Lectures on the physics of vacuum polarization: from GeV to TeV scale

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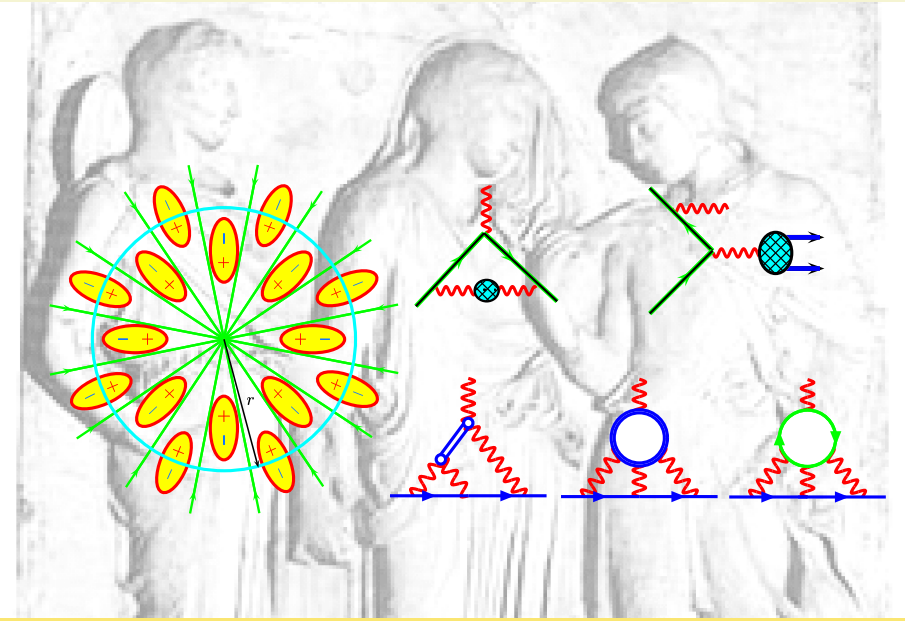
Outline of Lecture:

① **Introduction, theory tools, non-perturbative and perturbative aspects**

② **Vacuum Polarization in Low Energy Physics:** $g-2$

③ **Vacuum Polarization in High Energy Physics:** $\alpha(M_Z)$ **and** α **at ILC scales**

④ **Other Applications and Problems: new physics, future prospects, the role of DAFNE-2**



Memories to EURODAFNE, EURIDICE and all that

① Introduction, theory tools, non-perturbative and perturbative aspects

1. Intro SM
2. Some Basics in QFT
3. The Origin of Analyticity
4. Properties of the Form Factors
5. Dispersion Relations
6. Dispersion Relations and the Vacuum Polarization
7. Low Energies: $e^+e^- \longrightarrow$ hadrons
8. High Energies: the electromagnetic form factor in the SM

1. Introduction

Theory is the Standard Model:

$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ **local gauge theory**

broken to $SU(3)_{\rm QCD} \otimes U(1)_{\rm QED}$

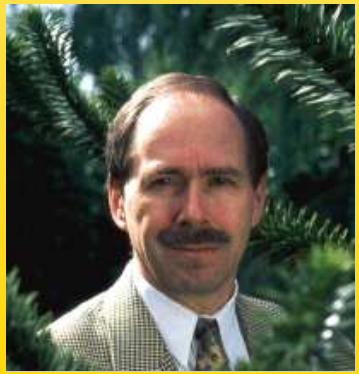
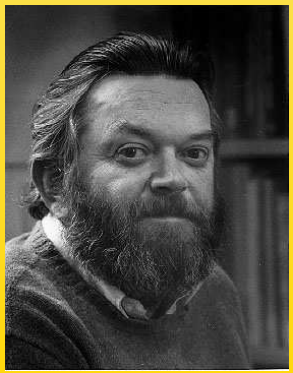
by the Higgs mechanism

Yang Mills 1954, Glashow 1961, Weinberg 1967, Gell-Mann, Fritzsch, Leutwyler 1973, Gross, Wilczek 1973, Politzer 1973, Higgs 1964, Englert, Brout 1964, ...

Nobel Prize 1999: G. 't Hooft, M. Veltman; 2003 Politzer, Gross, Wilczek

for elucidating the quantum structure of electroweak interactions in physics and for asymptotic freedom



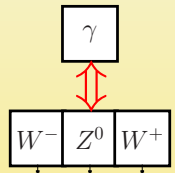
Constituents of matter:

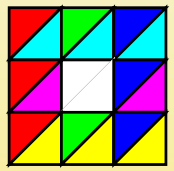
Spin 1/2 fermions

Carrier of Forces:

Spin 1 Gauge Bosons

Photon





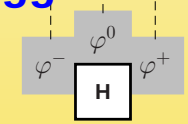
Octet of Gluons

Vector Bosons

Higgs Ghosts

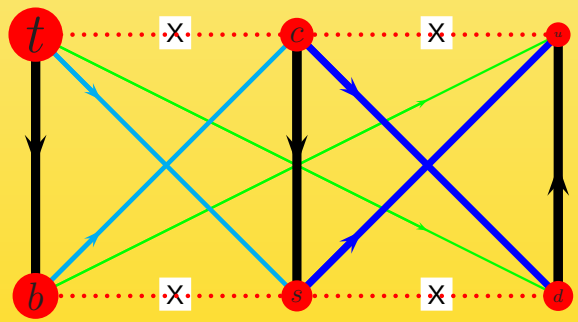
"Cooper Pairs":

Spin 0 Higgs Boson



Higgs Particle

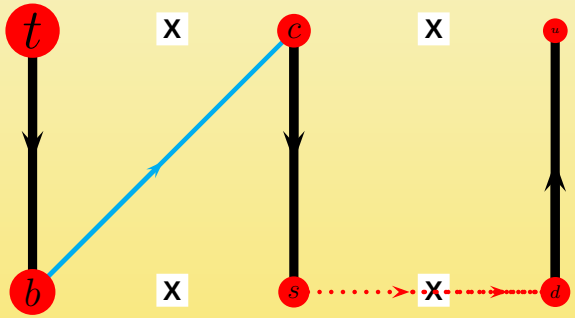
Also note quark decay pattern:

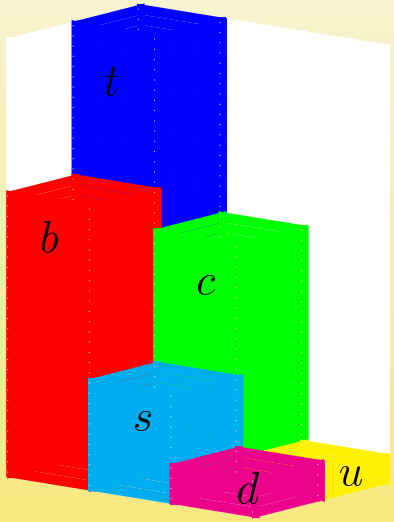


Note: The u quark is stable, the s and b quarks are metastable. Flavor changing neutral currents (FCNC) at tree level $\cdots\times\cdots$ are forbidden by the GIM–mechanism. Flavor changing neutral current transitions are allowed, however, as second (or higher) order transitions: e.g. $b \longrightarrow s$ is in fact $b \longrightarrow (u^*, c^*, t^*) \longrightarrow s$, where the asterix indicates "virtual transition".

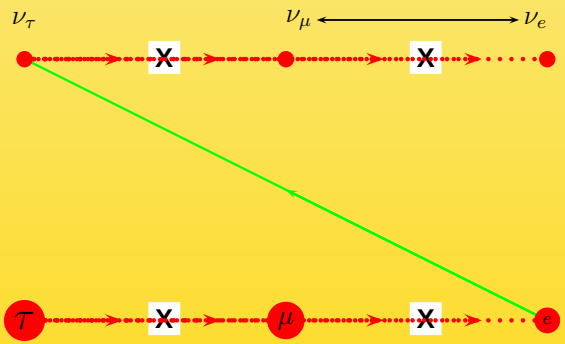
Most mysterious SM mass spectrum. Fermions:

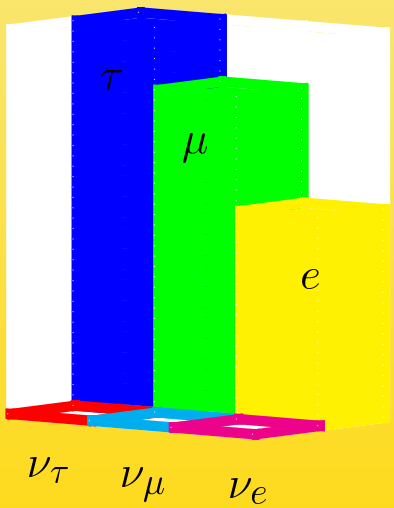
Quark mass hierarchy:





Lepton-neutrino mass hierarchy:





Hierarchy of quark (right top) and lepton masses (right bottom)
[$\sim \log m_f$ in arbitrary units]

SM tests:

- High Energy Frontier, e.g top discovery, triple gauge couplings [LEP,HERA,Tevatron,LHC]
- Rare Processes, e.g. $\mu \to e\gamma$ ($\not{L}_\mu$), neutrino physics [PSI,BNL,Tevatron,Kamiokande,Homestake,SNO]
- High Precision Frontier, e.g. m_t and m_H bound from LEP, $g-2$ [LEP,BNL,Harvard,ILC]

Many open questions in the Standard Model

Standard Model very good description

– Less satisfactory explanations

- ✗ **Why 3 families of fundamental fermions ?**
- ✗ **Why P, C and CP violations? Why not more $\not{C}P$?**
- ✗ **Why us ? Origin of matter–antimatter asymmetry ?**
- ✗ **What Dark Matter ? Why the Cosmological Constant ?**

- However: most extensions of the SM run into serious problems (FVNC's, CP): if not tuned!!! (R–parity etc.)

  impose MFV scheme! (Isidori et al.)

- another as serious issue: accidental??? custodial symmetry

  ρ–parameter constraint! (Veltman,Lynn+Nardi,FJ,Czakon et al.)

Cries for new physics

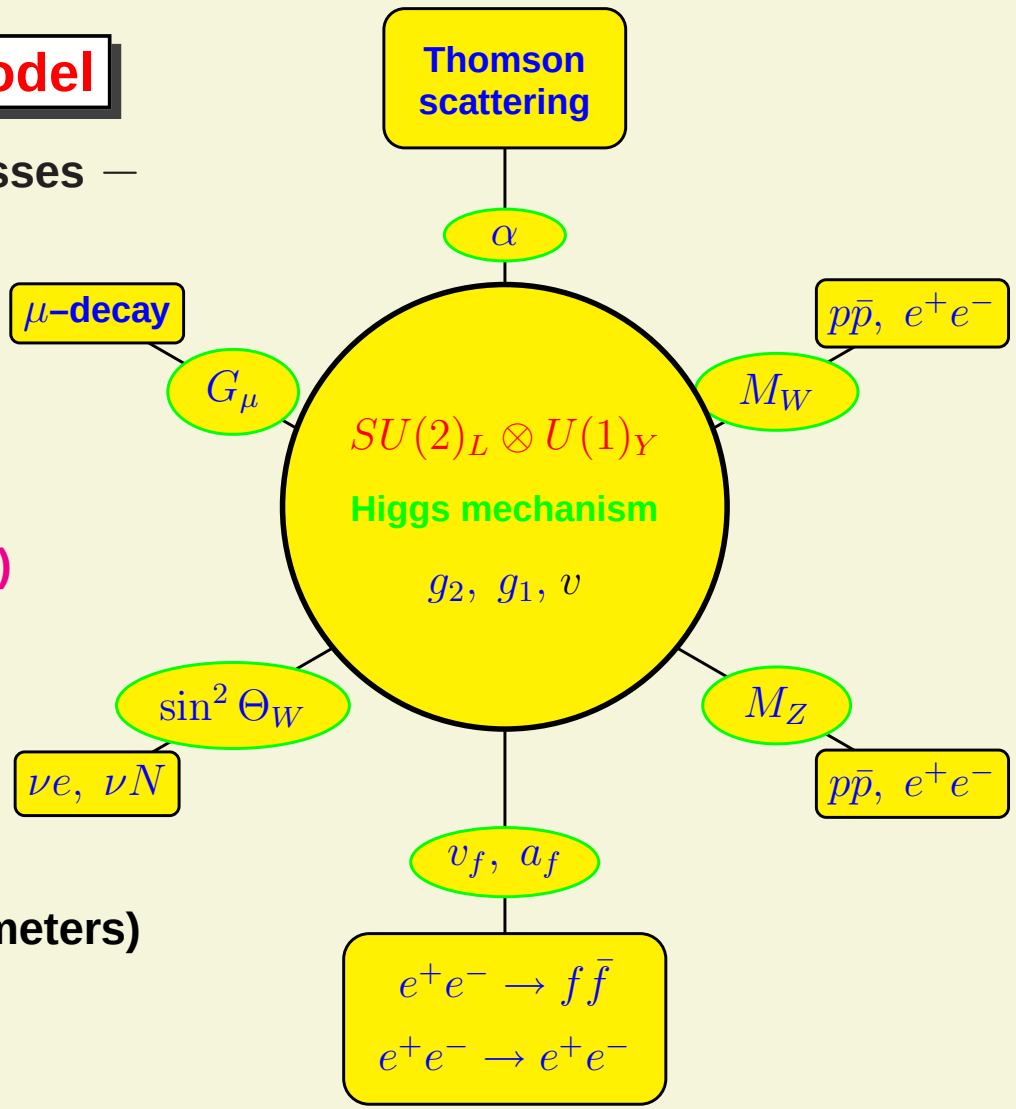
but, please,

don't touch at all what we have!

In general, SM instable against perturbations!

The Parameters of the Standard Model

— in four fermion and vector boson processes —



unlike in QED and QCD in SM (SBGT)

parameter interdependence

☞

only 3 independent quantities

(besides fermion masses and mixing parameters)

α, G_μ, M_Z

$\Downarrow$

parameter relationships between very precisely measurable quantities

precision tests, possible sign of new physics

Low Energy Parametrization and VB masses

— QED and four fermion processes at low energy—

- **Finestructure constant: e.g. Thomson scattering**

$$\alpha = \ (\ 137.0359895(61)\ )^{-1}$$

- **Fermi constant: μ–decay**

$$G_\mu = 1.166389(22) \times 10^{-5}\ \textbf{GeV}^{-2}$$

Higgs mechanism:

$$\sqrt{2}G_\mu = \frac{1}{v^2} \blacktriangleright v = 1/\sqrt{\sqrt{2}G_\mu} = 246.22\ \textbf{GeV}$$

Existence of the Higgs condensate verified!

- $\gamma - Z$ **mixing parameter: νN–scattering**

$$\sin^2\Theta_W = 0.231 \pm 0.006$$

➤ **Masses of vector bosons determined**

$$M_W = \frac{A_0}{\sin\Theta_W}, \quad M_Z = \frac{M_W}{\cos\Theta_W}; \quad A_0 = \sqrt{\frac{\pi\alpha}{\sqrt{2}G_\mu}}$$

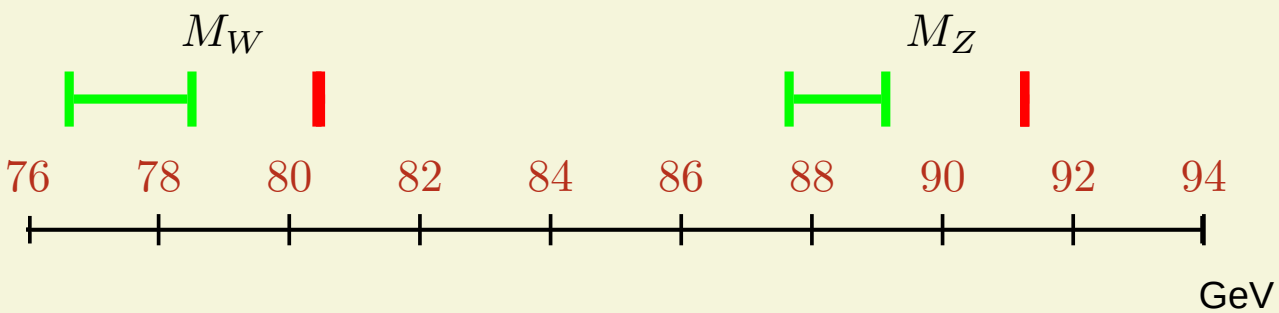
$M_W = 77.570 \pm 1.010$ GeV $\quad M_Z = 88.390 \pm 0.810$ GeV

Born level prediction!

$M_W = 80.403 \pm 0.029$ GeV $\quad M_Z = 91.1876 \pm 0.0021$ GeV

(UA2, CDF, LEP) (LEP, SLC)

Experimental result

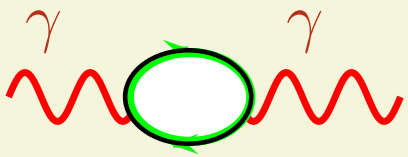


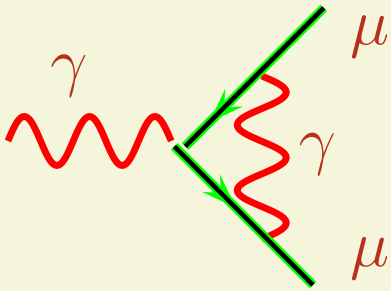
Born approximation “contradicts” data!

➤ Radiative corrections very important!

Test of quantum effects

Prototype QED:





vacuum polarization

$\longrightarrow$ Lamb shift

form factors

$\longrightarrow$ anomalous magnetic moment

LEP/SLC version of $g-2$:

$$\sqrt{2}G_\mu M_Z^2 \sin^2\Theta \cos^2\Theta = \pi\alpha\,(1+\delta)$$

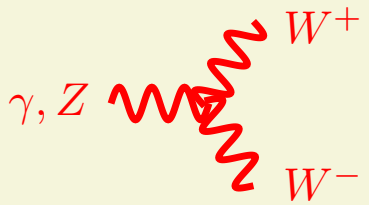
SM: renormalizable theory (’t Hooft 1971)

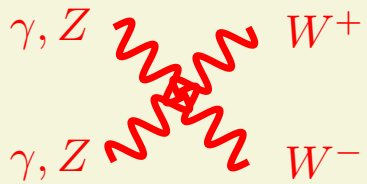
➤ δ uniquely calculable

in principle depends on unverified part of theory

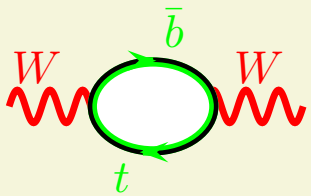
(couplings and masses)

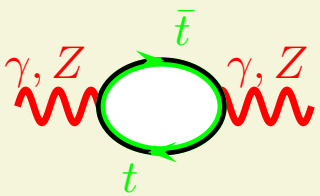
Gauge self couplings: (measured at LEP2)



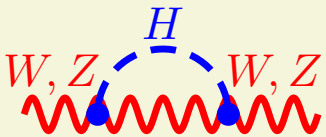


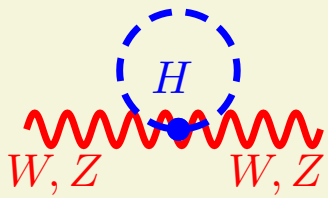
Top quark: (discovered at Tevatron some time ago)





Higgs boson: (the hunting is going on)





$$\delta \equiv \Delta r = \Delta r(\underbrace{\alpha, G_\mu, M_Z}_{\text{very precisely known}}, M_H, m_t, m_b, \cdots)$$

- M_H **unknown, bounds, free parameter**

  ($114\ \mathrm{GeV} < M_H < 186\ \mathrm{GeV}$ and $\notin [160, 170]$ **GeV)**

- m_t **rather precisely known now**

  ($m_t = 171.3 \pm 1.6\ \mathrm{GeV}$**)**

Precision measurement of Δr ➤

severe restrictions for "unknown physics" ➤

bounds on $\Delta r^{\text{new physics}}$ **?**

Important fact: subleading radiative corrections (beyond running $\Delta\alpha$ **and** $\Delta\rho_{\text{top}}$ **) are**

10 σ **effects**

e.g. on precision observables like $\sin^2\Theta^f_{\text{eff}}$**!!!**

Impact for $\Delta\alpha$**:**

Interference between precision test, new physics contribution and hadronic uncertainty:

- $\Delta\alpha = \Delta\alpha_{\rm lep} + \Delta\alpha_{\rm had}$
- $\Delta\alpha_{\rm had}$ **has substantial non-perturbative part from low energy hadrons**
- **Uncertainty** $\delta\Delta\alpha_{\rm had}$ **does seriously limit precision of predictions!**
- **Mandatory goal for all kinds of precision tests:**

precise determination of effective fine structure "constant"

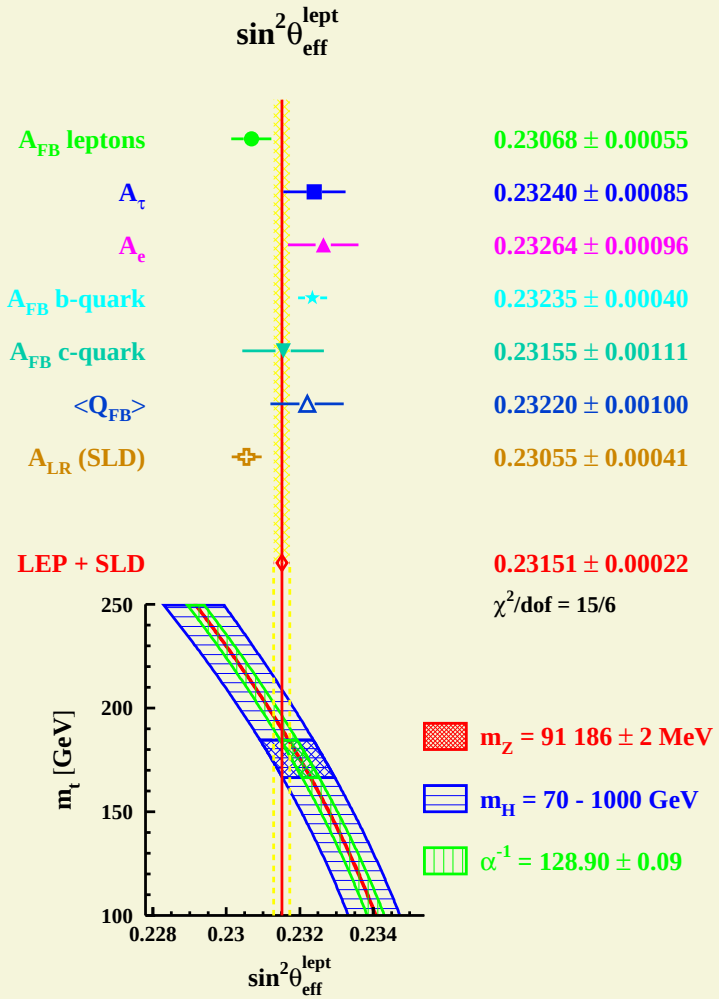
➠ **Reduction of hadronic uncertainty crucial!**

Some remarkable achievements:

TOP mass bound

$m_t^{\rm ind} = 170 \pm 10^{+17}_{-19}$ GeV LEP 1995

with 60 GeV $< m_H <$ 1000 GeV assumed

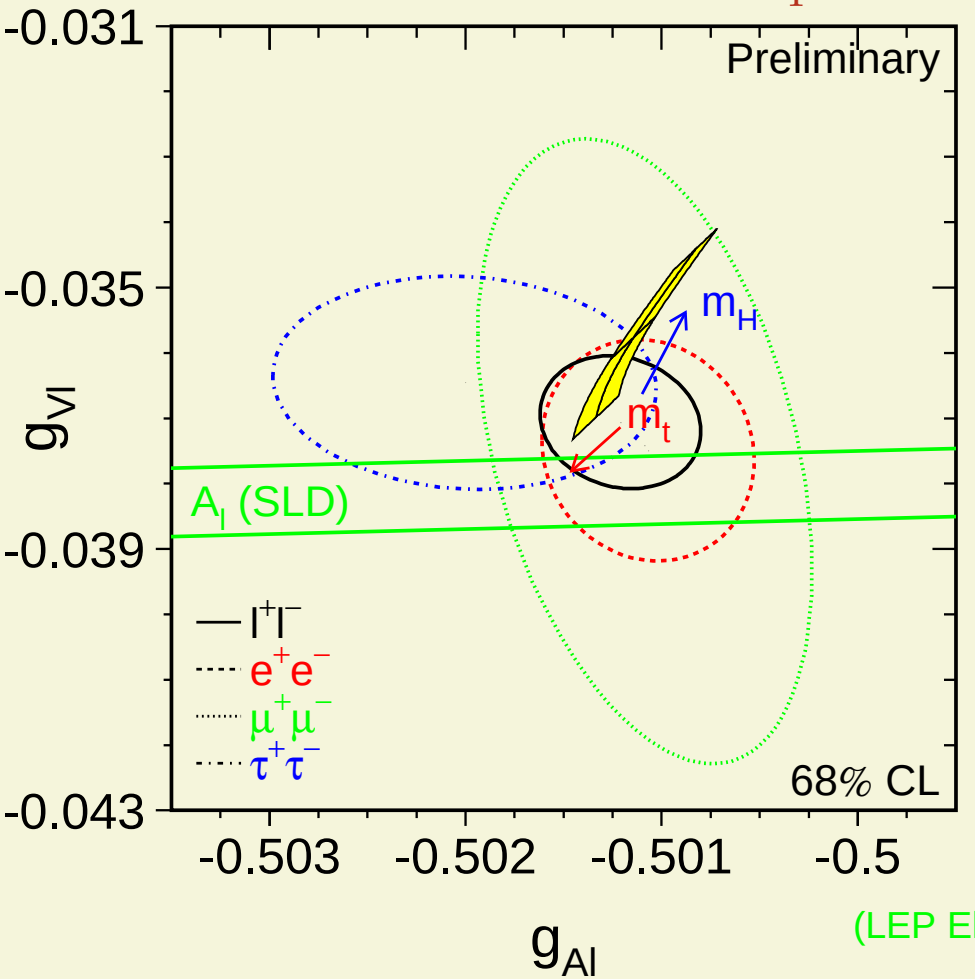


⇒ TOP discovery at Tevatron 1995 $m_t^{\rm dir} = 171.3 \pm 1.6$ GeV CDF/D0 2009

NC couplings

$$\rho^{\ell}_{\rm eff} = -2 g_{\rm Al}$$

$$\sin^2\Theta^{f}_{\rm eff} = \frac{1}{4}\left(1 - g_{\rm Vl}/g_{\rm Al}\right)$$



(LEP Electroweak Working Group: J. Alcaraz et al. 99)

Indirect

Higgs boson mass "measurement"

$m_H = 87^{+35}_{-26}$ **GeV**

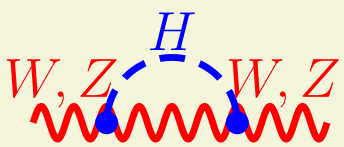
CDF/D0 exclude 160-170 GeV 95% C.L.

