with contributions from $W$, $Z$, photon and the fermions. For the "renormalized" virtual $W^\pm$ contribution one finds

$$\Delta\alpha_W(q^2) = \frac{\alpha}{16\pi s_\theta^2 c_\theta^2}\left\{a_0 + b_0 \ln\frac{|q^2|}{M_W^2} + a_1 \frac{q^2}{M_W^2}\left(\ln\frac{|q^2|}{\mu^2} - \frac{8}{3}\right)\right.$$
$$\left. - i\pi\theta(q^2 - 4M_W^2)\left(b_0 + a_1\frac{q^2}{M_W^2}\right)\right\}$$

in the $\overline{\mathrm{MS}}$ subtraction scheme. Introducing the notation $c_\theta^2 \doteq M_W^2/M_Z^2,\ s_\theta^2 \doteq 1 - c_\theta^2$ and $g(c_\theta^2) \doteq \sqrt{4c_\theta^2 - 1}\ \mathrm{arc\,ctg}\sqrt{4c_\theta^2 - 1}$, the coefficients are given by

$$\begin{aligned} a_0 &= -32c_\theta^6 - \frac{56}{3}c_\theta^4 + \frac{253}{6}c_\theta^2 + \frac{1}{2} \\ &\quad -(-32c_\theta^6 - \frac{64}{3}c_\theta^4 + \frac{134}{3}c_\theta^2 - \frac{22}{3} - \frac{1}{2}c_\theta^{-2})g(c_\theta^2) \\ b_0 &= \frac{5}{3}c_\theta^2 - \frac{19}{6} - \frac{1}{4}c_\theta^{-2} \\ a_1 &= \frac{1}{6}c_\theta^2 - \frac{1}{4}\ . \end{aligned}$$

The virtual $Z$ contribution reads

$$\Delta\alpha_Z(q^2) = \frac{\alpha}{16\pi s_\theta^2 c_\theta^2}\left\{a_0' + b_0' \ln\frac{|q^2|}{M_Z^2} + \frac{2}{3}b_0' \left(\mathrm{Sp}\left(1+\frac{q^2}{M_Z^2}\right) - \frac{\pi^2}{6}\right)\right. \\ \left. - i\pi\theta(q^2-4m_e^2)\, b_0'\right\}$$

which in contrast to the $W^\pm$ contribution is finite (i.e., $\mu$–independent). The coefficients are given by

$$\begin{aligned} a_0' &= -14c_\theta^4 + 21c_\theta^2 - \frac{35}{4} \\ b_0' &= 12c_\theta^4 - 18c_\theta^2 + \frac{15}{2} \ , \end{aligned}$$

and the Spence function Sp is asymptotically given by

$$\mathrm{Sp}\left(1+\frac{q^2}{M_Z^2}\right) - \frac{\pi^2}{6} \simeq \begin{cases} \frac{\pi^2}{6} - \frac{1}{2}\ln^2\frac{q^2}{M_Z^2} + i\pi\ln\frac{q^2}{M_Z^2}\ ; & q^2 \gg M_Z^2 \\ -\frac{\pi^2}{3} - \frac{1}{2}\ln^2\left(-\frac{q^2}{M_Z^2}\right)\ ; & -q^2 \gg M_Z^2\ . \end{cases}$$

The QED electron vertex $+$ self–energy contributions exhibit the well known infrared problem with soft and collinea logs which only become physical after combining them with the soft real photon radiation. Virtual $+$ soft QED corrections together are related to definition of the initial and/or final state and are therefore taken into account in a different way. They have nothing to do with the running of the charge or vacuum polarization effects. We therefore apply the convention to set $\Delta_\gamma = 0$ in the calculation of $\Delta\alpha$.

Another problem is due to $\gamma - Z$ mixing. At higher energies the mixing effects have to be taken into account. We h seen that this is crucial for the $W$–pair creation and re-absorption but in fact also applies to the fermion contributi once $\gamma \to Z \to \gamma$ transitions become relevant at sufficiently high energies, we must include

$$\Delta\alpha_f(q^2) = -2a\frac{M_Z^2}{q^2 - M_Z^2}\,(A_{1r}^{\gamma Z})_f - (ev)^2\frac{1}{q^2}\,(A_{1r}^{\gamma\gamma})_f$$

where

$$-(ev)^2\frac{1}{q^2}\,(A_{1r}^{\gamma\gamma})_f = \frac{\alpha}{3\pi}\sum_f Q_f^2 H_f\left(4m_f^2/q^2\right)$$

is the renormalized QED vacuum polarization and

$$(A_{1r}^{\gamma Z})_f = -\frac{\alpha}{24\pi s_\theta^2}\frac{q^2}{M_W^2}\sum_f (4a_f Q_f)\left[H_f\left(4m_f^2/q^2\right) - H_f\left(4m_f^2/M_Z^2\right)\right]$$

is the $\gamma - Z$ mixing contribution. The function $H_f$ is given by

$$H_f(y_f) = \frac{5}{3} + y_f + \left(1 + \frac{y_f}{2}\right)\,\sqrt{1-y_f}\,\ln\frac{\sqrt{1-y_f}-1}{\sqrt{1-y_f}+1}$$

and $a_f = -Q_f s_\theta^2 \pm \frac{1}{4}$ for the upper and lower components of the weak iso-doublets, respectively. The one–loop perturbative fermion formula also is appropriate to take into account the top quark contribution. At $M_Z$ we have

**included in $\alpha^{-1}(M_Z)$**

$$\Delta\alpha_{\rm top}(M_Z^2) = -0.76 \times 10^{-4} \ \ .$$

**Numerical results for the SM contributions in the singlet form–factor definition of the effective charge will be presented elsewhere.**