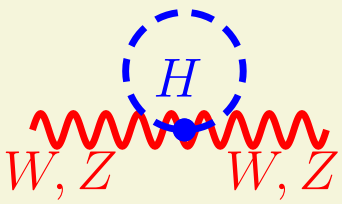
Direct lower bound:

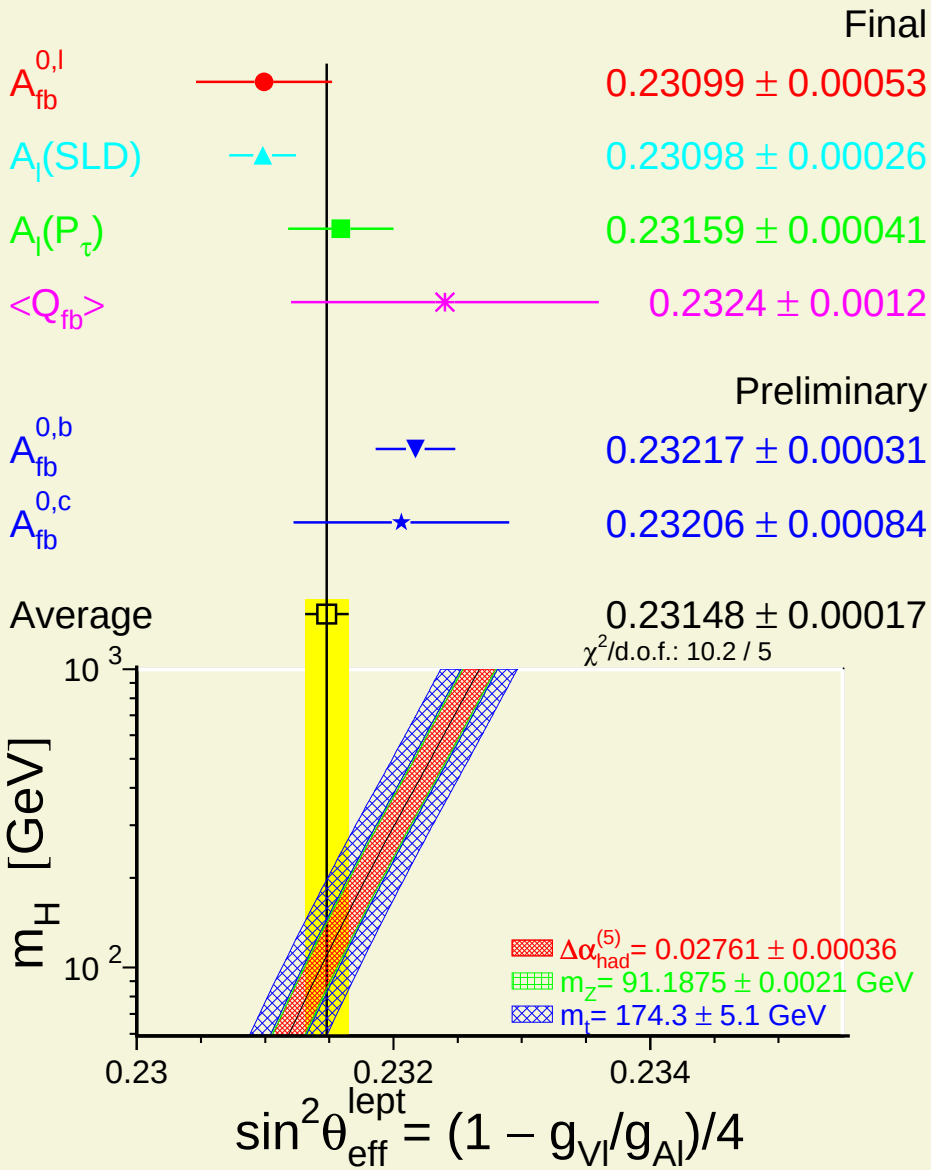
$m_H > 114$ **GeV at 95% CL**

Indirect upper bound:

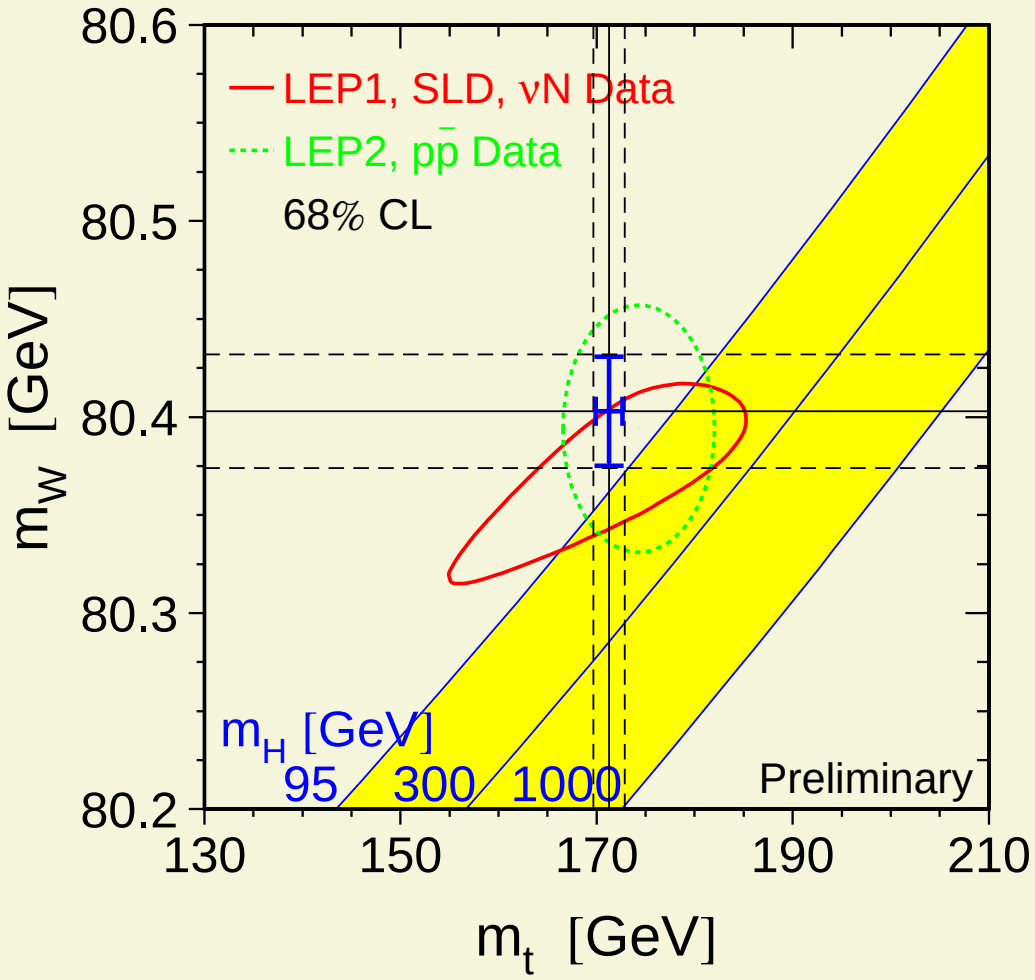
$m_H < 186$ **GeV at 95% CL**







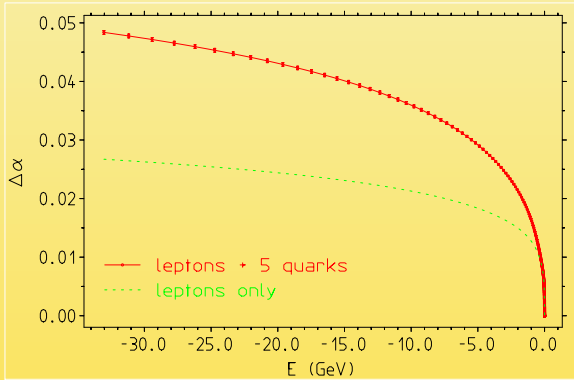
(LEP Electroweak Working Group: D. Abbaneo et al. 03)



(LEP Electroweak Working Group: J. Alcaraz et al. 99)

❑ Running parameters and the leading RC's

Like in QCD also in QED $\alpha_{\rm QED}$ is a running parameter. Most fundamental low energy parameters $\alpha = \alpha(0)$, $G_F = G_\mu$ very precise predictions of W and Z gauge boson properties, together with M_Z and m_t predict everything else. Large radiative corrections in $\alpha(E)$ while $G_\mu(M_Z) = G_\mu(0)$ with high accuracy (no running of VEV $v = 1/\sqrt{\sqrt{2}G_\mu}$ up to the electroweak scale $v \simeq 246$ GeV.



- e^+e^-: renormalized (running) charge $\alpha_{\rm QED}(s)$

$$e^2 \to e^2(s) = \frac{e^2}{1 + (\Pi'_\gamma(s) - \Pi'_\gamma(0))}$$

need photon vacuum polarization $\Pi'_\gamma(s)$; hadrons at low energy contribute

$\Rightarrow$ non–perturbative physics $\Rightarrow$ **dramatic loss in precision!**

- μ–decay: Fermi constant G_μ

$$\frac{G_\mu}{\sqrt{2}} = \frac{g_0^2}{8M_{W0}^2}\left\{\frac{1}{1-\frac{\Pi_W(0)}{M_W^2}} + \cdots\right\}$$

calculable subtraction (renormalization) constant $\Pi_W(0)$ $[m_\mu \ll M_W]$

Besides $\alpha(s)$ also the other gauge coupling $\alpha_2(s)$ cannot be calculated in pQCD, also here no direct evaluation in terms of experimental data possible. Way out: isospin and flavor separation (FJ 1968) The shift $\Delta\alpha_2$ in the $SU(2)_L$ coupling $\alpha_2 = \frac{g^2}{4\pi}$ is analogous to $\Delta\alpha$

$$\begin{aligned}\Delta\alpha_2 &= \Pi'_{3\gamma}(0) - \Pi'_{3\gamma}(M_Z^2) \\ &= \frac{\alpha_2}{12\pi}\Sigma_l|Q_l|(\ln\frac{M_Z^2}{m_l^2} - \frac{5}{3}) + \Delta\alpha^{(5)}_{2,\text{hadrons}}\end{aligned}$$

where the sum extends over the light leptons and

$$\Delta\alpha^{(5)}_{2,\text{hadrons}}(s) = 0.0567 \pm 0.0006 + 0.006184 \cdot \{\ln(s_0/s) + 0.005513 \cdot (s/s_0 - 1)\}$$

is the hadronic contribution of the 5 known light quarks u, d, s, c, b ($\sqrt{s_0} = 91.19$ GeV). Appears in the relation

$$\sin^2\Theta_e = \left\{\frac{1-\Delta\alpha_2}{1-\Delta\alpha} + \Delta_{\nu_\mu e,\text{vertex+box}} + \Delta\kappa_{e,\text{vertex}}\right\}\sin^2\Theta_{\nu_\mu e}$$

LEP $\sin^2\theta^{\text{lep}}_{\text{eff}}$ versus $\sin^2\Theta_{\nu_\mu e}$ of neutrino scattering

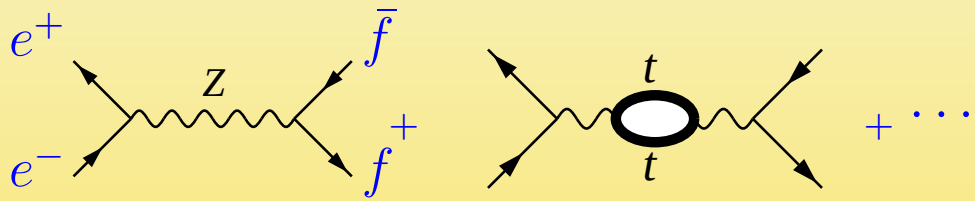
Other huge correction in SM: violation of custodial symmetry (weak isospin breaking) showing up in the ρ–parameter:

$$\rho \doteq \frac{G_{\text{NC}}}{G_\mu} = \frac{M_W^2}{M_Z^2\cos^2\theta_W} = 1 + \Delta\rho\,; \quad \Delta\rho = \frac{\sqrt{2}G_\mu}{16\pi^2}3|m_t^2 - m_b^2| \ .$$

$\longrightarrow$ LEP's m_t prediction Veltman's Nobel prize

Note on indirect m_t measurement by LEP

Another version of vacuum polarization effect: gauge boson propagators
Leading top mass effect $\propto m_t^2$ in LEP process $e^+e^- \to t^*\bar{t}^* \to f\bar{f}$ ($f \neq t$)?



Direct top corrections to $e^+e^- \to f\bar{f}$

In fact:

✗ **no m_t^2 effect**

✓ **is a renormalization effect, if very precisely known CC coupling G_μ used as an input**

➠ **m_t^2 effect via $\rho \doteq G_{\rm NC}/G_{CC} = 1 + \Delta\rho$ from gauge boson self-energies**

$$\Delta\rho \;=\; \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} + \text{ sub leading terms}$$

❑ **$\Pi_Z(0)$ and $\Pi_W(0)$ UV divergent but $\Delta\rho$ finite**

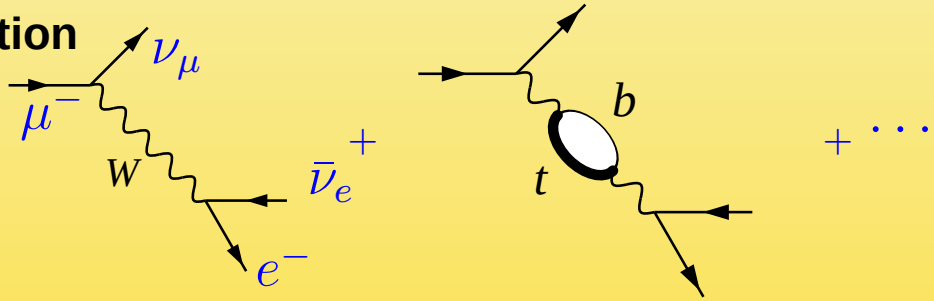
❑ **for $m_t = m_b$ one finds $\Delta\rho = 0$! (tb–doublet contribution)**

Veltman 1977 found that the doublet mass splitting $m_t \neq m_b$ to which the W self-energy is sensitive for large splitting becomes proportional to $|m_t^2 - m_b^2|$

$$\Delta\rho = \frac{\sqrt{2}G_\mu}{16\pi^2}\, 3\,|m_t^2 - m_b^2| + \text{ sub leading terms}$$

which measures the **weak isospin breaking** of the doublet.

W self–energy correction



Top corrections to $\mu^- \to e^-\bar{\nu}_e\nu_\mu$

Note: $G_\mu\, m_t^2 \propto \frac{m_t^2}{v^2} \propto y_t^2$

y_t top Yukawa coupling $\mathcal{L}_{\text{Yukawa}} = -\frac{m_t}{v}\bar{\psi}_t\psi_t H - \cdots$

- Leading top mass effect mediated by the longitudinal component of the W-field
- Slavonv-Taylor identities (expressing local gauge symmetry) in Feynman gauge reads $\partial_\mu W^{\pm\mu} + M_W\,\varphi^\pm = 0$
- $W^\pm$ couples with gauge coupling g via the CC fermion current

❖ **charged unphysical Higgs ghosts $\varphi^\pm$ couple to fermions via the Yukawa couplings like m_t/v or m_b/v**

Calculation in unitary gauge no ghosts, W transversal i.e. $\partial_\mu W^{\pm\mu} = 0$

It looks like a mystery how terms proportional to $(m_t/v)^2$ end up in the numerator in place of the g^2 ($W\bar{t}b$) coupling and t and b masses in the denominators via the quark propagators. Sometimes calculating a Feynman diagram can end up with a real surprise (Wtb very different from Ztt, Zbb).

In unitary gauge: Higgs H only scalar field (physical component)

$$\mathcal{L}_{\rm Yukawa} = -\sum_f m_f\, \bar{\psi}_f \psi_f \left(1 + \frac{H}{v}\right)$$

LEP has been able to measure the Top-Higgs coupling

in spite of the fact that the Higgs itself is not involved neither real nor virtual !

or do we see ghosts?

Remember large SM quantum corrections:

- $\Delta\alpha$ large due to large number of degrees of freedom (colored quarks,leptons) and large logs
- $\Delta\rho$ large single particle power correction,

Note: very important novel feature of SM in contrast to QED and QCD

the large m_t^2 isospin splitting effect is a new feature in QFT

➠ **such effect is forbidden in QED and QCD, where the usually anticipated Appelquist-Carrazone theorem holds**

In "normal" QFT Heavy particles decouple: $O(\mu/M)$ as $M \to \infty$ with $\mu =$energy scale or light mass (masses and couplings independent, masses in propagators only $\to$ damping effect)

➠ **non-decoupling effects direct consequence of fact that in SM masses are generate by SSB (Higgs mechanism)**

➠ **SSB implies mass-coupling relation: in fact significant non-decoupling effect $G_\mu m_t^2 \propto y_t^2$, y_t top Yukawa coupling $\to$ is strong coupling effect**

➠ **first observed by Veltman, measured at LEP and triggered top discovery**

2. Some Basics in QFT

Particle Physics is considered to be **fundamental physics**. What we mean is that we have some **general principles** which constrain the framework more than other physical theories. The most crucial step:

Special Relativity + Quantum Mechanics = Quantum Field Theory

Predicts matter $\longrightarrow$ antimatter! Science fiction turned out to be the reality!

A particularly predictive framework provided it is a **renormalizable** QFT. SM may be understood as kind of unique **minimal** renormalizable extension of QED + weak CC process (Fermi V-A).

❑ Space-time symmetries = symmetries of the dynamics

Quantum Mechanics

Galilei invariance, unitary symmetry, Hilbert space
Schrödinger [NP 1933] equation
Heisenberg uncertainty relation[NP 1932]
Pauli [NP 1933] equation ($SU(2)$-spin), exclusion principle

Newton Mechanics

rotations, space-translations
Galilei invariance

Maxwell Electromagnetism

+ Lorentz [NP 1902] boosts
+ time-translations
Lorentz/Poincaré invariance

Special Relativity

Einstein 1905 [NP 1921]
relativistic kinematics
$\mathcal{P}_+^\uparrow$ invariance

Quantum Field Theory

Dirac 1928 [NP 1933]
$SL(2,C)$ invariance
Wigner states
etc. (see next)
$\left[SL(2,C)' \sim \mathcal{P}_+^{\uparrow\,\prime} \right]$

❑ Quantum Field Theory: some general properties

❖ **Particle-antiparticle crossing symmetry:**

ANTIMATTER predicted 1928 by Dirac [NP 1933]

to each particle an antiparticle must exist charge conjugation C new symmetry !

electron $\longrightarrow$ positron [discovered 1932 by C. Anderson [NP 1936]]

❖ **Spin-statistics theorem:** (Fierz 1939, Pauli 1940)

fermions $=\frac{1}{2}$ odd integer, bosons=integer spin (consequence of Einstein causality)

❖ **CPT theorem:** (Schwinger 1951, Lüders, Pauli, Bell 1954)

product of C, P and T in any order universal symmetry of any local relativistic QFT

1948: Tomonaga, Schwinger and Feynman [NP 1965] QED renormalizable QFT

to all orders in perturbation theory mathematically well-defined

charge and mass as only free parameters [the ones in the Lagrangian],

very predictive [Lamb shift, $g-2$, etc.] 20 years after Dirac!

Besides relativistic covariance, main problem Dirac's Fermi sea [vacuum filled with electrons, positrons as holes in the sea] $\longrightarrow$ infinities physical; solution: QFT built on empty vacuum, positron are particles like an electron related by a symmetry (C). $\Rightarrow$ Dogma of an empty vacuum! (quantum fluctuations only)

Scheme: classical local Lagrangian specified by symmetries + quantization rules in conjunction with scattering theory setup (expansion in free fields). Key object: the *unitary* scattering matrix S which encodes the perturbative interaction processes and is given by

$$S = T\left(\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}^{(0)}_{\mathrm{int}}(x)}\right)\Big|_{\otimes} \quad .$$

$\otimes$ = omission of vacuum graphs (proper normalization of S)

Unitarity requires

$$SS^+ = S^+S = 1 \;\Leftrightarrow\; S^+ = S^{-1}\,,$$

infers conservation of quantum mechanical transition probabilities

Crucial: T–prescription = time ordering of all operators

$$T\left\{\phi(x)\phi(y)\right\} = \Theta(x^0 - y^0)\phi(x)\phi(y) \pm \Theta(y^0 - x^0)\phi(y)\phi(x)$$

Einstein locality ($v < c$) $\longrightarrow$ statistics: ❑ $\pm \begin{cases} \text{bosons} \\ \text{fermions} \end{cases}$

Implements scattering picture boundary condition ((LSZ reduction)): fields under the T product up to a possible sign (in case of Fermi statistics) always stand in the order of decreasing time arguments (from left to right). Fields to the left are associated with outgoing states those to the right are associated with incoming states, in relation to scattering matrix elements.

The full Green functions of the interacting fields like $A^\mu(x)$, $\psi(x)$, etc. can be expressed completely in terms of corresponding free fields via the Gell-Mann Low formula (interaction picture) e.g. electromagnetic vertex

$$\langle 0|T\left\{A_\mu(x)\psi_\alpha(y)\bar\psi_\beta(\bar y)\right\}|0\rangle =$$
$$\langle 0|T\left\{A^{(0)}_\mu(x)\psi^{(0)}_\alpha(y)\bar\psi^{(0)}_\beta(\bar y)\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x'\,\mathcal{L}^{(0)}_{\mathrm{int}}(x')}\right\}|0\rangle_\otimes = \sum_{n=0}^{N}\frac{\mathrm{i}^n}{n!}\int \mathrm{d}^4z_1\cdots\mathrm{d}^4z_n$$
$$\langle 0|T\left\{A^{(0)}_\mu(x)\psi^{(0)}_\alpha(y)\bar\psi^{(0)}_\beta(\bar y)\,\mathcal{L}^{(0)}_{\mathrm{int}}(z_1)\cdots\,\mathcal{L}^{(0)}_{\mathrm{int}}(z_n)\right\}|0\rangle_\otimes + O(\mathrm{e}^{N+1})$$

- $\mathcal{L}^{(0)}_{\mathrm{int}}(x)$ **the interaction part of the Lagrangian**
- **on right hand side free fields only, everything known**
- **expanding exponential $\Rightarrow$ perturbation expansion, works if coupling small**
- **formal series not well defined requires regularization and renormalization**
- **Feynman rules allow for straight forward evaluation.**

Note: path integral quantization

$$Z\{J,\bar\chi,\chi,\cdots\} = \int \mathcal{D}V_{\mu i}\mathcal{D}\psi\mathcal{D}\bar\psi e^{i\int(\mathcal{L}_{eff}+JV+\bar\chi\psi+\bar\psi\chi+\cdots)}$$

the generating functional of the time-ordered Green functions.

Crucial for non-Abelian gauge theories like the SM ('t Hooft) no splitting into free and interaction part (non-gaugeinvariant splitting)

Provides:

- ❑ non-perturbative definition of theory, basic for lattice QCD [discretized]
- ❑ relationship between Euclidean and Minkowski quantum field theory (Wick rotation)
- ❑ *Osterwalder–Schrader theorem* ascertains that the Euclidean correlation functions of fields can be analytically continued to Minkowski space
- ❑ full Minkowski QFT including its S–matrix, if it exists, can be reconstructed from the knowledge of the Euclidean correlation functions
- ❑ from a mathematical point of view the Minkowski and the Euclidean version of a QFT are completely equivalent, Special Relativity=O(4) rotation symmetry!

Technical aspects: ➠ non-perturbative approach to QFT, ❖ transparent approach to quantization and renormalization of non-Abelian gauge theories

❑ States vs Fields

Invert field decomposition in terms of creation and annihilation operators

Dirac field:

$$a(\vec{p},r) = \bar{u}(\vec{p},r)\gamma^0 \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px}\, \psi(x) \quad , \quad b^+(\vec{p},r) = \bar{v}(\vec{p},r)\gamma^0 \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}px}\, \psi(x) \ .$$

Photon field:

$$c(\vec{p},\lambda) = -\,\varepsilon^{\mu *}(\vec{p},\lambda)\, \mathrm{i} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px}\, \overleftrightarrow{\partial}_0\, A_\mu(x)$$

Since these operators create or annihilate scattering states, the above relations provide the bridge between the Green functions, the vacuum expectation values of time–ordered fields, and the scattering matrix elements.

S**–matrix elements are the residues of the one–particle poles of the time–ordered Green functions. Each external field has 4 possible interpretations as a state:**

- ❖ **incoming particle**
- ❖ **outgoing particle**
- ❖ **incoming antiparticle**
- ❖ **outgoing antiparticle**

One <u>analytic</u> function several physical processes. This is QFT!

Main consequences:

- **particle–antiparticle crossing, C–invariance of QED and QCD**
- **spin–statistics theorem: integer spin bosons, half-integer spin fermions**
- **CPT–invariance of any QFT $\Rightarrow$ CP-violation is T [maximal P $\overset{\sim}{\leftrightarrow}$ maximal C]**

What we finally need is T–matrix:

$$S = I + \mathrm{i}\,(2\pi)^4\,\delta^{(4)}(P_f - P_i)\,T \quad .$$

with unity(no-scattering) split off and the overall four–momentum conservation factored out.

Cross-sections

$$\mathrm{d}\sigma = \frac{(2\pi)^4\delta^{(4)}\left(P_f - P_i\right)}{2\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}}\;|\;T_{fi}\;|^2\;\mathrm{d}\mu(p_1')\mathrm{d}\mu(p_2')\cdots$$

Life-times $\tau = 1/\Gamma$:

$$\mathrm{d}\Gamma = \frac{(2\pi)^4\delta^{(4)}\left(P_f - P_i\right)}{2m_1}\;|\;T_{fi}\;|^2\;\mathrm{d}\mu(p_1')\mathrm{d}\mu(p_2')\cdots$$

etc.

3. The Origin of Analyticity

At the heart of analyticity is the causality. The time ordered Green functions which encode all information of the theory in perturbation theory are given by integrals over products of causal propagators ($z = x - y$)

$$\begin{aligned} \mathrm{i}S_{\mathrm{F}}(z) &= \langle 0|T\left\{\psi(x)\bar{\psi}(y)\right\}|0\rangle \\ &= \Theta(x^0 - y^0)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle - \Theta(y^0 - x^0)\langle 0|\bar{\psi}(y)\psi(x)|0\rangle \\ &= \Theta(z^0)\,\mathrm{i}S^+(z) + \Theta(-z^0)\,\mathrm{i}S^-(z) \end{aligned}$$

exhibiting a positive frequency part propagating forward in time and a negative frequency part propagating backward in time. The Θ function of time ordering makes the Fourier–transform to be analytic in a half–plane in momentum space. For $K(\tau = z^0) = \Theta(z^0)\mathrm{i}S^+(z)$, for example, we have

$$\tilde{K}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}\tau K(\tau)\,\mathrm{e}^{\mathrm{i}\omega\tau} = \int_{0}^{+\infty} \mathrm{d}\tau K(\tau)\,\mathrm{e}^{-\eta\tau}\mathrm{e}^{\mathrm{i}\xi\tau}$$

such that $\tilde{K}(\omega = \xi + \mathrm{i}\eta)$ is a regular analytic function in the **upper half** ω**–plane** $\eta > 0$. This of course only works because τ is restricted to be positive, which means causal.

In a relativistically covariant world, in fact, we always need two terms, a positive frequency part $\Theta(z^0 = t - t')\,S^+(z)$, corresponding to the particle propagating forward in time, and a negative frequency part $\Theta(-z^0 = t' - t)\,S^-(z)$, corresponding to the antiparticle propagating backward in time. The two terms correspond in momentum space to the two terms of exhibiting the simple poles.

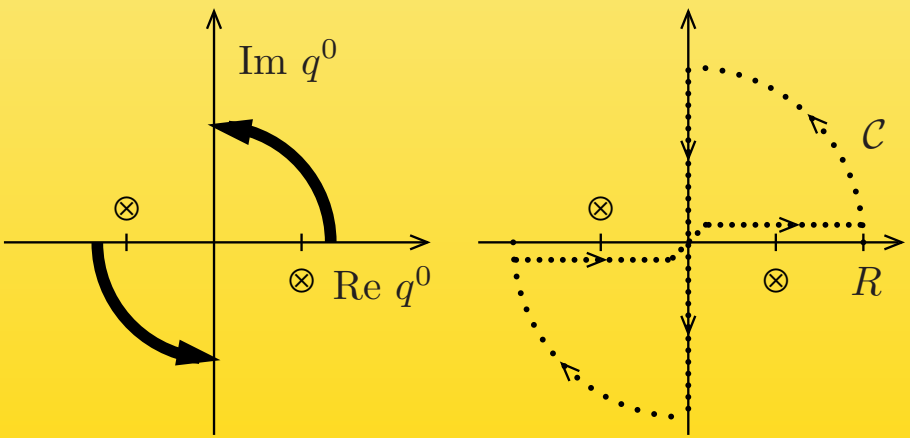
Of course, for a free Dirac field we know what the Stückelberg-Feynman propagator in momentum space looks like

$$\tilde{S}_{\rm F}(q) = \frac{\not{q} + m}{q^2 - m^2 + {\rm i}\varepsilon}$$

and its analytic properties are manifest: as

$$\begin{aligned}\frac{1}{q^2 - m^2 + {\rm i}\varepsilon} &= \frac{1}{q^0 - \sqrt{\vec{q}^{\,2} + m^2 - {\rm i}\varepsilon}}\,\frac{1}{q^0 + \sqrt{\vec{q}^{\,2} + m^2 - {\rm i}\varepsilon}} \\ &= \frac{1}{2\omega_p}\left\{\frac{1}{q^0 - \omega_p + {\rm i}\varepsilon} - \frac{1}{q^0 + \omega_p - {\rm i}\varepsilon}\right\}\end{aligned}$$

is an analytic function in q^0 with poles at $q^0 = \pm(\,\omega_p - {\rm i}\varepsilon)$. Is is the basis for the Wick rotation to the Euclidean region



Wick rotation in the complex q^0–plane. The poles of the Feynman propagator are indicated by $\otimes$'s. $\mathcal{C}$ is an integration contour, R is the radius of the arcs.

More precisely: analyticity of a function $\tilde{f}(q^0, \vec{q}\,)$ in q^0 implies that the **contour integral**

$$\oint_{\mathcal{C}(R)} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) \;= 0$$

for the closed path $C(R)$ vanishes. If the function $\tilde{f}(q^0, \vec{q}\,)$ falls off sufficiently fast at infinity, then the contribution from the two "arcs" goes to zero when the radius of the contour $R \to \infty$. In this case we obtain

$$\int\limits_{-\infty}^{\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) + \int\limits_{+\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) = 0$$

or

$$\int\limits_{-\infty}^{\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) = \int\limits_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \mathrm{d}q^d\, \tilde{f}(-\mathrm{i}q^d, \vec{q}\,) \;\;,$$

which is the Wick rotation. At least in perturbation theory, one can prove that the conditions required to allow us to perform a Wick rotation are fulfilled.

We notice that the Euclidean Feynman propagator obtained by the Wick rotation

$$\frac{1}{q^2 - m^2 + \mathrm{i}\varepsilon} \to -\frac{1}{\underline{q}^2 + m^2}$$

has no singularities (poles) and an $\mathrm{i}\varepsilon$–prescription is not needed any longer.

Analyticity is an extremely important basic property of a QFT and a powerful instrument which helps to solve seemingly purely "technical" problems as we will see. For example it allows us to perform a Wick rotation to Euclidean space and in Euclidean space a QFT looks like a classical statistical system and one can apply the methods of statistical physics to QFT. In particular the numerical approach to the intrinsically non–perturbative QCD via lattice QCD is based on analyticity. The objects which manifestly exhibit the analyticity properties and are providing the bridge to the Euclidean world are the *time ordered Green functions*.

Note that by far not all objects of interest in a QFT are analytic. For example, any solution of the homogeneous (no source) Klein-Gordon equation

$$(\Box_x + m^2) \, \Delta(x-y;m^2) = 0 \; ,$$

like the so called positive frequency part Δ^+ or the causal commutator Δ of a free scalar field $\varphi(x)$, defined by

$$\begin{aligned} < 0 \mid \varphi(x), \varphi(y) \mid 0 > &= \mathrm{i} \, \Delta^+(x-y;m^2) \\ [\varphi(x), \varphi(y)] &= \mathrm{i} \, \Delta(x-y;m^2) \; , \end{aligned}$$

which, given the properties of the free field, may easily be evaluated to have a representation

$$\begin{aligned} \Delta^+(z;m^2) &= -\mathrm{i} \, (2\pi)^{-3} \int \mathrm{d}^4p \, \Theta(p^0) \, \delta(p^2-m^2) \, \mathrm{e}^{-\mathrm{i}pz} \\ \Delta(z;m^2) &= -\mathrm{i} \, (2\pi)^{-3} \int \mathrm{d}^4p \, \epsilon(p^0) \, \delta(p^2-m^2) \, \mathrm{e}^{-\mathrm{i}pz} \; . \end{aligned}$$

Thus, in momentum space, as solutions of

$$(p^2 - m^2)\,\tilde{\Delta}(p) = 0\ ,$$

only singular ones exist. For the positive frequency part and the causal commutator they read

$$\Theta(p^0)\,\delta(p^2 - m^2) \quad \text{and} \quad \epsilon(p^0)\,\delta(p^2 - m^2)\ ,$$

respectively. The Feynman propagator, in contrast, satisfies an inhomogeneous (with point source) Klein-Gordon equation

$$(\,\Box_x + m^2\,)\,\Delta_F(x-y;m^2) = -\delta^{(4)}(x-y)\ .$$

The δ function comes from differentiating the Θ function factors of the T product. Now we have

$$\langle 0|T\,\{\varphi(x),\varphi(y)\}\,|0\rangle \quad = \quad \mathrm{i}\,\Delta_F(x-y;m^2)$$

with

$$\Delta_F(z;m^2) = (2\pi)^{-4}\int \mathrm{d}^4p\,\frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon}\,\mathrm{e}^{-\mathrm{i}pz}$$

and in momentum space

$$(p^2 - m^2)\,\tilde{\Delta}_F(p) = 1\ ,$$

obviously has analytic solutions, a particular one being the scalar Feynman propagator

$$\frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon} = \mathcal{P}\left(\frac{1}{p^2 - m^2}\right) - \mathrm{i}\,\pi\,\delta(p^2 - m^2)\ .$$

The $\mathrm{i}\varepsilon$ prescription used here precisely correspond to the boundary condition imposed by the time ordering prescription T in configuration space. The symbol $\mathcal{P}$ denotes the principal value; the right hand side exhibits the splitting into real and imaginary part.

Note:

✓ $1/(p^2 - m^2)$ makes sense in Minkowski and Euclidean region

✗ while $\delta(p^2 - m^2) \equiv 0 \,\forall\, p^2 < 0$ does not

Note: Causality is fundamental in any physical theory (predictability of the future) [unlike in economics] i.e. also non-relativistic theories are causal

- non-relativistic causality ➠ analyticity in half-plain
- relativistic causality particle (forward) antiparticle (backward) yields analyticity in entire energy plane
- only in relativistic theory Wick rotation is possible (equivalence Euclidean vs Minkowski)

4. Properties of the Form Factors

We again consider the interaction of a lepton in an external field: the relevant T–matrix element is

$$T_{fi} = e\mathcal{J}^{\mu}_{fi}\tilde{A}^{\rm ext}_{\mu}(q)$$

with

$$\mathcal{J}^{\mu}_{fi} = \bar{u}_2\Gamma^{\mu}u_1 = \langle f|j^{\mu}_{\rm em}(0)|i\rangle = \langle \ell^-(p_2)|j^{\mu}_{\rm em}(0)|\ell^-(p_1)\rangle \ .$$

By the crossing property we have the following channels:

- Elastic ℓ^- scattering: $s = q^2 = (p_2 - p_1)^2 \leq 0$
- Elastic ℓ^+ scattering: $s = q^2 = (p_1 - p_2)^2 \leq 0$
- Annihilation (or pair–creation) channel: $s = q^2 = (p_1 + p_2)^2 \geq 4m_\ell^2$

The domain $0 < s < 4m_\ell^2$ is unphysical. A look at the unitarity condition

$$\mathrm{i}\left\{T^*_{if} - T_{fi}\right\} = \sum\!\!\!\!\!\!\int_n (2\pi)^4\,\delta^{(4)}(P_n - P_i)\,T^*_{nf}T_{ni}\ ,$$

which derives from unitarity of S and the definition of the T–matrix, taking $\langle f|S^+S|i\rangle$ and inserting a complete set of intermediate states, tells us that for $s < 4m_\ell^2$ there is no physical state $|n\rangle$ allowed by energy and momentum conservation and thus

$$T_{fi} = T^*_{if} \quad \text{for} \quad s < 4m_\ell^2\ ,$$

which means that the current matrix element is hermitian. As the electromagnetic potential $A_\mu^{\rm ext}(x)$ is real its Fourier transform satisfies

$$\tilde{A}_\mu^{\rm ext}(-q) = \tilde{A}_\mu^{*\,\rm ext}(q)$$

and hence

$$\mathcal{J}_{fi}^\mu = \mathcal{J}_{if}^{\mu *} \quad \text{for} \quad s < 4m_\ell^2 \ .$$

If we interchange initial and final state the four–vectors p_1 and p_2 are interchanged such that q changes sign: $q \to -q$. The unitarity relation for the form factor decomposition of $\bar{u}_2\, \Pi^\mu_{\gamma\ell\ell}\, u_1$ thus reads ($u_i = u(p_i, r_i)$)

$$\begin{aligned}
& \bar{u}_2 \left(\gamma^\mu F_{\rm E}(q^2) + {\rm i}\sigma^{\mu\nu}\frac{q_\nu}{2m} F_{\rm M}(q^2)\right) u_1 \\
= \quad & \left\{\bar{u}_1 \left(\gamma^\mu F_{\rm E}(q^2) - {\rm i}\sigma^{\mu\nu}\frac{q_\nu}{2m} F_{\rm M}(q^2)\right) u_2\right\}^* \\
= \quad & u_2^+ \left(\gamma^{\mu +} F_{\rm E}^*(q^2) + {\rm i}\sigma^{\mu\nu +}\frac{q_\nu}{2m} F_{\rm M}^*(q^2)\right) \bar{u}_1^+ \\
= \quad & \bar{u}_2 \left(\gamma^\mu F_{\rm E}^*(q^2) + {\rm i}\sigma^{\mu\nu}\frac{q_\nu}{2m} F_{\rm M}^*(q^2)\right) u_1 \ .
\end{aligned}$$

The last equality follows using $u_2^+ = \bar{u}_2\gamma^0,\ \bar{u}_1^+ = \gamma^0 u_1,\ \gamma_5^+ = \gamma_5,\ \gamma^0\gamma_5\gamma^0 = -\gamma_5,\ \gamma^0\gamma^{\mu +}\gamma^0 = \gamma^\mu$ and $\gamma^0\sigma^{\mu\nu +}\gamma^0 = \sigma^{\mu\nu}$. Unitarity thus implies that the form factors are real

$$\text{Im}\, F(s)_i = 0 \quad \text{for} \quad s < 4m_e^2$$

below the threshold of pair production $s = 4m_e^2$. For $s \geq 4m_e^2$ the form factors are complex; they are analytic in the complex s–plane with a cut along the positive axis starting at $s = 4m_e^2$. In the annihilation channel ($p_- = p_2,\ p_+ = -p_1$)

$$\langle 0|j^\mu_{\rm em}(0)|p_-,\ p_+\rangle = \sum\!\!\!\!\!\!\int_n \langle 0|j^\mu_{\rm em}(0)|n\rangle\langle n|p_-,\ p_+\rangle\ ,$$

where the lowest state $|n\rangle$ contributing to the sum is an e^+e^- state at threshold : $E_+ = E_- = m_e$ and $\vec{p_+} = \vec{p_-} = 0$ such that $s = 4m_e^2$. Because of the causal $\mathrm{i}\varepsilon$–prescription in the time–ordered Green functions the amplitudes change sign when $s \longrightarrow s^*$ and hence

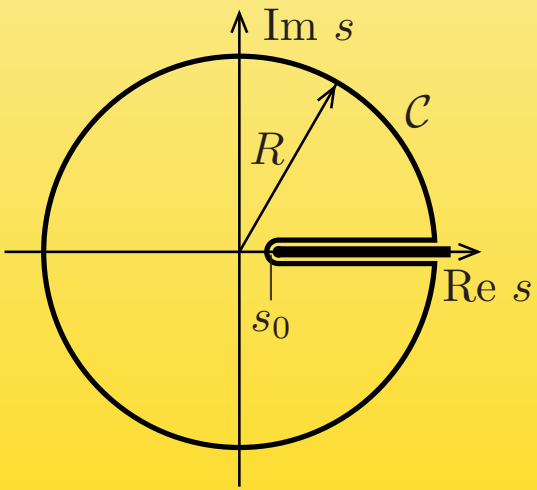
$$\boxed{F_i(s^*) = F_i^*(s)\ ,}$$

which is the Schwarz reflection principle.

5. Dispersion Relations

Causality together with unitarity imply analyticity of the form factors in the complex s–plane except for the cut along the positive real axis starting at $s \geq 4m_\ell^2$. Cauchy's integral theorem tells us that the contour integral, for the contour $\mathcal{C}$ shown in the Figure, satisfies

$$F_i(s) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\mathrm{d}s' F(s')}{s' - s} \quad .$$



Analyticity domain and Cauchy contour $\mathcal{C}$ for the lepton form factor (vacuum polarization). $\mathcal{C}$ is a circle of radius R with a cut along the positive real axis for $s > s_0 = 4m^2$ where m is the mass of the lightest particles which can be pair–produced

Since $F^*(s) = F(s^*)$ the contribution along the cut may be written as

$$\lim_{\varepsilon\to 0}\left(F(s+\mathrm{i}\varepsilon) - F(s-\mathrm{i}\varepsilon)\right) = 2\,\mathrm{i}\,\mathrm{Im}\,F(s)\;;\quad s\ \text{real}\,,\ s>0$$

and hence for $R\to\infty$

$$F(s) = \lim_{\varepsilon\to 0} F(s+\mathrm{i}\varepsilon) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int\limits_{4m^2}^{\infty}\mathrm{d}s'\,\frac{\mathrm{Im}\,F(s')}{s'-s-\mathrm{i}\varepsilon} + \mathcal{C}_\infty\ .$$

In all cases where $F(s)$ falls off sufficiently rapidly as $|s|\to\infty$ the boundary term $\mathcal{C}_\infty$ vanishes and the integral converges. This may be checked order by order in perturbation theory. In this case the “un–subtracted” dispersion relation (DR)

$$F(s) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int\limits_{4m^2}^{\infty}\mathrm{d}s'\,\frac{\mathrm{Im}\,F(s')}{s'-s-\mathrm{i}\varepsilon}$$

uniquely determines the function by its imaginary part. A technique based on DRs is frequently used for the calculation of Feynman integrals, because the calculation of the imaginary part is simpler in general. The real part which actually is the object to be calculated is given by the principal value ($\mathcal{P}$) integral

$$\mathrm{Re}\,F(s) = \frac{1}{\pi}\,\mathcal{P}\!\!\!\!\!\!\int\limits_{4m^2}^{\infty}\mathrm{d}s'\,\frac{\mathrm{Im}\,F(s')}{s'-s}\ ,$$

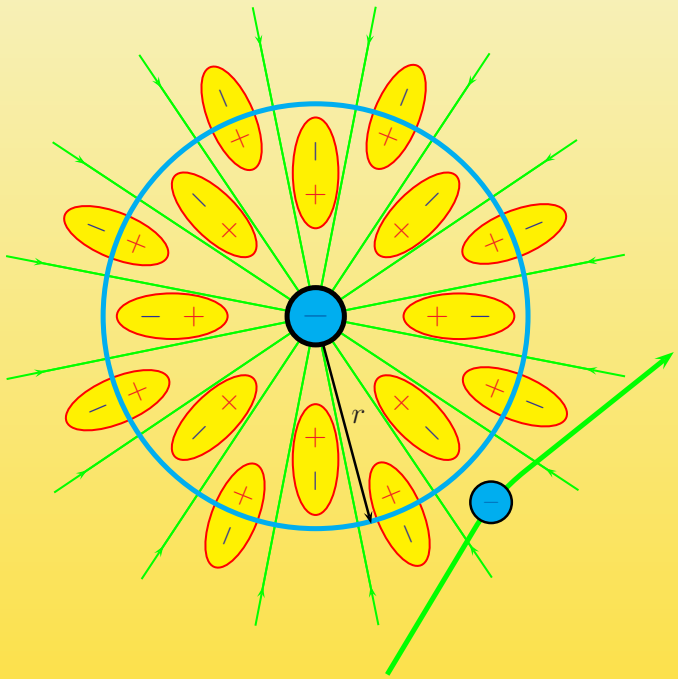
which is also known under the name Hilbert transform.

For our form factors the fall off condition is satisfied for the Pauli form factor $F_{\rm M}$ but not for the Dirac form factor $F_{\rm E}$. In the latter case the fall off condition is not satisfied because $F_{\rm E}(0) = 1$ (charge renormalization condition = subtraction condition). However, performing a subtraction of $F_{\rm E}(0)$, one finds that $(F_{\rm E}(s) - F_{\rm E}(0))/s$ satisfies the "subtracted" dispersion relations

$$\frac{F(s) - F(0)}{s} = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{4m^2}^{\infty} {\rm d}s' \, \frac{{\rm Im}\, F(s')}{s'(s' - s - {\rm i}\varepsilon)} \, ,$$

which exhibits one additional power of s' in the denominator and hence improves the damping of the integrand at large s' by one additional power. Order by order in perturbation theory the dispersion integral is convergent for the Dirac form factor. A very similar relation is satisfied by the vacuum polarization amplitude which we will discuss in the following section.

6. Dispersion Relations and the Vacuum Polarization



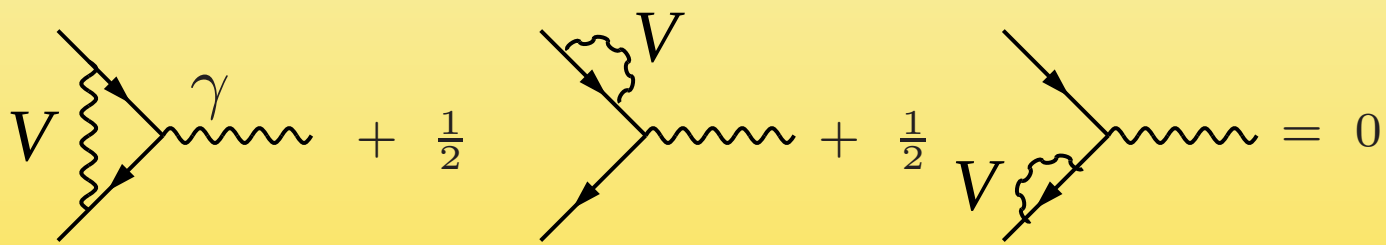
Vacuum polarization causing charge screening by virtual pair creation and re–annihilation

Dispersion relations play an important role for taking into account the photon propagator contributions. The related photon self–energy, obtained from the photon propagator by the amputation of the external photon lines, is given by the correlator of two electromagnetic currents and may be interpreted as vacuum polarization for the following reason:

The em Ward-Takahashi identity (current conservation) in QED reds $Z_f \;\; \Gamma^\mu(p,p)|_{\rm on-shell} = -e\gamma^\mu$ or

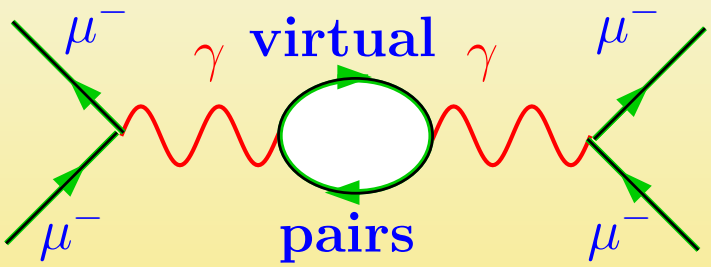
$$-e\gamma^\mu \delta Z_f + \Gamma^{'\mu}(p,p)\Big|_{\rm on-shell} = 0$$

where the prime denotes the non–trivial part of the vertex function. This relation tells us that some of the diagrams directly cancel. For example, we have ($V = \gamma$)



implies that in QED (not in the SM) charge renormalization is caused solely by the photon self–energy correction; the fundamental electromagnetic fine structure constant α in fact is a function of the energy scale $\alpha \to \alpha(E)$ of a process due to charge screening. The latter is a result of the fact that a naked charge is surrounded by a cloud of virtual particle–antiparticle pairs (dipoles mostly) which line up in the field of the central charge and such lead to a vacuum polarization which screens the central charge. This is illustrated in the figure. From long distances (classical charge) one thus sees less charge than if one comes closer, such that one sees an increasing charge with increasing energy.

Diagrammatic representation of a vacuum polarization effect:



$$\gamma^* \to e^+e^-, \mu^+\mu^-, \tau^+\tau^-, u\bar{u}, d\bar{d}, \cdots \to \gamma^*$$

Feynman diagram describing the vacuum polarization in muon scattering

The vacuum polarization affects the photon propagator in that the full or dressed propagator is given by the geometrical progression of self–energy insertions $-\mathrm{i}\Pi_\gamma(q^2)$. The corresponding Dyson summation implies that the free propagator is replaced by the dressed one

$$\mathrm{i}D_\gamma^{\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2 + \mathrm{i}\varepsilon} \to \mathrm{i}D_\gamma^{'\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2 + \Pi_\gamma(q^2) + \mathrm{i}\varepsilon}$$

modulo unphysical gauge dependent terms. By $U(1)_{\mathrm{em}}$ gauge invariance the photon remains massless and hence we have $\Pi_\gamma(q^2) = \Pi_\gamma(0) + q^2\,\Pi'_\gamma(q^2)$ with $\Pi_\gamma(0) \equiv 0$. As a result we obtain

$$\mathrm{i}D_\gamma^{'\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2\,(1 + \Pi'_\gamma(q^2))} + \text{ gauge terms}$$

where the “gauge terms” will not contribute to gauge invariant physical quantities, and need not be considered further.

Including a factor e^2 and considering the renormalized propagator (wave function renormalization factor Z_γ) we have

$$\mathrm{i}\, e^2\, D_\gamma^{'\mu\nu}(q) = \frac{-\mathrm{i} g^{\mu\nu}\, e^2\, Z_\gamma}{q^2\, \left(1 + \Pi'_\gamma(q^2)\right)} + \text{ gauge terms}$$

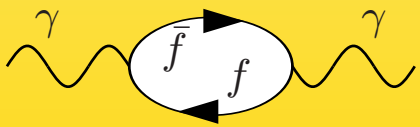
which in effect means that the charge has to be replaced by a **running charge**

$$e^2 \to e^2(q^2) = \frac{e^2 Z_\gamma}{1 + \Pi'_\gamma(q^2)} \quad .$$

The wave function renormalization factor Z_γ is fixed by the condition that at $q^2 \to 0$ one obtains the classical charge (charge renormalization in the *Thomson limit*. Thus the renormalized charge is

$$e^2 \to e^2(q^2) = \frac{e^2}{1 + (\Pi'_\gamma(q^2) - \Pi'_\gamma(0))}$$

where the lowest order diagram in perturbation theory which contributes to $\Pi'_\gamma(q^2)$ is



and describes the virtual creation and re–absorption of fermion pairs

$$\gamma^* \to e^+e^-, \mu^+\mu^-, \tau^+\tau^-, u\bar{u}, d\bar{d}, \cdots \to \gamma^*.$$

In terms of the fine structure constant $\alpha = \frac{e^2}{4\pi}$ reads

$$\alpha(q^2) = \frac{\alpha}{1-\Delta\alpha} \quad ; \quad \Delta\alpha = -\mathrm{Re}\,\left(\Pi'_\gamma(q^2) - \Pi'_\gamma(0)\right) \quad .$$

The various contributions to the shift in the fine structure constant come from the leptons (lep = e, μ and τ) the 5 light quarks (u, d, s, c, and b and the corresponding hadrons = had) and from the top quark:

$$\Delta\alpha = \Delta\alpha_{\rm lep} + \Delta^{(5)}\alpha_{\rm had} + \Delta\alpha_{\rm top} + \cdots$$

Also W–pairs contribute at $q^2 > M_W^2$. While the other contributions can be calculated order by order in perturbation theory the hadronic contribution $\Delta^{(5)}\alpha_{\rm had}$ exhibits low energy strong interaction effects and hence cannot be calculated by perturbative means. Here the dispersion relations play a key role.

The leptonic contributions are calculable in perturbation theory. Using our result for the renormalized photon self–energy, at leading order the free lepton loops yield

$$\begin{aligned}
\Delta\alpha_{\rm lep}(q^2) = & \\
= & \sum_{\ell=e,\mu,\tau} \frac{\alpha}{3\pi}\left[-\frac{5}{3} - y_\ell + (1+\frac{y_\ell}{2})\sqrt{1-y_\ell}\,\ln\left(\left|\frac{\sqrt{1-y_\ell}+1}{\sqrt{1-y_\ell}-1}\right|\right)\right] \\
= & \sum_{\ell=e,\mu,\tau} \frac{\alpha}{3\pi}\left[-\frac{8}{3} + \beta_\ell^2 + \frac{1}{2}\beta_\ell(3-\beta_\ell^2)\ln\left(\left|\frac{1+\beta_\ell}{1-\beta_\ell}\right|\right)\right] \\
= & \sum_{\ell=e,\mu,\tau} \frac{\alpha}{3\pi}\left[\ln\left(|q^2|/m_\ell^2\right) - \frac{5}{3} + O\left(m_\ell^2/q^2\right)\right] \text{ for } |q^2| \gg m_\ell^2 \\
\simeq & \; 0.03142 \text{ for } q^2 = M_Z^2
\end{aligned}$$

where $y_\ell = 4m_\ell^2/q^2$ and $\beta_\ell = \sqrt{1-y_\ell}$ are the lepton velocities[a]. This leading contribution is affected by small electromagnetic corrections only in the next to leading order. The leptonic contribution is actually known to three loops at which it takes the value

$$\Delta\alpha_{\rm leptons}(M_Z^2) \;\simeq\; 314.98 \;\times\; 10^{-4}.$$

As already mentioned, in contrast, the corresponding free quark loop contribution gets substantially modified by low energy strong interaction effects, which cannot be calculated reliably by perturbative QCD. The evaluation of the hadronic contribution will be discussed later.

Vacuum polarization effects are large when large scale changes are involved (large logarithms) and because of the large number of light fermionic degrees of freedom as we infer from the asymptotic form in perturbation theory

$$\Delta\alpha^{\rm pert}(q^2) \;\simeq\; \frac{\alpha}{3\pi}\sum_f Q_f^2 N_{cf}\left(\ln\frac{|q^2|}{m_f^2} - \frac{5}{3}\right) \;\;;\;\; |q^2| \gg m_f^2 \;\;.$$

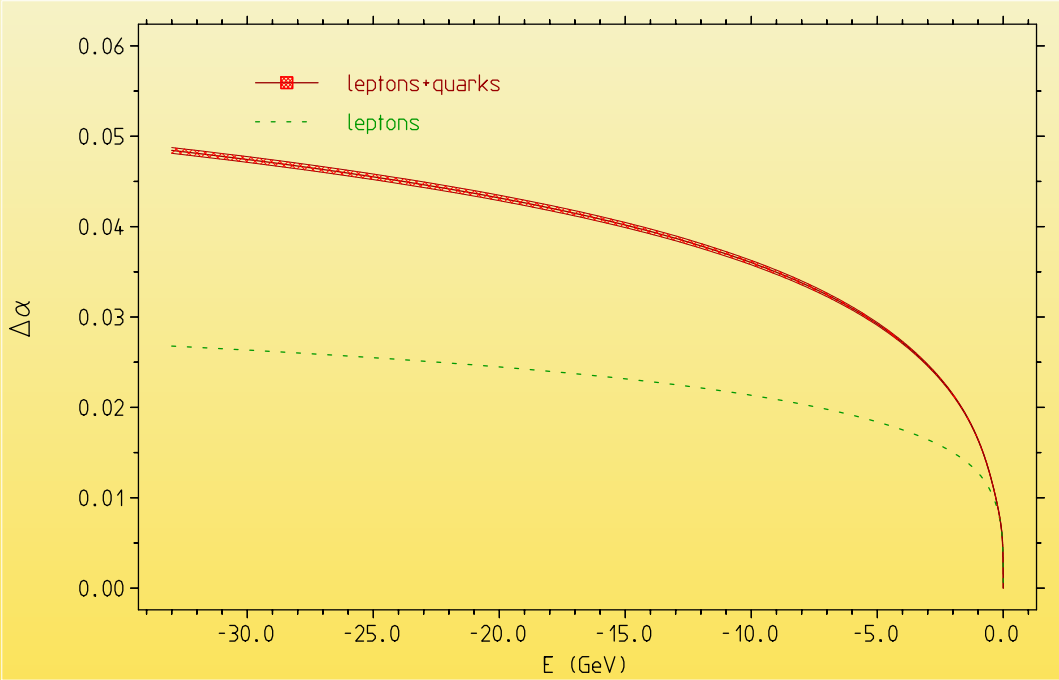
The figure illustrates the running of the effective charges at lower energies in the space–like region. Typical values are $\Delta\alpha(5\text{GeV}) \sim 3\%$ and $\Delta\alpha(M_Z) \sim 6\%$, where about $\sim 50\%$ of the contribution comes from leptons and

[a] The final result for the renormalized photon vacuum polarization for a lepton of mass m then reads

$$\Pi'_{\gamma\,\rm ren}(q^2) = \frac{\alpha}{3\pi}\left\{\frac{5}{3} + y - 2\,(1+\frac{y}{2})\,(1-y)\,G(y)\right\}$$

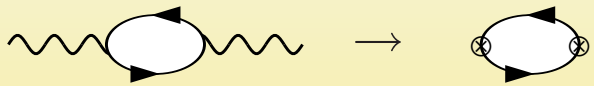
with $y = 4m^2/q^2$ and $G(y) = \frac{1}{2\sqrt{1-y}}\,\ln\frac{\sqrt{1-y}+1}{\sqrt{1-y}-1}$.

about $\sim 50\%$ from hadrons. Note the sharp increase of the screening correction at relatively low energies.



Shift of the effective fine structure constant $\Delta\alpha$ as a function of the energy scale in the space–like region $q^2 < 0$ ($E = -\sqrt{-q^2}$). The vertical bars at selected points indicate the uncertainty

Note alternative interpretation of VP: ❖ photon self–energy ❖ vacuum expectation value of the time ordered product of two electromagnetic currents:



One may represent the current correlator as a Källen-Lehmann representation in terms of spectral densities. To this end, let us consider first the Fourier transform of the vacuum expectation value of the product of two currents. Using translation invariance and inserting a complete set of states n of momentum p_n[b], satisfying the completeness relation

$$\int \frac{\mathrm{d}^4 p_n}{(2\pi)^3} \sum\!\!\!\!\!\!\int_n |n\rangle\langle n| = 1$$

where $\sum\!\!\!\!\!\!\int_n$ includes, for fixed total momentum p_n, integration over the phase space available to particles of all

[b] Note that the intermediate states are multi–particle states, in general, and the completeness integral includes an integration over p_n^0, since p_n is not on the mass shell $p_n^0 \neq \sqrt{m_n^2 + \vec{p}_n^2}$. In general, in addition to a possible discrete part of the spectrum we are dealing with a continuum of states.

possible intermediate physical states $|n\rangle$, we have

$$
\begin{aligned}
& \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle 0|j^\mu(x)\, j^\nu(0)|0\rangle \\
= \quad & \mathrm{i}\int \frac{\mathrm{d}^4p_n}{(2\pi)^3}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(q-p_n)x}\sum\!\!\!\!\!\!\int_n \langle 0|j^\mu(0)|n\rangle\langle n|j^\nu(0)|0\rangle \\
= \quad & \mathrm{i}\int \frac{\mathrm{d}^4p_n}{(2\pi)^3}\sum\!\!\!\!\!\!\int_n (2\pi)^4\, \delta^{(4)}(q-p_n)\langle 0|j^\mu(0)|n\rangle\langle n|j^\nu(0)|0\rangle \\
= \quad & \mathrm{i}\, 2\pi \sum\!\!\!\!\!\!\int_n \langle 0|j^\mu(0)|n\rangle\langle n|j^\nu(0)|0\rangle|_{p_n=q} \quad .
\end{aligned}
$$

In our case, for the conserved electromagnetic current, only the transversal amplitude is present and, taking into account the physical spectrum condition $p^2 \geq 0, p^0 \geq 0$ we may write the Källen-Lehmann representation

$$
\begin{aligned}
& \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle 0|T j^\mu_{\mathrm{em}}(x)\, j^\nu_{\mathrm{em}}(0)|0\rangle \\
= \quad & \int_0^\infty \mathrm{d}m^2\, \rho(m^2)\left(m^2\, g^{\mu\nu} - q^\mu q^\nu\right)\, \frac{1}{q^2 - m^2 + \mathrm{i}\varepsilon} \\
= \quad & -\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\, \hat{\Pi}'_\gamma(q^2)
\end{aligned}
$$

where $\hat{\Pi}'_\gamma(q^2)$ up to a factor e^2 is the **photon vacuum polarization function** introduced before[c]:

$$\Pi'_\gamma(q^2) = e^2 \hat{\Pi}'_\gamma(q^2) \ .$$

With this bridge to the photon self–energy function Π'_γ we can get its imaginary part by substituting

$$\frac{1}{q^2 - m^2 + \mathrm{i}\varepsilon} \to -\pi\,\mathrm{i}\,\delta(q^2 - m^2)$$

. Thus contracting with $2\Theta(q^0) g_{\mu\nu}$ and dividing by $g_{\mu\nu}(q^2\, g^{\mu\nu} - q^\mu q^\nu) = 3q^2$ we obtain

$$\begin{aligned} 2\Theta(q^0)\,\mathrm{Im}\,\hat{\Pi}'_\gamma(q^2) &= \Theta(q^0)\,2\pi\,\rho(q^2) \\ &= -\frac{1}{3q^2}\,2\pi \sum\!\!\!\!\!\!\int_n \langle 0|j^\mu_{\mathrm{em}}(0)|n\rangle\langle n|j_{\mu\,\mathrm{em}}(0)|0\rangle\big|_{p_n=q} \ . \end{aligned}$$

Again causality implies analyticity and the validity of a dispersion relation. In fact the electromagnetic current

[c] In case of a conserved current, where $\rho_0 \equiv 0$, we may formally derive that $\rho_1(s)$ is real and positive $\rho_1(s) \geq 0$. To this end we consider the element ρ^{00}

$$\begin{aligned} \rho^{00}(q) &= \sum\!\!\!\!\!\!\int_n \langle 0|j^0(0)|n\rangle\langle n|j^0(0)|0\rangle\big|_{q=p_n} \\ &= \sum\!\!\!\!\!\!\int_n |\langle 0|j^0(0)|n\rangle|^2_{q=p_n} \geq 0 \\ &= \Theta(q^0)\,\Theta(q^2)\,\vec{q}^{\,2}\,\rho_1(q^2) \end{aligned}$$

from which the statement follows.

correlator exhibits a logarithmic UV singularity and thus requires one subtraction such that we find

$$\Pi'_\gamma(q^2) - \Pi'_\gamma(0) = \frac{q^2}{\pi} \int_0^\infty \mathrm{d}s \, \frac{\mathrm{Im}\,\Pi'_\gamma(s)}{s\,(s - q^2 - \mathrm{i}\varepsilon)} \quad .$$

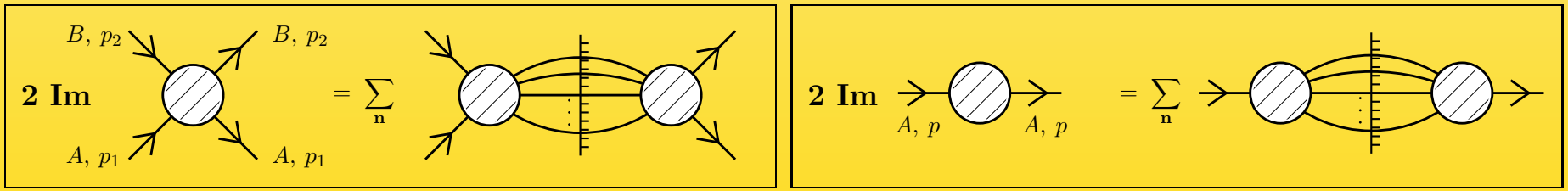
Here, we need the optical theorem which derives from the unitarity of S translated to T–matrix elements

$$\mathrm{i}\left\{T^*_{if} - T_{fi}\right\} = \sum\!\!\!\!\!\!\int_n (2\pi)^4\, \delta^{(4)}(P_n - P_i)\, T^*_{nf} T_{ni} \ ,$$

and the *optical theorem*, is obtained from this relation in the limit of elastic forward scattering $|f\rangle \longrightarrow |i\rangle$ where

$$2\,\mathrm{Im}\, T_{ii} = \sum\!\!\!\!\!\!\int_n (2\pi)^4\, \delta^{(4)}(P_n - P_i)\, |T_{ni}|^2 \quad .$$

Graphically, this relation may be represented by



Optical theorem for scattering and propagation.

It tells us that the imaginary part of the photon propagator is proportional to the total cross section

$\sigma_{\rm tot}(e^+e^- \to \gamma^* \to {\rm anything})$ ("anything" means any possible state). The precise relationship reads

$$\begin{aligned} \text{Im}\,\hat{\Pi}'_\gamma(s) &= \frac{1}{12\pi} R(s) \\ \text{Im}\,\Pi'_\gamma(s) &= e(s)^2\,\text{Im}\,\hat{\Pi}'_\gamma(s) = \frac{s}{e(s)^2}\,\sigma_{\rm tot}(e^+e^- \to \gamma^* \to {\rm anything}) = \frac{\alpha(s)}{3} R(s) \end{aligned}$$

where

$$R(s) = \sigma_{\rm tot} / \frac{4\pi\alpha(s)^2}{3s} \quad .$$

The normalization factor is the point cross section (tree level) $\sigma_{\mu\mu}(e^+e^- \to \gamma^* \to \mu^+\mu^-)$ in the limit $s \gg 4m_\mu^2$. Taking into account the mass effects the $R(s)$ which corresponds to the production of a lepton pair reads

$$R_\ell(s) = \sqrt{1 - \frac{4m_\ell^2}{s}} \left(1 + \frac{2m_\ell^2}{s}\right) \;, (\ell = e, \mu, \tau)$$

which may be read of from the imaginary part. This result provides an alternative way to calculate the renormalized vacuum polarization function, namely, via the DR which now takes the form

$$\Pi'^{\ell}_{\gamma\,\rm ren}(q^2) = \frac{\alpha q^2}{3\pi} \int_{4m_\ell^2}^{\infty} {\rm d}s \, \frac{R_\ell(s)}{s(s - q^2 - {\rm i}\varepsilon)}$$

yielding the vacuum polarization due to a lepton–loop.

In contrast to the leptonic part, the hadronic contribution cannot be calculated analytically as a perturbative series,

but it can be expressed in terms of the cross section of the reaction $e^+e^- \to$ hadrons, which is known from experiments. Via

$$R_{\rm had}(s) = \sigma(e^+e^- \to \text{hadrons}) / \frac{4\pi\alpha(s)^2}{3s} \quad .$$

we obtain the relevant hadronic vacuum polarization

$$\Pi'^{\rm had}_{\gamma\,\rm ren}(q^2) = \frac{\alpha q^2}{3\pi} \int_{4m_\pi^2}^{\infty} {\rm d}s \, \frac{R_{\rm had}(s)}{s(s-q^2-{\rm i}\varepsilon)} \quad .$$

At low energies, where the final state necessarily consists of two pions, the cross section is given by the square of the electromagnetic form factor of the pion (undressed from VP effects),

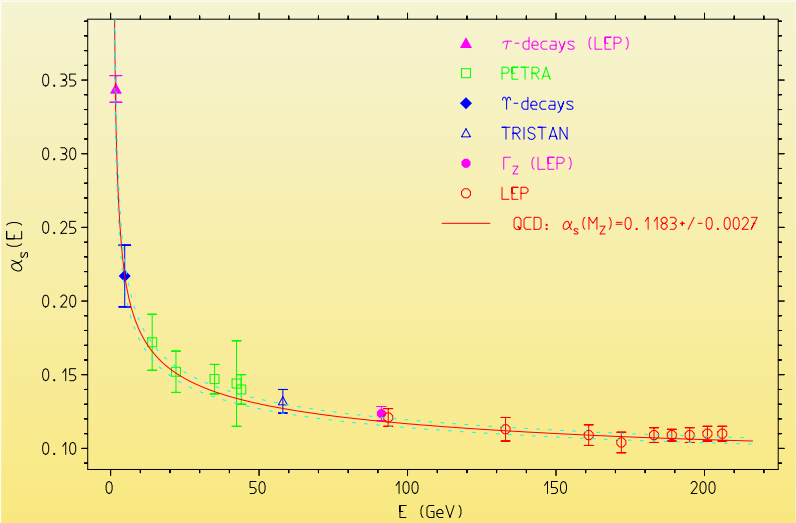
$$R_{\rm had}(s) = \frac{1}{4}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} 1|F_\pi^{(0)}(s)|^2 \quad , \qquad s < 9\,m_\pi^2 \quad ,$$

which directly follows from the corresponding imaginary part of the photon vacuum polarization. There are three differences between the pionic loop integral and those belonging to the lepton loops:

- the masses are different
- the spins are different
- the pion is composite – the Standard Model leptons are elementary

The compositeness manifests itself in the occurrence of the form factor $F_\pi(s)$, which generates an enhancement: at the ρ peak, $|F_\pi(s)|^2$ reaches values about 45, while the quark parton model would give about 7. The

remaining difference in the expressions for the quantities $R_\ell(s)$ and $R_h(s)$, respectively, originates in the fact that the leptons carry spin $\frac{1}{2}$, while the spin of the pion vanishes. Near threshold, the angular momentum barrier suppresses the function $R_h(s)$ by three powers of momentum, while $R_\ell(s)$ is proportional to the first power. The suppression largely compensates the enhancement by the form factor – by far the most important property is the mass.



Compilation of α_s measurements (Bethke 04).

$\alpha_s(M_\tau) = 0.322 \pm 0.030$ **at** $M_\tau = 1.78$ **GeV**

pQCD not only fails due to strong coupling (non-convergence of expansion) but because of spontaneous chiral symmetry breaking (100% non-perturbative) responsible for the existence of pions and quark condensates, which are missing to all orders in pQCD, i.e. pQCD fails to correctly describe the low energy structure of QCD

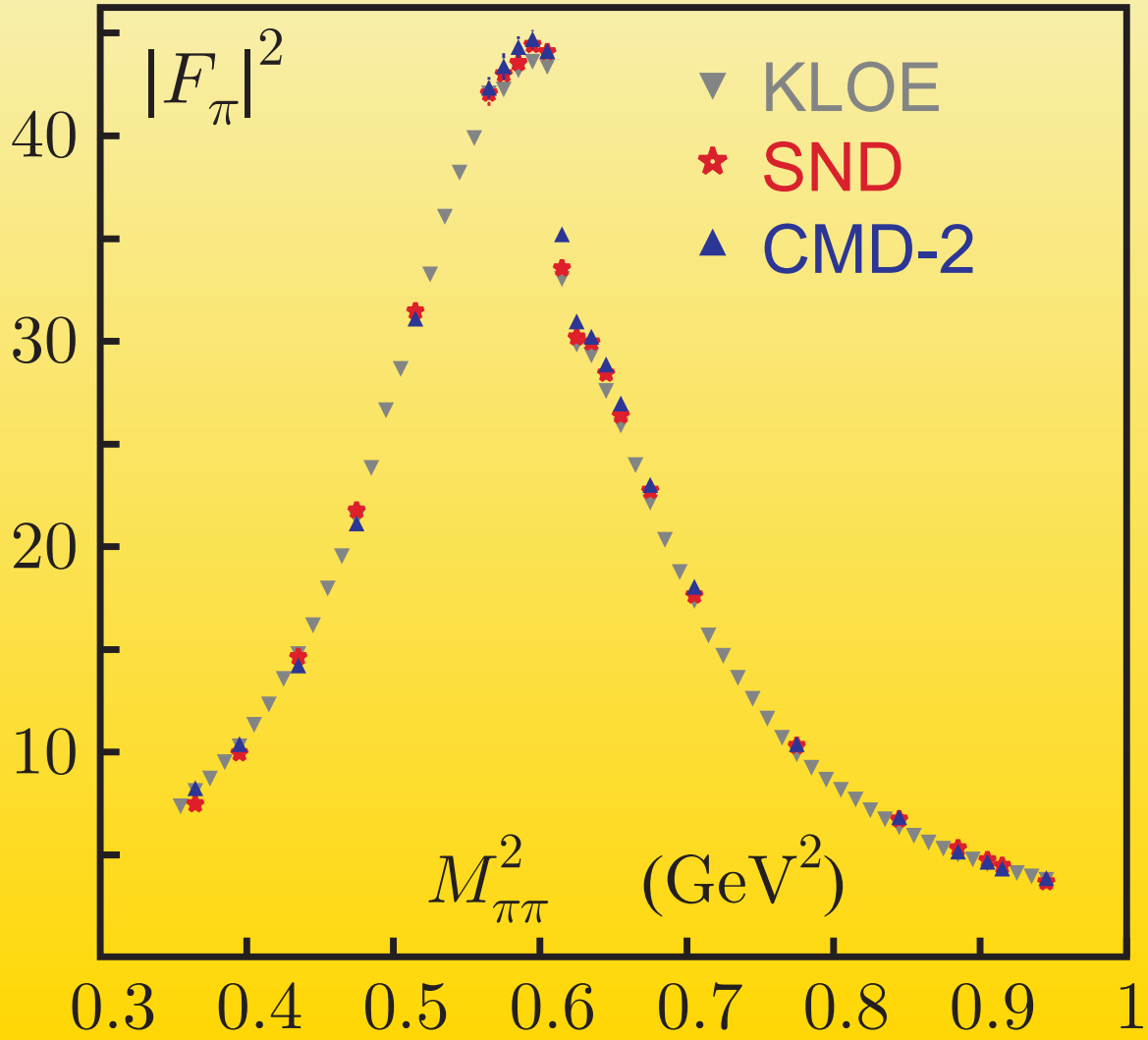
Must use non-perturbative methods:

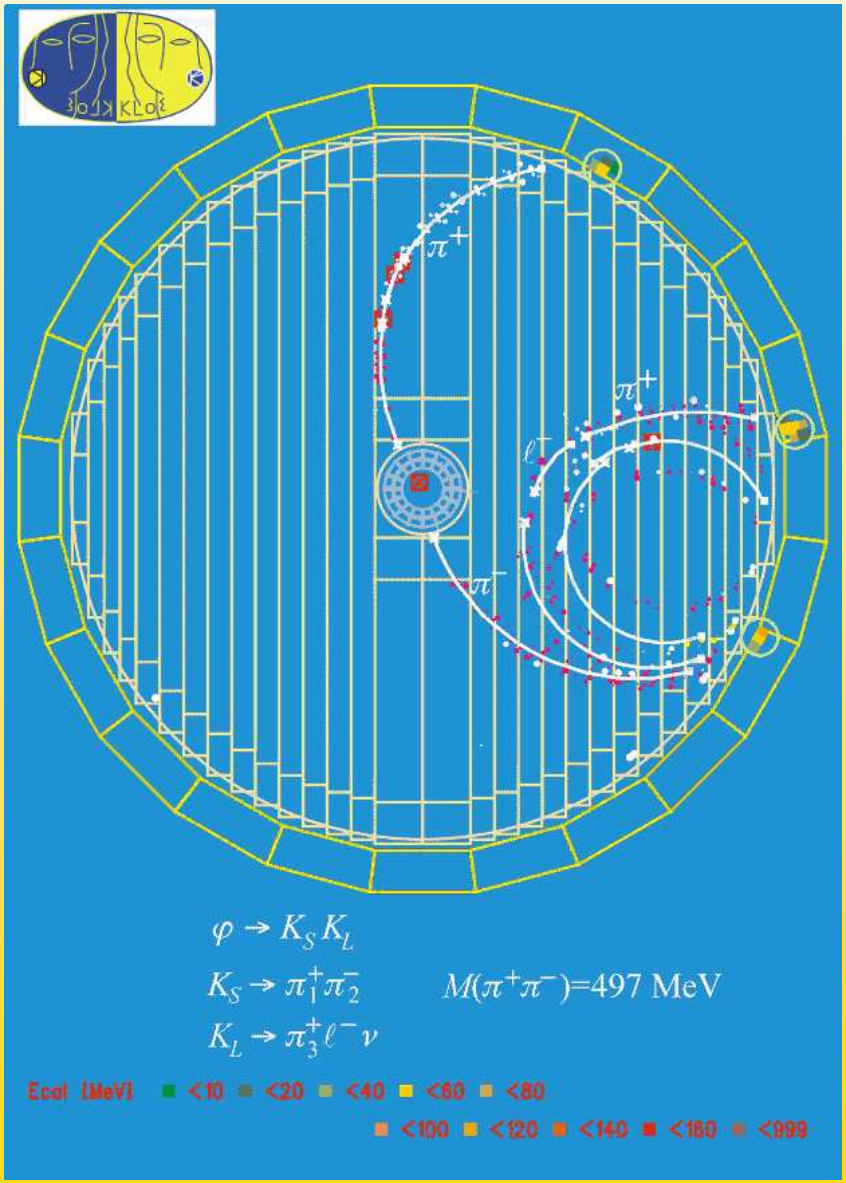
- **Dispersion relations, sum rules**
- **Chiral perturbation theory extended to include spin 1 vector states**
- **QCD inspired models: extended Nambu Jona-Lasinio model (ENJL), hidden local symmetry (HLS) model**
- **large N_c QCD approach (dual to infinite series of narrow vector resonances)**
- **lattice QCD in future**

❑ Low energy: the ρ peak

"Outstanding" non-perturbative low energy hadron spectrum: $e^+e^- \to \pi^+\pi^-$

A precise new KLOE measurement of $|F_\pi|^2$ **with ISR events**





$\pi\pi$ event with KLOE detector.

7. Hadron-production in e^+e^-–annihilation

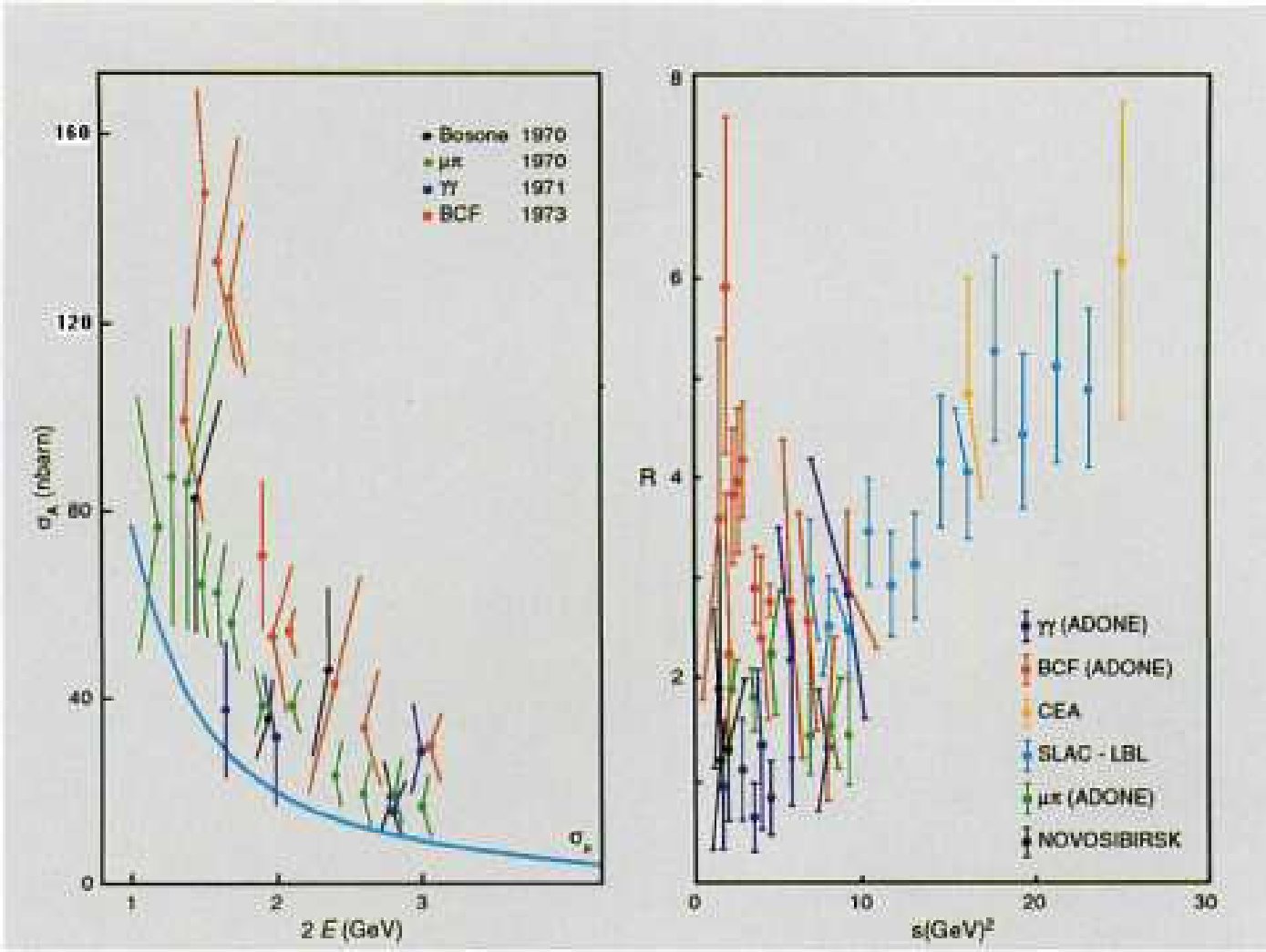
- at dawn of QCD: 1974 hadron physics in crisis
- since 1969: $e^+e^- \to$ hadrons cross-sections with storage ring ADONE at Frascati much too high for theoretical expectations at that time.

These experiments measured the ratio

$$R(s) = \frac{\sigma_{\rm tot}(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma_{\mu\mu}^{(m_\mu=0)}}$$

of the hardonic cross section in units of the $\mu^+\mu^-$ pair production cross section, and R values in the range from 1 up to 6 were actually measured.

- in 1973: R measured at the Cambridge Electron Accelerator (CEA) at Harvard: $R \sim 5[6]$ at 4[5] GeV
- London Conference in 1974: conclusion R raising smoothly form 2 at 2 GeV to about 6 at 5 GeV
- on the theory side confusing: about 22 different models were proposed, among them also QCD [Fritzsch+Leutwyler], as a solution of the “unitarity crisis”.
- Unitarity predicts a decrease of the total cross–section like $1/s$ up to logs at high energies.

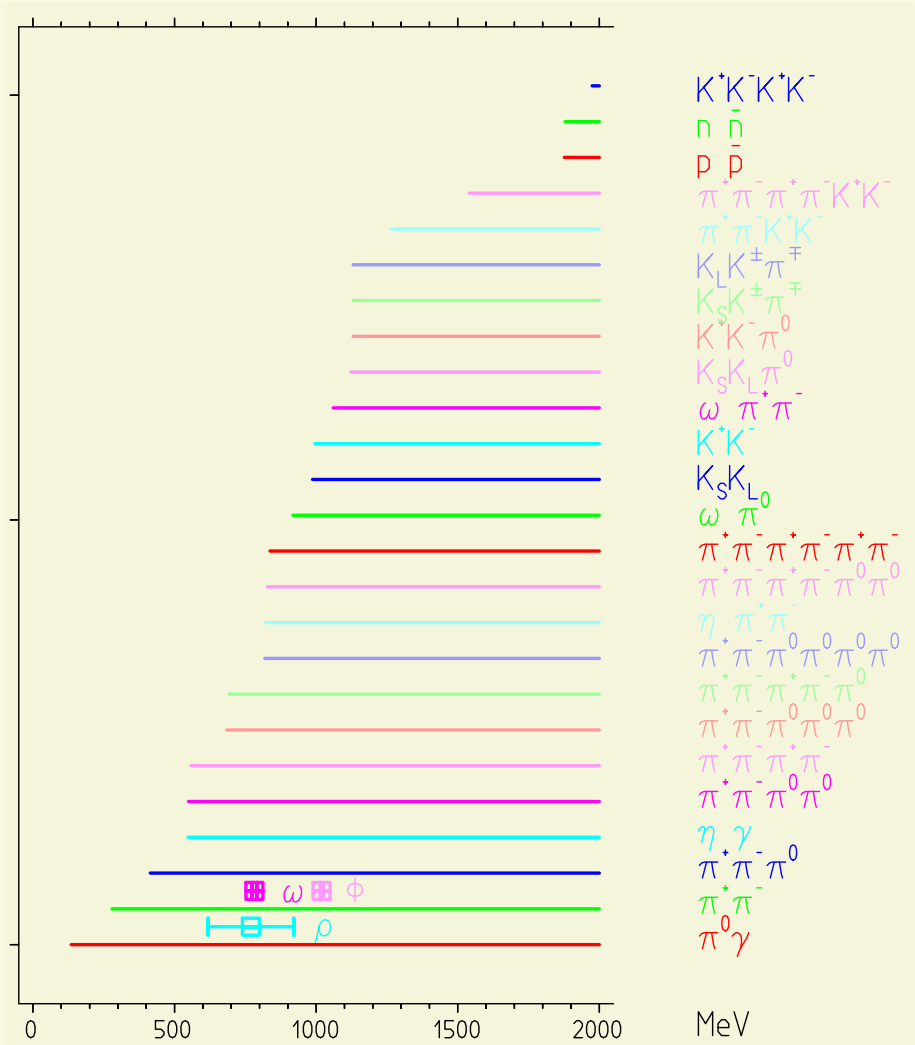


The ratio R as of July 1974.

The solution was QCD, providing a factor 3 from color degrees of freedom, and a new species of quark charm, which completed the 2nd family of fermions.

Chronology of e^+e^- facilities

Year	Accelerator	$E_{\max}$ (GeV)	Experiments	Laboratory
1961-1962	AdA	0.250		LNF Frascati (Italy)
1965-1973	ACO	0.6-1.1	DM1	Orsay (France)
1967-1970	VEPP-2	1.02-1.4	'spark chamber'	Novosibirsk (Russia)
1967-1993	ADONE	3.0	BCF,$\gamma\gamma$, $\gamma\gamma 2$, MEA, $\mu\pi$, FENICE	LNF Frascati (Italy)
1971-1973	CEA	4,5		Cambridge (USA)
1972-1990	SPEAR	2.4-8	MARK I, CB, MARK 2	SLAC Stanford (USA)
1974-1992	DORIS	-11	ARGUS, CB, DASP 2, LENA, PLUTO	DESY Hamburg (D)
1975-1984	DCI	3.7	DM1,DM2,M3N,B$\overline{\mathrm{B}}$	Orsay (France)
1975-2000	VEPP-2M	0.4-1.4	OLYA, CMD, CMD-2, ND,SND	Novosibirsk (Russia)
1978-1986	PETRA	12-47	PLUTO, CELLO, JADE, MARK-J, TASSO	DESY Hamburg (D)
1979-1985	VEPP-4	-11	MD1	Novosibirsk (Russia)
1979-	CESR	9-12	CLEO, CUSB	Cornell (USA)
1980-1990	PEP	-29	MAC, MARK-2	SLAC Stanford (USA)
1987-1995	TRISTAN	50-64	AMY, TOPAZ, VENUS	KEK Tsukuba (Japan)
1989	SLC	90 GeV	SLD	SLAC Stanford (USA)
1989-2001	BEPC	2.0-4.8	BES, BES-II	IHEP Beijing (China)
1989-2000	LEP I/II	110/210	ALEPH, DELPHI, L3, OPAL	CERN Geneva (CH)
1999-2007	DAΦNE	Φ factory	KLOE	LNF Frascati (Italy)
1999-	PEP-II	B factory	BaBar	SLAC Stanford (USA)
1999-	KEKB	B factory	Belle	KEK Tsukuba (Japan)



Thresholds for exclusive multi particle channels below 2 GeV

❑ General $O(\alpha^2)$ framework for predicting $R(s)$

Quarks are charged particles and couple to the photon via the quark contribution to the electromagnetic current:

$$J_\mu^\gamma = \sum_i \left(\frac{2}{3}\,\bar{u}_i \gamma_\mu u_i - \frac{1}{3}\,\bar{d}_i \gamma_\mu d_i \right)$$

where $i = 1, 2, 3$ labels the three quark doublets:

$$\begin{pmatrix} u_i \\ d_i \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} .$$

The Lagrangian density for strong and electromagnetic interactions is given by

$$\mathcal{L}_{\text{strong\&electromagnetic}} = \mathcal{L}_{\text{QCD}} + e J_\mu^\gamma A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} .$$

We are interested in the S–matrix element

$$\langle X | S | e^+ e^- \rangle = \langle X | T \left\{ \mathrm{e}^{\mathrm{i} \int \mathrm{d}^4 x \, (\mathcal{L}_{\text{QCD,int}} + \mathcal{L}_{\text{QED,int}})} \right\} | e^+ e^- \rangle$$

of the process

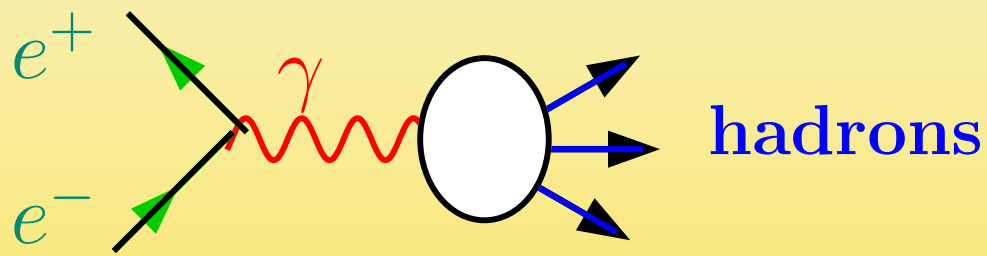
$$e^+ e^- \to X \text{ specifically } X = \text{ any hadronic state of appropriate QN's} .$$

➠ $\alpha = \frac{e^2}{4\pi} = (137.036)^{-1}$ small $\longrightarrow$ one–photon exchange approximation + perturbative corrections

Leading $O(\alpha^2)$ approximation

$$\langle X|S|e^+e^-\rangle \quad = \quad -\mathrm{i}\, e^2\, (2\pi)^4\, \delta^{(4)}(p_X - p_+ - p_-)\, \tilde{D}_{\mu\nu}(q) \times \langle X|J^\mu_{\rm had}(0)|0\rangle\langle 0|J^\nu_{\rm lep}(0)|e^+e^-\rangle \ .$$

where $D_{\mu\nu}$ denotes the photon propagator



Hadron production in electron–positron annihilation.

Current conservation of the leptonic current $q_\nu J^\nu_{\rm lep} = 0$ and the hadronic one $q_\mu J^\mu_{\rm had} = 0$ implies that only the $g_{\mu\nu}$–part of the photon propagators contributes to the matrix element:

$$\tilde{D}_{\mu\nu}(q) \to -g_{\mu\nu}\, \frac{1}{q^2 + \mathrm{i}\varepsilon} \ .$$

For the relevant T–matrix element, defined by $S = \mathbf{1} + \mathrm{i}\,(2\pi)^4\, \delta^{(4)}(\sum p_i)\, T$ we find

$$T_{e^+e^-\to X} = \frac{e^2}{q^2}\, \langle X|J^\mu_{\rm had}(0)|0\rangle\langle 0|J_{\mu\,\rm lep}(0)|e^+e^-\rangle$$

where the leptonic matrix element is given at leading order by

$$\langle 0|J_{\mu\,\rm lep}(0)|e^+e^-\rangle = -\bar{v}(p_+, s_+)\, \gamma_\mu\, u(p_-, s_-)$$

and therefore

$$T(e^+e^- \to \gamma^* \to X) = -\frac{e^2}{q^2}\,\bar{v}(p_+,s_+)\,\gamma_\mu\,u(p_-,s_-)\;\cdot\;\langle X|J^\mu_{\rm had}(0)|0\rangle\;\;.$$

The total cross–section is

$$\sigma_h(s) = \sum\!\!\!\!\!\!\int_\Gamma \mathrm{d}\sigma(e^+e^- \to X,\; s=(p_+ + p_-)^2)$$

with the differential one given by (denoting $\mathrm{d}\mu(p) = \frac{\mathrm{d}^3p}{(2\pi)^3\,2\,\omega(p)}$, $\omega(p) = \sqrt{m^2+\vec{p}^{\,2}}$)

$$\mathrm{d}\sigma = \frac{(2\pi)^4\,\delta^{(4)}(p_+ + p_- - p_X)}{2\sqrt{\lambda(s,m_e^2,m_e^2)}}\,|T|^2\,\mathrm{d}\mu(p_1')\cdots$$

For unpolarized e^+e^--beams, the square matrix element is given by the average over the initial spins of e^- and e^+ and can easily be calculated by standard techniques:

$$\overline{|T|^2} = \frac{1}{4}\sum_{s_\pm}|T|^2 = \frac{e^4}{q^4}\,\ell_{\mu\nu}\,h^{\mu\nu}$$

where $\ell_{\mu\nu} = \frac{1}{2}\left\{q_\mu q_\nu - q^2\,g_{\mu\nu} - (p_+-p_-)_\mu(p_+-p_-)_\nu\right\}$ and $h^{\mu\nu} = \langle 0|J^\mu(0)|X\rangle\langle X|J^\nu(0)|0\rangle\;\;.$
One obtains

$$\sigma_h(s) = \frac{\alpha^2}{2s^3}\,\frac{16\pi^2}{\sqrt{1-4m_e^2/s}}\,\ell_{\mu\nu}\sum\!\!\!\!\!\!\int_X (2\pi)^4\,\delta^{(4)}(p_+ + p_- - p_X)\,h^{\mu\nu}$$

where summation/integration is over the hadronic final states. The leptonic tensor $\ell_{\mu\nu}$ describes the initial state

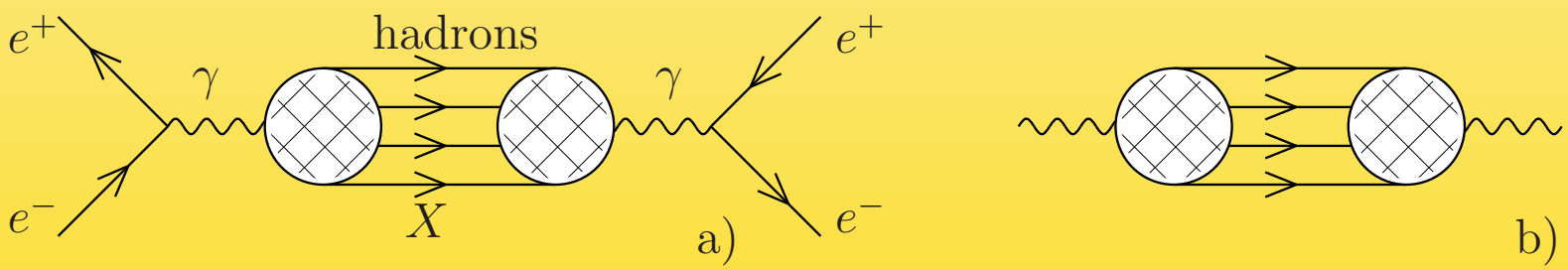
and is fixed. For what concerns the hadronic part we try to be general before we will resort to pQCD which only applies at sufficiently high energy. At this point the optical theorem helps to proceed at the non-perturbative level. As shown above we have

$$\mathrm{Im}\Pi'_\gamma(s) = \frac{s}{4\pi\alpha}\,\sigma_{\rm tot}(e^+e^- \to \text{anything}) \ .$$

In first place, applying the optical theorem to e^+e^-–annihilation we have

$$2\,\mathrm{Im}\,T(e^+e^- \to e^+e^-)_{\rm forward} = (2\pi)^4 \sum_{QN's} \int \Pi_{i=1}^n \,\mathrm{d}\mu(p'_i)\,\delta^{(4)}(P'_n - P_i)\,\cdot\,|T(p'_1,\cdots,p'_n|p_+,p_-)|^2$$

which in the given one–photon exchange approximation has the structure shown



The optical theorem in e^+e^-–annihilation: a) including the QED part, b) the QCD part isolated.

The isolated QCD part, representing $(2\pi)^4\,\delta^{(4)}(p_+ + p_- - p_X)\,h^{\mu\nu}$, thus corresponds to the hadronic excitations (hadronic vacuum polarization) in the photon propagator. The hadronic part corresponds to the photon propagator after amputation of the external photons $A^\mu(x) \to J^\mu(x)$, where only the hadronic part $J^\mu_{\rm had}(x)$ is

of interest here. The hadronic "blob" b) thus is given by the hadronic current correlator

$$\begin{aligned}\hat{\Pi}^{\mu\nu}(q) &= \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}\,qx}\langle 0|J^{\mu}_{\rm had}(x)J^{\nu}_{\rm had}(0)|0\rangle \\ &= -\left(g^{\mu\nu}\,q^2 - q^{\mu}q^{\nu}\right)\,\hat{\Pi}'_{\gamma}(q^2)\ .\end{aligned}$$

Note that $\hat{\Pi}'_{\gamma}(q^2)$ times e^2 is the usual photon self-energy. We have used transversality which is a consequence of the current conservation. The hadronic photon vacuum polarization thus is given by a single scalar amplitude $\Pi'_{\gamma}(q^2) = e^2\,\hat{\Pi}'_{\gamma}(q^2)$.

The physical thresholds of hadron production in $e^+e^- \to X$ are $X = \pi^+\pi^-, \rho, \pi^0\pi^+\pi^-, \omega, K^+K^-, K^0\overline{K^0}, \phi, \cdots$ with lowest threshold at $s = 4\,m^2_{\pi^\pm}$ [a]. The vector mesons ρ, ω and ϕ correspond to $\pi^+\pi^-$, $\pi^0\pi^+\pi^-$ and $K\bar{K}$ resonance peaks, respectively.

We now may continue our cross–section calculation using

$$(2\pi)^4\,\delta^{(4)}(p_+ + p_- - p_X)\,h^{\mu\nu} = 2\,\mathrm{Im}\,\hat{\Pi}^{\mu\nu}(q) = 2\,\left(q^{\mu}q^{\nu} - q^2\,g^{\mu\nu}\right)\,\mathrm{Im}\,\hat{\Pi}'_{\gamma}(q^2)\,,$$

[a] If we would include higher order electromagnetic interaction, like final state radiation (FSR) from hadrons, the lowest possible state exhibiting a hadron would be $\pi^0\gamma$ with threshold at $s = m^2_{\pi^0}$. In our counting this is a $O(\alpha^3)$ effect not included in our leading order consideration.

such that, working out kinematics

$$\begin{aligned}\sigma_h(s) &= \frac{\alpha^2}{s}\frac{16\pi^2}{\sqrt{1-\frac{4m_e^2}{s}}}\left(1+\frac{2m_e^2}{s}\right)\,\mathrm{Im}\,\hat{\Pi}'_\gamma(s)\\ &\simeq \frac{16\pi^2\alpha^2}{s}\,\mathrm{Im}\,\hat{\Pi}'_\gamma(s)\ \text{since}\ \ s\geq 4m_\pi^2 \gg m_e^2\ .\end{aligned}$$

It is common use to represent the hadronic production cross–section $\sigma_h(s)$ in units of the leptonic point cross section. This leads to the definition of the so called R–ratio function

$$\begin{aligned}R(s) &\doteq \frac{\sigma_{\rm tot}(e^+e^-\to \text{hadrons})}{\sigma_0(e^+e^-\to\mu^+\mu^-)}\\ &= \frac{\sigma_h(s)}{\left(\frac{4\pi\alpha^2}{3s}\right)} = \frac{16\pi^2\alpha^2}{4\pi\alpha^2}\cdot 3\,s\,\frac{1}{s}\,\mathrm{Im}\,\hat{\Pi}'_\gamma(s)\\ &= 12\pi\,\mathrm{Im}\,\hat{\Pi}'_\gamma(s) = 12\pi^2\,\rho_1(s)\end{aligned}$$

where $\rho_1(s)=\frac{1}{\pi}\,\mathrm{Im}\,\hat{\Pi}'_\gamma(s)$ is the spectral density used in the Källen-Lehmann representation earlier.

As we will see below the function $R(s)$ has the asymptotic property

$$R(s)\to\ \text{constant}\ \ ;\ \ s\to\infty$$

which means that the imaginary part of the vacuum polarization function $\hat{\Pi}'_\gamma(s)$ tends to a non–vanishing constant at time–like infinity.

Exploiting analyticity:

The DR, derived earlier, determines the real part of the vacuum polarization amplitude in terms of its imaginary part. In fact the electromagnetic current correlator exhibits a logarithmic UV singularity and thus requires one subtraction such in place of the unsubtracted DR

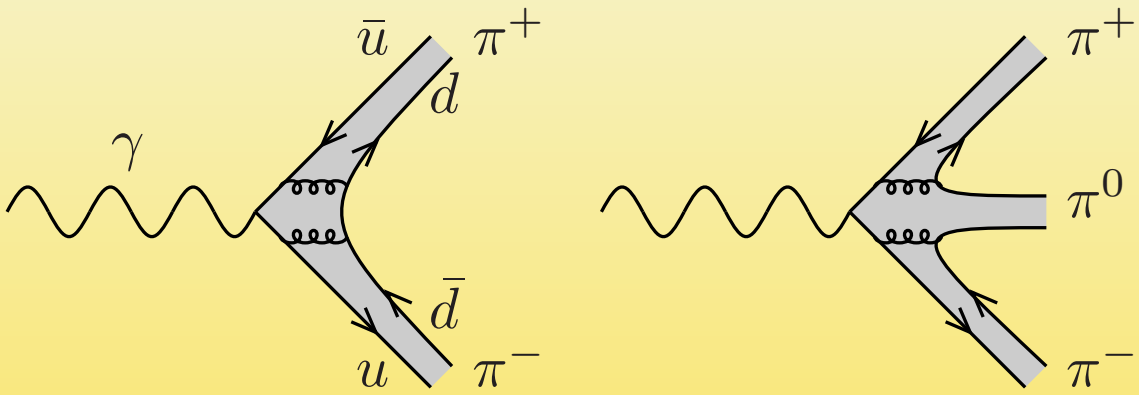
$$\hat{\Pi}'_\gamma(q^2) = \frac{1}{\pi}\int_0^\infty \mathrm{d}s\,\frac{\mathrm{Im}\,\hat{\Pi}'_\gamma(s)}{s-q^2-\mathrm{i}\varepsilon}\ ,$$

we find

$$\hat{\Pi}'_\gamma(q^2) - \hat{\Pi}'_\gamma(0) = \frac{q^2}{\pi}\int_0^\infty \mathrm{d}s\,\frac{\mathrm{Im}\,\hat{\Pi}'_\gamma(s)}{s\,(s-q^2-\mathrm{i}\varepsilon)} = -\Delta\alpha(s)\ \ .$$

In a renormalizable QFT UV divergencies in physical amplitudes are at most quadratic. Therefore at most two subtractions (mass– and wavefunction–renormalization, or vertex renormalization) are needed to get any DR finite. In low energy effective theories the number of subtractions needed corresponds to the number of low energy constants at the given order of the expansion (powers in p/Λ; p process specific momentum, Λ, typically Λ_{QCD} or M_p the proton mass).

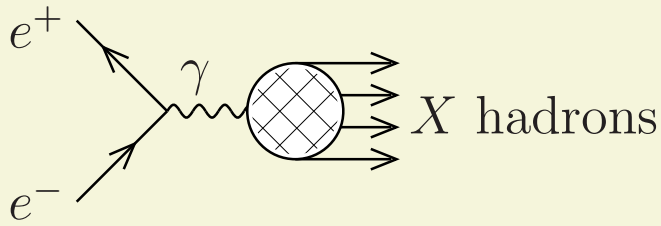
While in perturbative QCD colored, fractionally charged $q\bar{q}$-pairs are produced, in reality only hadrons are realized as states via non-perturbative hadronization:



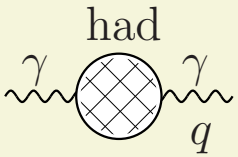
Hadron production in low energy e^+e^-–annihilation: the primarily created quarks must hadronize. The shaded zone indicates strong interactions via gluons which confine the quarks inside hadrons.

Summary: In the one photon exchange approximation, i.e. at $O(\alpha^2)$, the total hadronic production cross–section in electron–positron annihilation

$$\sigma_h(e^+e^- \to \gamma^* \to \text{hadrons}) :$$



determines the imaginary part of the photon vacuum polarization amplitude



$$: \quad \hat{\Pi}^{\mu\nu}(q) = \left(q^\mu q^\nu - q^2\, g^{\mu\nu}\right)\, \hat{\Pi}'_{\gamma\,\text{had}}(q^2)$$

$$\sigma_h = \frac{16\pi^2\alpha^2}{s}\, \text{Im}\, \hat{\Pi}'_{\gamma\,\text{had}} \quad .$$

Usually one represents the $e^+e^- \to \text{hadrons}$ data in terms of

$$R(s) = \frac{\sigma_h(s)}{\sigma_{\mu\mu}^{(m_\mu=0)}} = 12\,\pi\, \text{Im}\, \hat{\Pi}'_{\gamma\,\text{had}} \quad .$$

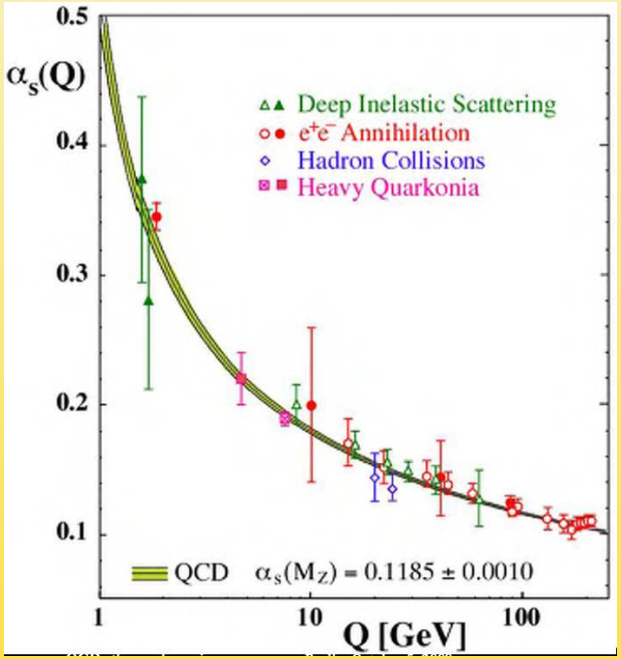
The hadronic shift of the effective fine structure constant at scale M is given by

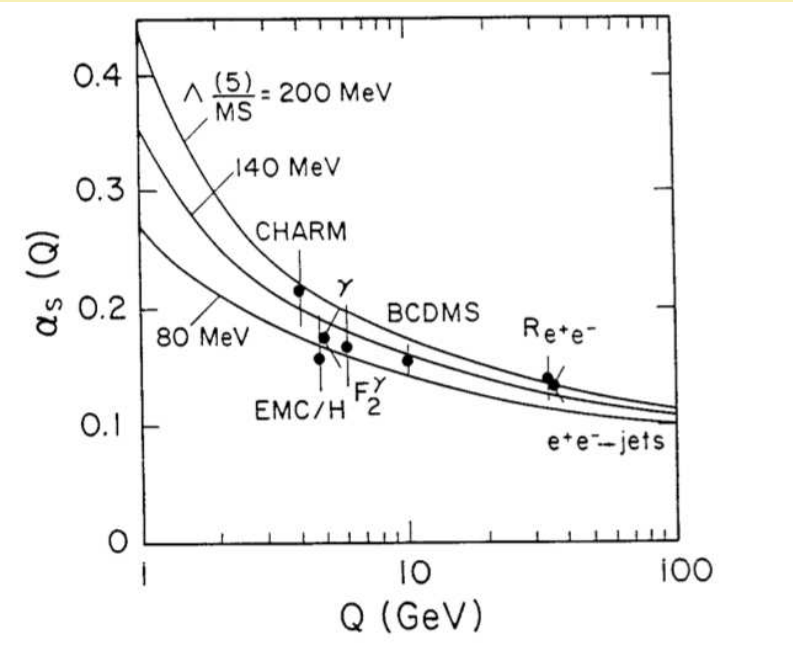
$$\Delta\alpha_{\text{had}}(M^2) = -\frac{\alpha\, M^2}{3\pi}\, \text{Re} \int\limits_{4m_\pi^2}^{\infty} \frac{\text{d}s\, R(s)^{e^+e^-\to\gamma^*\to\text{had}}}{s\,(s - M^2 - \text{i}\varepsilon)} \quad .$$

(Cabibbo-Gatto Roma 1960!)

❑ High energy hadron-production in e^+e^-–annihilation

❖ Asymptotic freedom: weak coupling at high energies





Status of α_s [Bethke 2009](left) compared with 1989 pre LEP status (right) $\alpha_s^{(5)}(M_Z) = 0.11 \pm 0.01$ (corresponding to $\Lambda^{(5)}_{\overline{\rm MS}} = 140 \pm 60$ MeV).[Altarelli 89].

❖ perturbative QCD (pQCD) applies for $M_\tau \leq \sqrt{s}$

❖ photon at short distances directly couples to the quarks $q = u, d, s, c, b, t$

❖ $e^+e^- \longrightarrow$ hadrons via $\gamma^* \longrightarrow q\bar{q}$

- observed hadrons only (confinement!) and sharp spikes of vector bosons resonances: J/ψ series, Υ series
- main issue hadronization: no detailed understanding
- meaning of quark parton cross sections ruled by "quark–hadron duality"[a]

Quark Hadron Duality (QHD): for sufficiently large s the average non–perturbative hadron cross–section equals the perturbative quark cross–section:

$$\overline{\sigma(e^+e^- \to \text{hadrons})}(s) \simeq \sum_q \sigma(e^+e^- \to q\bar{q})(s) \ ,$$

where the averaging extends from threshold up to the given s value which must lie far enough above a threshold (global duality). Approximately, such duality relations then would hold for energy intervals which start just below the last threshold passed up to s.

[a] Quark–hadron duality was first observed phenomenologically for the structure function in deep inelastic electron–proton scattering (Bloom&Gilman 1970)

Status of quark hadron duality:

- qualitatively QHD is clearly seen in the data for appropriate regions of s
- it is not yet possible to quantify the accuracy of the conjecture
- a quantitative check would require much more precise cross-section measurements
- applicability of QHD as a tool in precision physics? not really under control
- in large-N_c QCD, $N - c \to \infty$ QHD becomes exact (infinite series of narrow vector states)

Note: this is one of the most important testing grounds for pQCD.

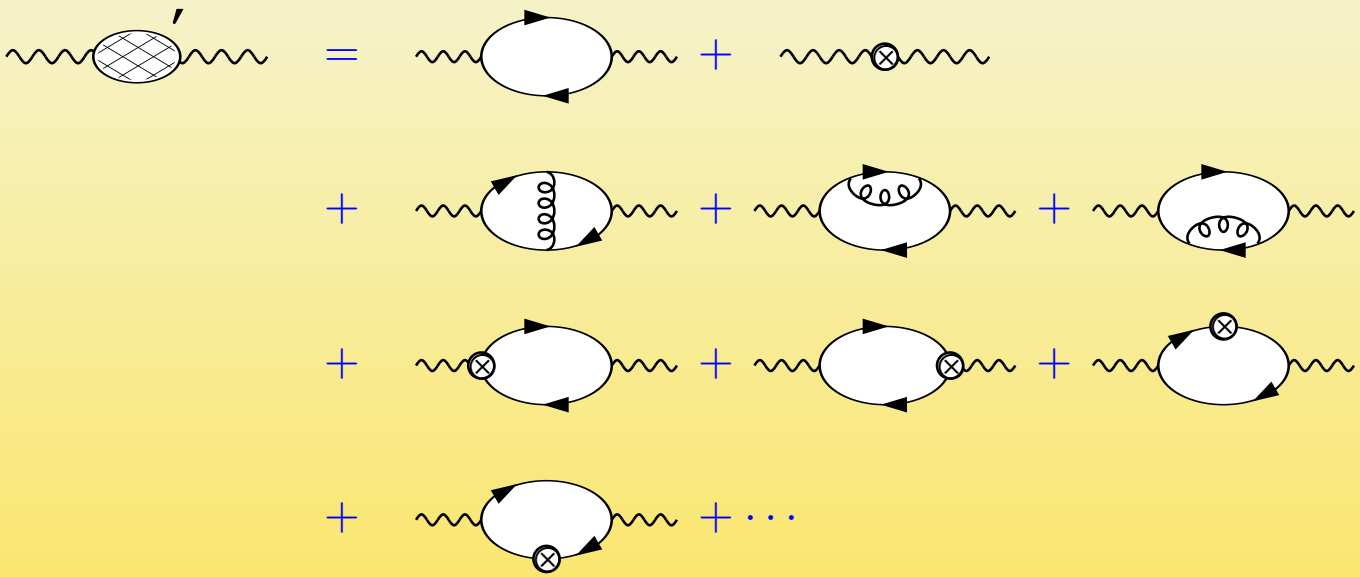
On theory side: calculate

$$R(s) = 12\,\pi\,\mathrm{Im}\,\hat{\Pi}'_{\gamma\,\mathrm{had}}(s) \;\; ; \quad \hat{\Pi}_{\gamma\,\mathrm{had}}(q) :$$

,

where the “blob” is the hadronic one, in pQCD represented by valence quark loops dressed in all possible ways by gluons and possible sea quark loops. The leading plus next to leading QCD perturbation expansion

diagrammatically is given by



Lowest order is $(\alpha_s)^0$:

➠ proportional to free quark–antiquark production cross-section

This approximation = $\boxed{\text{Quark Parton Model (QPM)}}$,

Free quarks with the strong interaction turned off. By AF the QPM is expected to provide a good approximation in

the high energy limit of QCD.

Calculation yields

$$\text{Im}\,\hat{\Pi}'_{\gamma\,\text{had}}(q^2) = N_c \sum_f \frac{Q_f^2}{12\pi^2}\,\left(1+2m_f^2/q^2\right)\,\text{Im}\,B_0(m_f,m_f;q^2)$$

and the imaginary part of scalar two-point function integral

$$B_0(m_1,m_2;q^2) = \text{Reg} - \int_0^1 \mathrm{d}x\,\ln(m_1\,x + m_2\,(1-x) - x\,(1-x)q^2)$$

with Im $B_0 = \pi\,\sqrt{1-4m_f^2/q^2}$. The mass dependent threshold factor is a function of the center of mass quark velocity

$$v_f = \left(1 - \frac{4m_f^2}{s}\right)^{1/2}$$

and, we may write

$$\begin{aligned} R(s) &= N_c \sum_f \frac{Q_f^2}{12\pi^2}\,12\pi^2\,\left(1+2m_f^2/s\right)\,\sqrt{1-4m_f^2/s}\quad [s \geq 4m_f^2] \\ &= N_c \sum_f Q_f^2\,\frac{v_f}{2}\,\left(3-v_f^2\right)\,\Theta(s-4m_f^2)\ . \end{aligned}$$

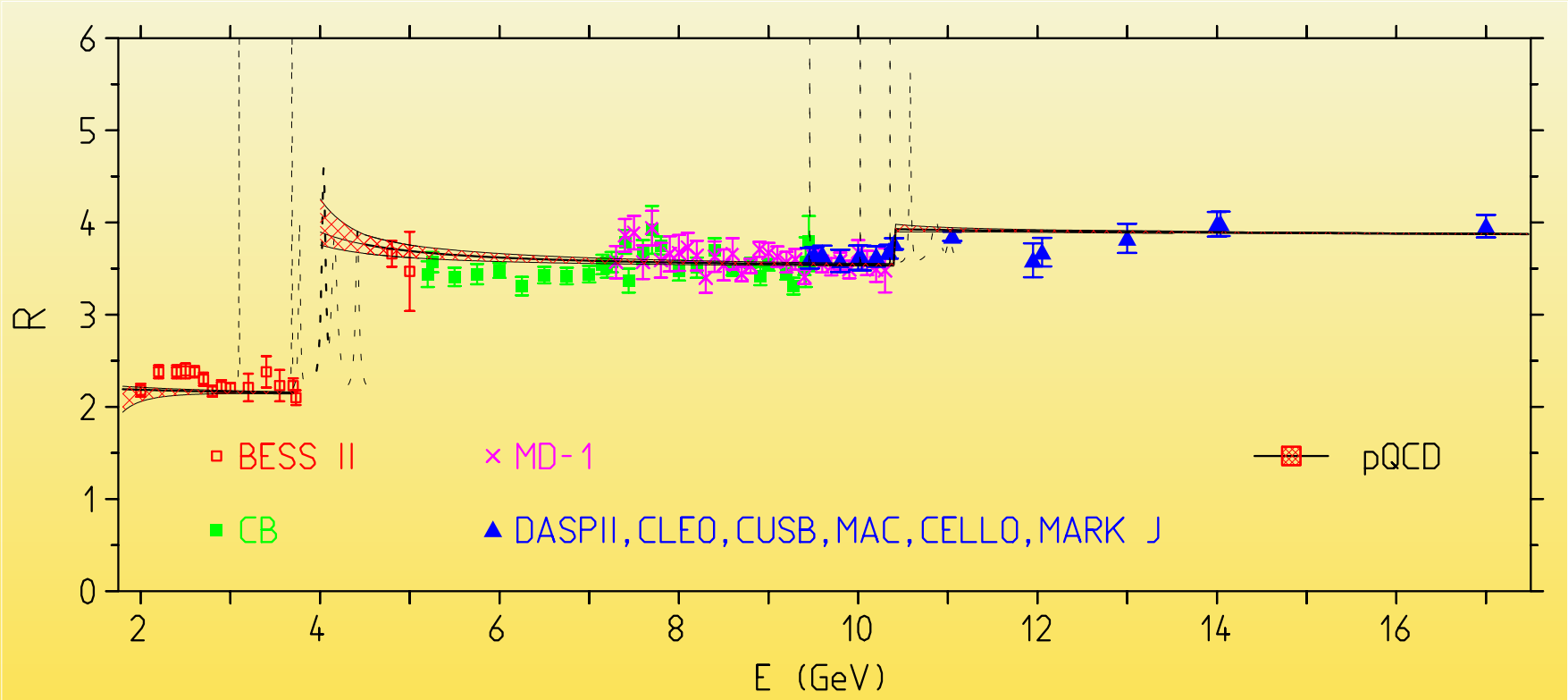
The other possibility to get this result is to calculate the Born cross section for the production of a quark-antiquark

pair $\sigma(e^+e^- \to \bar{q}q)$, directly, by taking free quarks $X = \bar{q}q$ in the matrix element. We then have

$$\langle X|J^\mu_{\rm had}(0)|0\rangle = Q_q\,\bar{u}_q(p_q,s_q)\,\gamma^\mu\,v_q(p_{\bar{q}},s_{\bar{q}})$$

and $h^{\mu\nu}$ in is given by an expression like $\ell_{\mu\nu}$. Like μ-pair production with adjusted charge and color factor $N_c = 3$. In quark–parton model which, with $N_c = 3$, $Q_{u,c,t} = \frac{2}{3}$ and $Q_{d,s,b} = -\frac{1}{3}$, predicts the following constant R–values:

N_f	3	4	5	6
quarks	uds	$udsc$	$udscb$	$udscbt$
R	2	$3\,\frac{1}{2}$	$3\,\frac{2}{3}$	5
range	1.8 - 3.73 GeV	4.8 - 10.52 GeV	11.20 - (2 m_t-10.0) GeV	(2 m_t+10.0) GeV - ∞



Test of pQCD prediction of R with some more recent data in the non-resonant regions. The spikes show the sharp J/ψ ($\bar{c}c$) and Υ ($\bar{b}b$) resonances.

Including the presently known higher order terms the perturbative result for $R(s)$ is given by

$$R(s)^{\rm pert} = N_c \sum_f Q_f^2 \, \frac{v_f}{2} \, \left(3 - v_f^2\right) \, \Theta(s - 4m_f^2)$$
$$\times \left\{1 + a c_1(v_f) + a^2 c_2 + a^3 c_3 + a^4 c_4 + \cdots\right\}$$

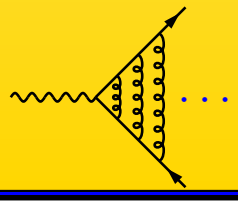
where $a = \alpha_s(s)/\pi$ and, assuming $4m_f^2 \ll s$, i.e. in the massless approximation

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= C_2(R)\left\{-\frac{3}{32}C_2(R) - \frac{3}{4}\beta_0\zeta(3) - \frac{33}{48}N_f + \frac{123}{32}N_c\right\} \\
&= \frac{365}{24} - \frac{11}{12}N_f - \beta_0\zeta(3) \simeq 1.9857 - 0.1153 N_f \\
c_3 &= -6.6368 - 1.2002 N_f - 0.0052 N_f^2 - 1.2395\,(\sum_f Q_f)^2/(3\sum_f Q_f^2) \\
c_4 &= 135.8 - 34.4\,N_f + 1.88\,N_f^2 - 0.010\,N_f^3 - \pi^2\beta_0^2\left(1.9857 - 0.1153\,N_f + \frac{5\beta_1}{6\beta_0}\right)\,,
\end{aligned}
$$

with $\beta_0 = (11 - 2/3\,N_f)/4$, $\beta_1 = (102 - 38/3\,N_f)/16$. All results are in the $\overline{MS}$ scheme. $N_f = \sum_{f:4m_f^2 \le s} 1$ is the number of active flavors. There is a mass dependent threshold factor in front of the curly brackets and the exact mass dependence of the first correction term

$$c_1(v) = \frac{2\pi^2}{3v} - (3+v)\left(\frac{\pi^2}{6} - \frac{1}{4}\right)$$

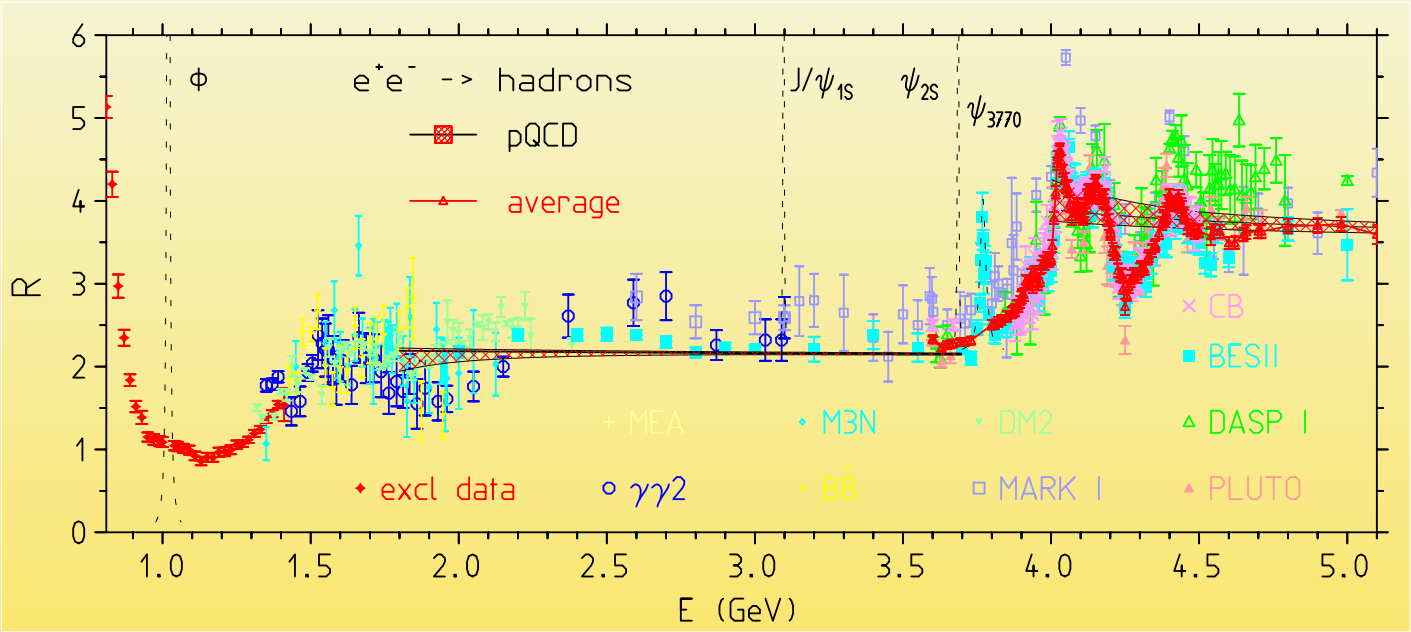
is singular (Coulomb singularity due to soft gluon final state interaction) at threshold. The singular terms of the n–gluon ladder diagrams

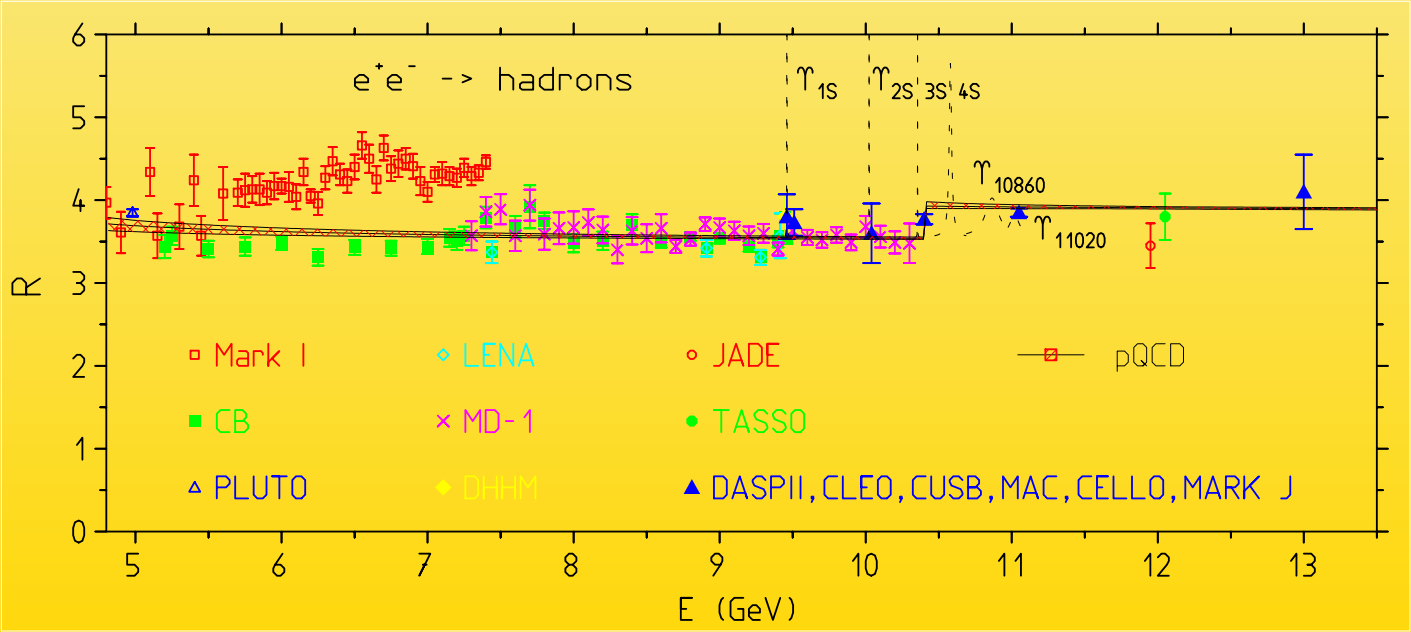


exponentiate and thereby remove the singularity : $1+x \to \frac{2x}{1-\mathrm{e}^{-2x}}$; $x = \frac{2\pi\alpha_s}{3v}$

$$\left(1 + c_1(v)\,\frac{\alpha_s}{\pi} + \cdots\right) \quad \to \quad \left(1 + c_1(v)\,\frac{\alpha_s}{\pi} - \frac{2\pi\alpha_s}{3v}\right)\,\frac{4\pi\alpha_s}{3v}\,\frac{1}{1-\exp\left\{-\frac{4\pi\alpha_s}{3v}\right\}} \quad .$$

Since pQCD in any case should not be applied near threshold, the above remark is somewhat academic. In contrast to QED, where Coulomb interaction problems are there and have to be cured appropriately.


e+e- -> hadrons
pQCD
average
φ
J/ψ1S
ψ2S
ψ3770
CB
BESII
DASP I
MEA
M3N
DM2
excl data
γγ2
BB̄
MARK I
PLUTO
R
E (GeV)

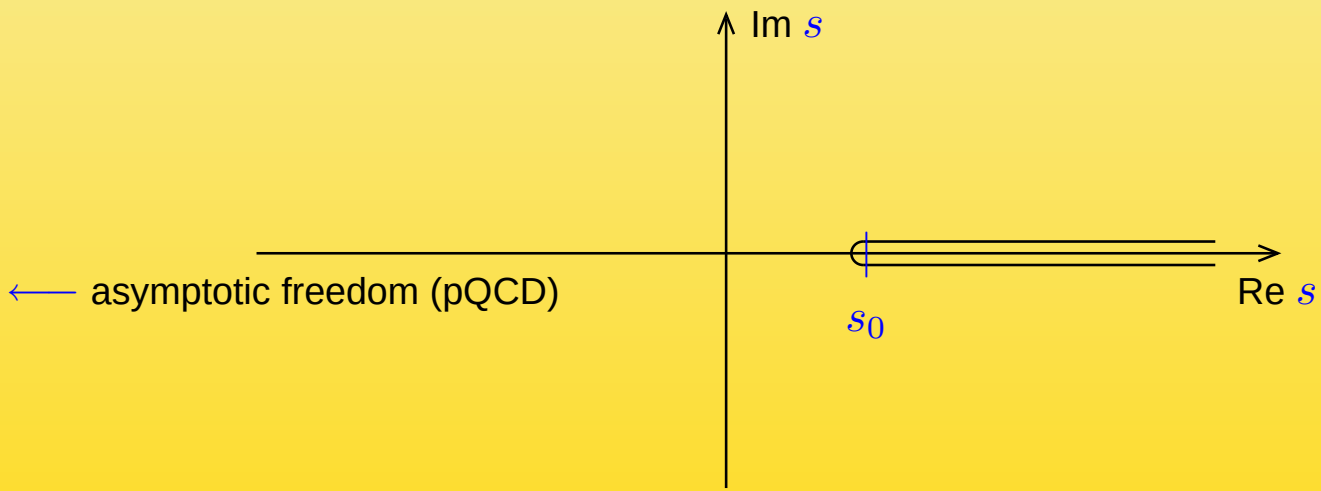

e+e- -> hadrons
ϒ1S
ϒ2S
3S
4S
ϒ10860
ϒ11020
Mark I
LENA
JADE
pQCD
CB
MD-1
TASSO
PLUTO
DHHM
DASPII,CLEO,CUSB,MAC,CELLO,MARK J
R
E (GeV)

Calculations: the coupling α_s and the masses m_q have to be understood as running parameters

$$R\left(\frac{m_{0f}^2}{s_0}, \alpha_s(s_0)\right) = R\left(\frac{m_f^2(\mu^2)}{s}, \alpha_s(\mu^2)\right) \;\; ; \;\; \mu = \sqrt{s} \;\; .$$

where $\sqrt{s_0}$ is a reference energy. RG resummation dramatically improves the convergence of perturbative approximations.

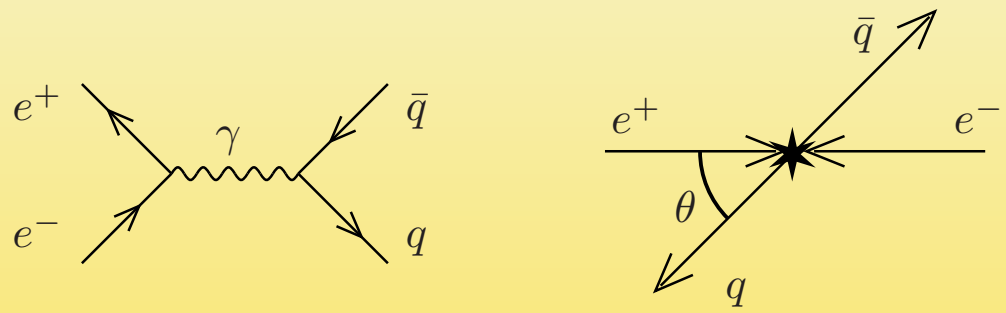
Perturbative QCD is supposed to work best in the deep Euclidean region[a] away from the physical region characterized by the cut in the analyticity plane:



Analyticity domain for the photon vacuum polarization function. In the complex s–plane there is a cut along the positive real axis for $s > s_0 = 4m^2$ where m is the mass of the lightest particles which can be pair–produced.

[a] This is directly accessible in Deep Inelastic electron–proton Scattering (DIS).

More “parton physics” in e^+e^-–annihilation: elementary hard process tells us that in a high energy collision of positrons and electrons (in the center of mass frame) q and $\bar{q}$ are produced with high momentum in opposite directions (back–to–back),

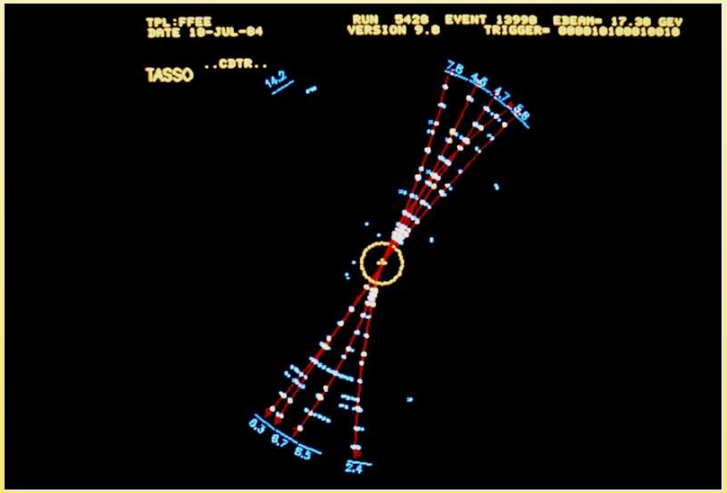


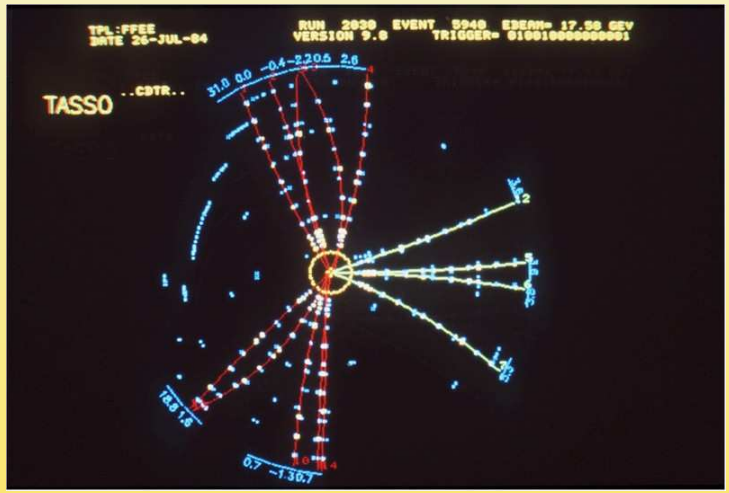
Fermion pair production in e^+e^-–annihilation. The lowest order Feynman diagram (left) and the same process in the c.m. frame (right). The arrows represent the spatial momentum vectors and θ is the production angle of the quark relative to the electron in the c.m. frame.

The differential cross–section, up to a color factor the same as for $e^+e^- \to \mu^+\mu^-$, reads

$$\frac{d\sigma}{d\Omega}(e^+e^- \to q\bar{q}) = \frac{3}{4}\frac{\alpha_s^2}{s}\sum Q_f^2 \; \left(1+\cos^2\theta\right)$$

typical for an angular distribution of a spin 1/2 particle. Indeed, the quark and the antiquark seemingly hadronize individually in that they form $\boxed{\text{jets}}$ = bunches of hadrons which concentrate in a relatively narrow angular cone.





Two and three jet event first seen by TASSO at DESY in 1979.

It is one of the spectacular and at the beginning completely unexpected predictions of QCD, that the energy get collimated more and more into narrower and narrower jets as the energy increases. However, as the energy increases also the number of primary partons increases such that the energy distributes among a larger number of jest, which then brings back a more isotropic distribution.

❑ Non–Perturbative Effects, Operator Product Expansion

Chiral Symmetry

$$U(N_f)_{\rm V} \otimes U(N_f)_{\rm A}$$

key role of Left–handed and Right–handed massless fields

Nambu 1960

chiral symmetry of strong interaction must be broken spontaneously!

only $SU(N_f)_V$ hadrons exist [parity partners missing]

$\Longrightarrow$ pions as Nambu-Goldstone bosons!

(non-perturbative)

if unbroken: nucleons must be massless, parity doubling! contradicting observation!

- ❑ SBB $\Rightarrow$ associated quark condensates
- ❑ how do they contribute to $R(s)$
- ❑ such non–perturbative (NP) effects may be included via Operator Product Expansion (OPE)[a]

[a] The operator product expansion (Wilson short distance expansion) is a formal expansion of the product of two local field operators $A(x)\,B(y)$ in powers of the distance $(x-y) \to 0$ in terms of singular coefficient functions and regular composite operators:

$$A(x)\,B(y) \simeq \sum_i C_i(x-y)\,\mathcal{O}_i(\frac{x+y}{2})$$

OPE of the electromagnetic current correlator yields

$$\begin{aligned}\hat{\Pi}'^{\rm NP}_\gamma(Q^2) &= \frac{4\pi\alpha}{3}\sum_{q=u,d,s} Q_q^2 N_{cq}\cdot\left[\frac{1}{12}\left(1-\frac{11}{18}a\right)\frac{\langle\frac{\alpha_s}{\pi}GG\rangle}{Q^4}\right.\\ &+ 2\left(1+\frac{a}{3}+\left(\frac{11}{2}-\frac{3}{4}l_{q\mu}\right)a^2\right)\frac{\langle m_q\bar{q}q\rangle}{Q^4}\\ &+ \left.\left(\frac{4}{27}a+\left(\frac{4}{3}\zeta_3-\frac{257}{486}-\frac{1}{3}l_{q\mu}\right)a^2\right)\sum_{q'=u,d,s}\frac{\langle m_{q'}\bar{q}'q'\rangle}{Q^4}\right]+\cdots\end{aligned}$$

where $a\equiv\alpha_s(\mu^2)/\pi$ and $l_{q\mu}\equiv\ln(Q^2/\mu^2)$. $\langle\frac{\alpha_s}{\pi}GG\rangle$ and $\langle m_q\bar{q}q\rangle$ are the scale–invariantly defined condensates. Sum rule estimates of the condensates yield typically (large uncertainties) $\langle\frac{\alpha_s}{\pi}GG\rangle\sim\ (0.389\ \text{GeV})^4$, $\langle m_q\bar{q}q\rangle\sim-(0.098\ \text{GeV})^4\ \text{ for }\ q=u,d$, and $\langle m_q\bar{q}q\rangle\sim-(0.218\ \text{GeV})^4$ for $q=s$. Note that the above expansion is just a parametrization of the high energy tail of NP effects associated with the existence of non–vanishing condensates. The dilemma with the OPE in our context is that it works for large enough Q^2 only and in this form fails do describe NP physics at lower Q^2. Once it starts to be numerically relevant pQCD starts to fail because of the growth of the strong coupling constant.

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where the operators $\mathcal{O}_i(\frac{x+y}{2})$ represent a complete system of local operators of increasing dimensions. The coefficients may be calculated formally by normal perturbation theory by looking at the Green functions

$$\langle 0|TA(x)\,B(y)\,X|0\rangle=\sum_{i=0}^{N}C_i(x-y)\,\langle 0|T\mathcal{O}_i(\frac{x+y}{2})\,X|0\rangle+R_N(x,y)\ .$$

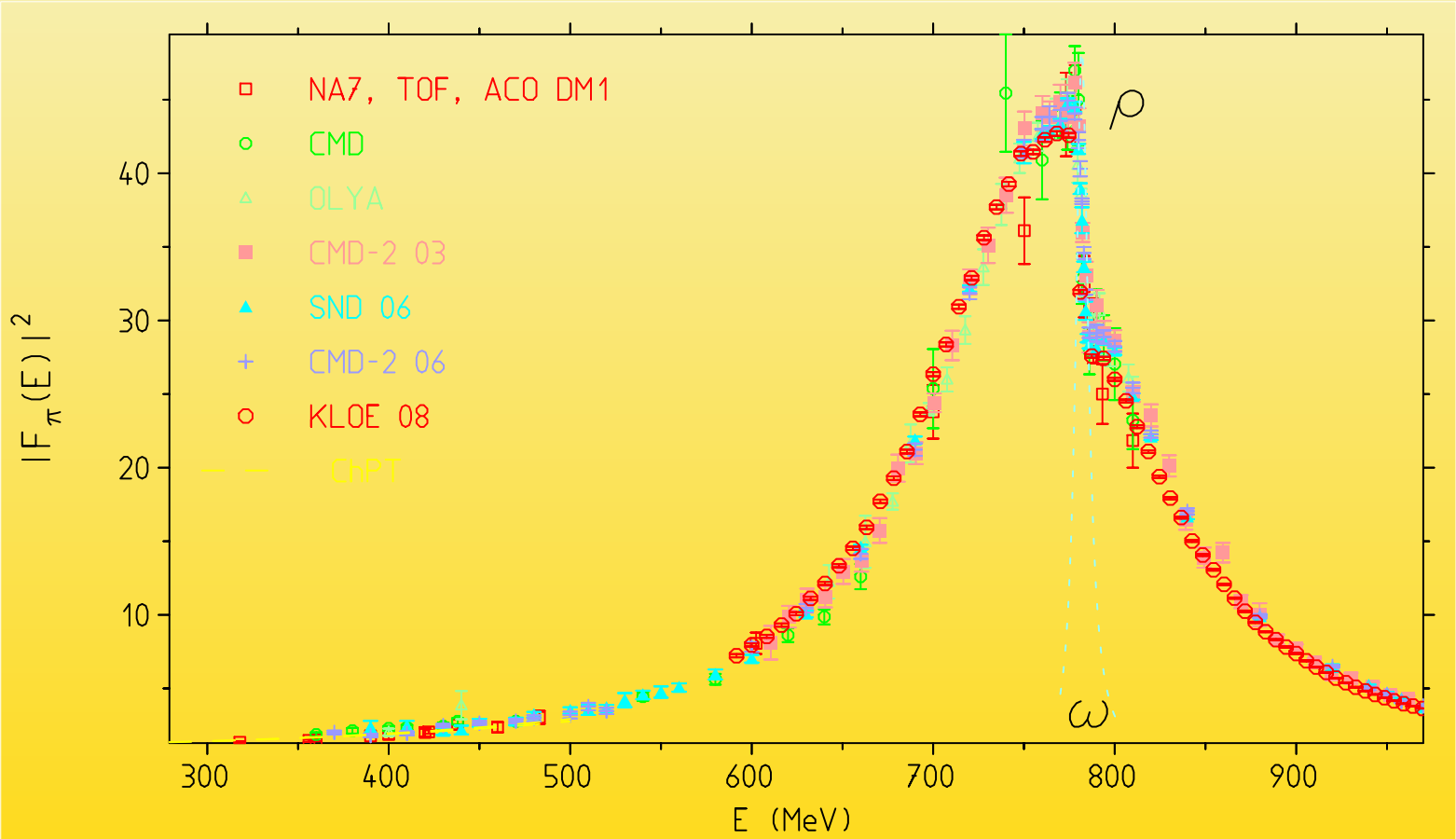
Note: condensates break dilatation (conformal) invariance

$$\Theta^{\mu}_{\mu} = \frac{\beta(g_s)}{2g_s}\, GG + (1+\gamma(g_s))\left\{m_u\, \bar{u}u + m_d\, \bar{d}d + \cdots\right\}$$

where $\beta(g_s)$ and $\gamma(g_s)$ are the RG coefficients

❑ Some considerations on hadron production at low energy

Certainly, quark-antiquark pair production is far from describing experimental cross–section data at low energies, where “infrared slavery” (confinement) is the dominating feature.



The Fig. shows a compilation of the low energy data in terms of the pion form factor. The latter is defined by

$$\sigma_{\pi\pi}(s) = \frac{\pi\alpha^2}{3s}\,(\beta_\pi)^3|F_\pi(s)|^2 \quad \text{or} \quad |F_\pi(s)|^2 = 4\,R_{\pi\pi}(s)\,(\beta_\pi)^{-3} \ .$$

- also low energy effective theory of QCD: chiral perturbation theory (CHPT) fails (has no ρ)
- looks natural to apply vector-meson dominance (VMD); proper QCD implementation

Resonance Lagrangian Approach (RLA)

- in fact photon mixes with hadronic vector–mesons like the ρ^0
- also nearby resonances like ρ^0 and ω are mixing (distorting Breit-Wigner shape)

❑ Pions and Scalar QED

- Pions can be seen in a particle detector ➠ behave like point particles (relatvistic Wigner state)
- on pion level coupling to photon fixed by gauge invariance

charged pion field = complex scalar field φ free Lagrangian

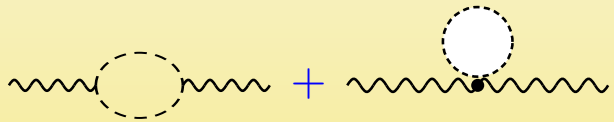
$$\mathcal{L}_\pi^{(0)} \quad = \quad (\partial_\mu\varphi)(\partial^\mu\varphi)^* - m_\pi^2\varphi\varphi^*$$

via *minimal substitution* $\partial_\mu\varphi \longrightarrow D_\mu\varphi = (\partial_\mu + \mathrm{i}eA_\mu(x))\ \varphi$, which replaces the ordinary by the *covariant derivative*, and which implies the scalar QED (sQED) Lagrangian

$$\mathcal{L}_\pi^{\mathrm{sQED}} \quad = \quad \mathcal{L}_\pi^{(0)} - \mathrm{i}e(\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*)A^\mu + e^2 g_{\mu\nu}\varphi\varphi^* A^\mu A^\nu \ .$$

In sQED the contribution of a pion loop to the photon VP is given by

$$-\mathrm{i}\,\Pi_\gamma^{\mu\nu\,(\pi)}(q) =$$



and the renormalized transversal photon self–energy reads

$$\Pi'^{(\pi)}_{\gamma\mathrm{ren}}(q^2) = \frac{\alpha}{6\pi}\left\{\frac{1}{3} + (1-y) - (1-y)^2\,G(y)\right\}$$

where $y = 4m^2/q^2$ and $G(y)$ was given before. For $q^2 > 4m^2$ there is an imaginary or absorptive part given by substituting $G(y) \to \mathrm{Im}\,G(y) = -\frac{\pi}{2\sqrt{1-y}}$ such that

$$\mathrm{Im}\,\Pi'^{(\pi)}_\gamma(q^2) = \frac{\alpha}{12}\,(1-y)^{3/2}$$

and for large q^2 is 1/4 of the corresponding value for a lepton. According to the *optical theorem* the absorptive part may be written in terms of the $e^+e^- \to \gamma^* \to \pi^+\pi^-$ production cross–section $\sigma_{\pi^+\pi^-}(s)$ as

$$\mathrm{Im}\,\Pi'^{\,\mathrm{had}}_\gamma(s) = \frac{s}{4\pi\alpha}\,\sigma_{\mathrm{had}}(s)$$

which hence we can read off to be

$$\sigma_{\pi^+\pi^-}(s) = \frac{\pi\alpha^2}{3s}\,\beta_\pi^3$$

$\beta_\pi = \sqrt{(1 - 4m_\pi^2/s)}$ pion velocity ➠ sQED predicts

$$F_\pi(s) = 1 \text{ independent of } s \ ,$$

in view of the ρ-resonance sitting there a complete failure up to $\boxed{50:1}$ results.

Thus sQED only works in conjunction with vector meson dominance model (VDM) or improvements of it

$$\Rightarrow e \to e\, F_\pi(q^2),\ e^2 \to e^2\, |F_\pi(q^2)|^2$$

taking care of bound-state nature of pion ($\longrightarrow$ form-factor)

Mandatory constraint: electromagnetic current conservation $\leftrightarrow$ $\boxed{F_\pi(0) = 1}$

Still sQED used in controlling real photon radiation issues: final state radiation (FSR) (Bloch-Nordsieck prescription), Kinoshita-Lee-Nauenberg theorem and all that.

Urgently need precise experimental investigation of FSR from hadrons (Venanzoni et al.)

❑ The Vector Meson Dominance Model

❖ phenomenological fact: photon and ρ meson exhibit direct coupling $\leftrightarrow \gamma - \rho^0$–mixing

❖ implementation: VMD model and extensions like RLA

❖ naive VMD model: replacing the photon propagator as

$$\frac{\mathrm{i}\, g^{\mu\nu}}{q^2} + \cdots \to \frac{\mathrm{i}\, g^{\mu\nu}}{q^2} + \cdots - \frac{\mathrm{i}\,(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2})}{q^2 - m_\rho^2} = \frac{\mathrm{i}\, g^{\mu\nu}}{q^2}\,\frac{m_\rho^2}{m_\rho^2 - q^2} + \cdots\,,$$

where the ellipses stand for the gauge terms.

❖ no change as $q^2 \to 0$

❖ changes asymptotics $1/q^2 \to 1/q^4$

❖ directly exhibits ρ^0-resonance

However, the naive VMD model does not respect chiral symmetry properties.

More precisely, VMD relates

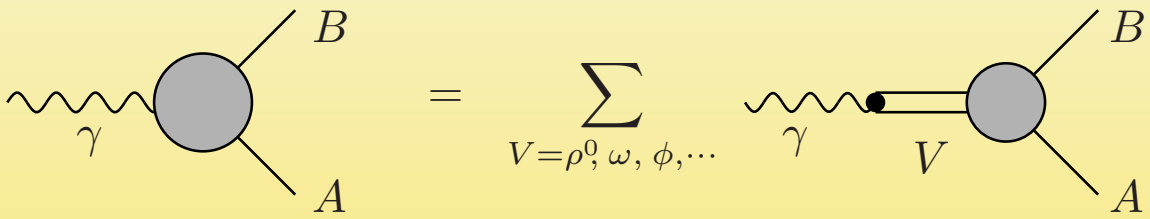
❑ hadronic part of the electromagnetic current $j_\mu^{\mathrm{had}}(x)$

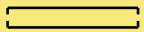
❑ source density $J^{(\rho)}(x)$ of the neutral vector meson ρ^0 by

$$\langle B|j_\mu^{\mathrm{had}}(0)|A\rangle = -\frac{M_\rho^2}{2\gamma_\rho}\,\frac{1}{q^2 - M_\rho^2}\,\langle B|J_\mu^{(\rho)}(0)|A\rangle$$

where $q = p_B - p_A$, p_A and p_B the four momenta of the hadronic states A and B, respectively, M_ρ is the

mass of the ρ meson. So far our VMD ansatz only accounts for the isovector part, but the isoscalar contributions mediated by the ω and the ϕ mesons may be included in exactly the same manner:



 $= \dfrac{M_V^2}{2\gamma_V}$;  $= \dfrac{-1}{q^2 - M_V^2}$.

The vector meson dominance model. A and B denote hadronic states.

The key idea is to treat the vector meson resonances like the ρ as elementary fields in a first approximation. Free massive spin 1 vector bosons are described by a Proca field $V_\mu(x)$ satisfying the Proca equation $(\Box + M_V^2)\, V_\mu(x) - \partial_\mu\, (\partial_\nu V^\nu) = 0$, which is designed such that it satisfies the Klein-Gordon equation and at the same time eliminates the unwanted spin 0 component: $\partial_\nu V^\nu = 0$. In the interacting case this equation is replaced by a *current–field identity* (CFI)

$$(\Box + M_V^2)\, V_\mu(x) - \partial_\mu\, (\partial_\nu V^\nu) = g_V\; J_\mu^{(V)}(x)$$

where the r.h.s. is the source mediating the interaction of the vector meson and g_V the coupling strength. The

current should be conserved $\partial^\mu J_\mu^{(V)}(x) = 0$. The CFI then implies

$$\langle B|V_\mu(0)|A\rangle = -\frac{g_V}{q^2 - M_V^2}\,\langle B|J_\mu^{(V)}(0)|A\rangle$$

where terms proportional to q^μ have dropped due to current conservation. The VMD assumes that the hadronic electromagnetic current is saturated by vector meson resonances[a]

$$j_\mu^{\rm had}(x) = \sum_{V=\rho^0,\,\omega,\,\phi,\cdots} \frac{M_V^2}{2\gamma_V}\,V_\mu(x)$$

in particular

$$\boxed{\langle\rho(p)|j_\mu^{\rm had}(0)|0\rangle = \varepsilon(p,\lambda)_\mu\,\frac{M_\rho^2}{2\gamma_\rho}\ ,\ \ p^2 = M_\rho^2\ \ .}$$

- M_V^2 required for dimensional reasons
- γ_V defines coupling constant (convention)
- VMD relation derives from the CFI and ansatz

The VMD model is known to describe the gross features of the electromagnetic properties of hadrons quite well,

[a] In large N_c QCD all hadrons become infinitely narrow, since all widths are suppressed by powers of $1/N_c$, and the VMD model becomes exact with an infinite number of narrow vector meson states. The large-N_c expansion attempts to approach QCD ($N_c = 3$) by an expansion in $1/N_c$. In leading approximation in the $SU(\infty)$ theory $R(s)$ would have the form

$$R(s) = \frac{9\pi}{\alpha^2}\sum_{i=0}^{\infty}\Gamma_i^{ee}M_i\,\delta(s - M_i^2)\ \ .$$

most prominent example are the nucleon form factors.

A way to incorporate vector–mesons ρ, ω, ϕ, $\ldots$ in accordance with the basic symmetries of QCD is the Resonance Lagrangian Approach (RLA), an extended version of CHPT, which also implements VMD in a consistent manner. In the flavor $SU(3)$ sector, similar to the pseudoscalar field $\Phi(x)$, the $SU(3)$ gauge bosons conveniently may be written as a 3×3 matrix field

$$V_\mu(x) \quad = \quad \sum_i T_i V_{\mu i} = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & \rho^+ & K^{*+} \\ \rho^- & \frac{-\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & K^{*0} \\ K^{*-} & \overline{K}^{*0} & -2\,\frac{\omega_8}{\sqrt{6}} \end{pmatrix}_\mu$$

in order to keep track of the appropriate $SU(3)$ weight factors.

❑ $\rho^0 - \gamma$ –Mixing

The unstable spin 1 vector mesons are described by a propagator exhibiting a pole

$$\tilde{M}_\rho^2 \equiv \left(q^2\right)_{\rm pole} = M_\rho^2 - {\rm i}\, M_\rho\, \Gamma_\rho$$

in the complex q^2-plane with correspondence

physical mass	$\Longleftrightarrow$	real part of location of propagator pole
width	$\Longleftrightarrow$	imaginary part of the location of the pole .

The simple relation between the full propagator and the irreducible self-energy only holds if there is no mixing, like for the charged $\rho^{\pm}$. In the neutral sector, because of $\gamma - \rho^0$ mixing, we cannot consider the ρ^0 and γ propagators separately. They form a 2×2 matrix propagator, so that the pole condition is modified into

$$s_P - m^2_{\rho^0} - \Pi_{\rho^0\rho^0}(s_P) - \frac{\Pi^2_{\gamma\rho^0}(s_P)}{s_P - \Pi_{\gamma\gamma}(s_P)} = 0 \ ,$$

with $s_P = \tilde{M}^2_{\rho^0}$[b].

Note: such mixing is not present in the charged channel (e.g. in τ–decay)

[b] The simplest way to treat this problem is to start from the inverse propagator given by the irreducible self-energies (sum of 1pi diagrams). Again we restrict ourselves to a discussion of the transverse part and we take out a trivial factor $-\mathrm{i}\, g_{\mu\nu}$ in order to keep notation as simple as possible. With this convention we have for the inverse $\gamma - \rho$ propagator the symmetric matrix

$$\hat{D}^{-1} = \begin{pmatrix} k^2 + \Pi_{\gamma\gamma}(k^2) & \Pi_{\gamma\rho}(k^2) \\ \Pi_{\gamma\rho}(k^2) & k^2 - M^2_\rho + \Pi_{\rho\rho}(k^2) \end{pmatrix}$$

Using 2×2 matrix inversion

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Rightarrow M^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

❑ Vector–Meson Production Cross–Sections

The cross–section for spin 1 boson production in e^+e^-–annihilation is described in full detail in Sect. 14 of my Lausanne Lectures. For massive spin 1 boson the polarization vectors $\varepsilon(p,\lambda)$ satisfy the completeness relation

$$\sum_\lambda \varepsilon(p,\lambda)_\mu\, \varepsilon^*(p,\lambda)_\nu = \left(-g_{\mu\nu} - \frac{p_\mu p_\nu}{M_\rho^2}\right)$$

and the hadronic tensor reads

$$h_{\mu\nu} = \frac{1}{3}\sum_\lambda \varepsilon(p,\lambda)_\mu\, \frac{M_\rho^2}{2\gamma_\rho}\varepsilon^*(p,\lambda)_\nu\, \frac{M_\rho^2}{2\gamma_\rho} = \left(-g_{\mu\nu} - \frac{p_\mu p_\nu}{M_\rho^2}\right)\frac{M_\rho^4}{2\gamma_\rho^2}\,,$$

and one easily calculates the spin average $\overline{|T|^2}$.

The result for the $e^+e^- \to \rho^0$ cross–section and some useful approximations are the following:

we find for the propagators

$$\begin{aligned}
D_{\gamma\gamma} &= \frac{1}{k^2 + \Pi_{\gamma\gamma}(k^2) - \frac{\Pi_{\gamma\rho}^2(k^2)}{k^2 - M_\rho^2 + \Pi_{\rho\rho}(k^2)}} \simeq \frac{1}{k^2 + \Pi_{\gamma\gamma}(k^2)} \\
D_{\gamma\rho} &= \frac{-\Pi_{\gamma\rho}(k^2)}{(k^2 + \Pi_{\gamma\gamma}(k^2))(k^2 - M_\rho^2 + \Pi_{\rho\rho}(k^2)) - \Pi_{\gamma\rho}^2(k^2)} \simeq \frac{-\Pi_{\gamma\rho}(k^2)}{k^2\,(k^2 - M_\rho^2)} \\
D_{\rho\rho} &= \frac{1}{k^2 - M_\rho^2 + \Pi_{\rho\rho}(k^2) - \frac{\Pi_{\gamma\rho}^2(k^2)}{k^2 + \Pi_{\gamma\gamma}(k^2)}} \simeq \frac{1}{k^2 - M_\rho^2 + \Pi_{\rho\rho}(k^2)}\,.
\end{aligned}$$

These expressions sum correctly all the reducible bubbles. The approximations indicated are the one-loop results. The extra terms are higher order contributions.

❖ Breit-Wigner resonance: field theory version

The field theoretic form of a Breit-Wigner resonance obtained by the Dyson summation of a massive spin 1 transversal part of the propagator in the approximation that the imaginary part of the self–energy yields the width by Im $\Pi_V(M_V^2) = M_V\Gamma_V$ near resonance.

$$\sigma_{BW}(s) = \frac{12\pi}{M_R^2}\frac{\Gamma_{e^+e^-}}{\Gamma}\frac{s\Gamma^2}{(s-M_R^2)^2+M_R^2\Gamma^2} \quad .$$

❖ Breit-Wigner resonance

The resonance cross–section from a classical non–relativistic Breit-Wigner resonance is given by

$$\sigma_{BW}(s) = \frac{3\pi}{s}\frac{\Gamma\Gamma_{e^+e^-}}{(\sqrt{s}-M_R)^2+\frac{\Gamma^2}{4}} \quad .$$

❖ Narrow width resonance

The narrow width approximation for a zero width resonance reads

$$\sigma_{\rm NW}(s) = \frac{12\pi^2}{M_R}\Gamma_{e^+e^-}\delta(s-M_R^2) \quad .$$

More realistic refined approaches including isospin breaking from $m_d \neq m_u$ and multiple resonances

- Gounaris-Sakurai model
- Resonance Lagrangian models
- Analytic approach à la Omnès (Colangelo, Leutwyler et al.)