\[ \alpha_{\text{QED}}(M_Z) \] and future prospects with low energy \( e^+e^- \) collider data

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Outline of Talk:

- Motivation
- The Role of $\alpha_{\text{QED, eff}}$ in Precision Physics
- Evaluation of $\alpha_{\text{QED}}(M_Z^2)$
- Reducing uncertainties via the Euclidean split trick: Adler function controlled pQCD
- Prospects for future improvements
- Addendum 1: Using $\tau$–decay spectra + isospin breakings
- Addendum 2: The coupling $\alpha_2$, $M_W$ and $\sin^2 \Theta_f$
1. Motivation

The Parameters of the Standard Model

– in four fermion and vector boson processes –

in addition QCD coupling $\alpha_s, \gamma_t$ vs. $M_t, \lambda_H$ vs. $M_H$ etc.

unlike in QED and QCD in SM (SBGT)

parameter interdependence

only 3 independent quantities
(besides fermion masses and mixing parameters)

$\alpha, G_\mu, M_Z \Rightarrow \alpha_{\text{eff}}(M_Z^2) \Rightarrow \text{large hadronic correction}$

$$\sin^2 \Theta_i \cos^2 \Theta_i = \frac{\pi \alpha}{\sqrt{2} G_\mu M_Z^2} \frac{1}{1 - \Delta r_i}; \quad \Delta r_i = \Delta r_i(\alpha, G_\mu, M_Z, m_H, m_{f\neq t}, m_t)$$

parameter relationships between very precisely measurable quantities

precision tests, possible sign of new physics

non-perturbative $\Delta \alpha_{\text{had}}^{(5)}(M_Z^2)$ is limiting precision predictions

Note: 30 SD disagreement between SM prediction and experiment when subleading corrections are dropped!
Gauge coupling unification?

Precise SM predictions require to determine the $U(1)_Y \otimes SU(2)_L \otimes SU(3)_c$ SM gauge couplings $\alpha_{em}$, $\alpha_2$ and $\alpha_s \equiv \alpha_3$ (QCD) as accurately as possible.

**** a theory can not be better than its input parameters ****

⇒ precision limitations due to non-perturbative hadronic contributions ⇐

Beyond SM physics?

\[ \alpha_s = 0.1183 \pm 0.0027 \quad \text{vs} \quad \pm 0.0009 \]
SM extrapolation up to Planck scale?

After Higgs discovery: Higgs vacuum stability issue!

⇒ Need very precise SM parameters: $g', g, g_s, y_t, \lambda$

The SM dimensionless couplings in the $\overline{\text{MS}}$ scheme as a function of the renormalization scale for $M_H = 124 - 127$ GeV.
perturbation expansion works up to the Planck scale!
no Landau pole or other singularities, Higgs potential likely remains stable!

- $U(1)_Y$ screening (IR free), $SU(2)_L$, $SU(3)_c$ antiscreening (UV free): $g_1, g_2, g_3$
  
  as expected (standard wisdom)

- Top Yukawa $y_t$ and Higgs $\lambda$: screening if standalone (IR free, like QED)
  
  as part of SM, transmutation from IR free to UV free

As SM couplings are as they are: QCD dominance in top Yukawa RG requires $g_3 > \frac{3}{4} y_t$, top Yukawa dominance in Higgs RG requires $\lambda < \frac{3(\sqrt{5}-1)}{2} y_t^2$ in the gaugeless ($g_1, g_2 = 0$) limit.

In the focus:
- does Higgs self-coupling stay positive $\lambda > 0$ up to $\Lambda_{\text{Pl}}$?
- the key question/problem concerns the size of the top Yukawa coupling $y_t$
  
  decides about stability of our world! — [$\lambda = 0$ would be essential singularity!]

Will be decided by:

- more precise input parameters
- better established EW matching conditions
F.J., Kalmykow, Kniehl, On-Shell vs $\overline{\text{MS}}$ parameter matching

- the big issue is the very delicate conspiracy between SM couplings:
  precision determination of parameters more important than ever $\Rightarrow$
  the challenge for LHC and ILC/FCC: precision values for $\lambda$, $y_t$ and $\alpha_s$,
  and for low energy hadron facilities: more precise hadronic cross sections to reduce hadronic uncertainties in $\alpha(M_Z)$ and $\alpha_2(M_Z)$

New gate to precision cosmology of the early universe!
Shaposnikov et al., Degrassi et al. matching

- the big issue is the very delicate conspiracy between SM couplings:
- precision determination of parameters more important than ever ⇒
- the challenge for LHC and ILC/FCC: precision values for $\lambda$, $y_t$ and $\alpha_s$,
- and for low energy hadron facilities: more precise hadronic cross sections to reduce hadronic uncertainties in $\alpha(M_Z)$ and $\alpha_2(M_Z)$

New gate to precision cosmology of the early universe!
2. $\alpha(M_Z^2)$ in precision physics (precision physics limitations)

Uncertainties of hadronic contributions to effective $\alpha$ are a problem for electroweak precision physics:

$\alpha, G_\mu, M_Z$ most precise input parameters $\Rightarrow$ precision predictions

$\sin^2 \Theta_f, v_f, a_f, M_W, \Gamma_Z, \Gamma_W, \cdots$

$\alpha(M_Z), G_\mu, M_Z$ best effective input parameters for VB physics $(Z,W)$ etc.

$$\frac{\delta \alpha}{\delta G_\mu G_\mu} \sim 0.9 \div 1.6 \times 10^{-4} \quad \text{(present : lost } 10^5 \text{ in precision!)}$$

$$\frac{\delta \alpha(M_Z)}{\alpha(M_Z)} \sim 5.3 \times 10^{-5} \quad \text{(ILC requirement)}$$

**LEP/SLD:** $\sin^2 \Theta_{\text{eff}} = (1 - g_V / g_A) / 4 = 0.23148 \pm 0.00017$

$$\delta \Delta \alpha(M_Z) = 0.00020 \quad \Rightarrow \quad \delta \sin^2 \Theta_{\text{eff}} = 0.00007$$

affects Higgs mass bounds, precision tests and new physics searches!!!

For pQCD contributions very crucial: precise QCD parameters $\alpha_s, m_c, m_b, m_t \Rightarrow$ Lattice-QCD
3. Evaluation of $\alpha(M_Z^2)$

Non-perturbative hadronic contributions $\Delta\alpha_{\text{had}}^{(5)}(s) = -\left(\Pi'_{\gamma}(s) - \Pi'_{\gamma}(0)\right)$ can be evaluated in terms of $\sigma(e^+e^- \rightarrow \text{hadrons})$ data via dispersion integral:

$$\Delta\alpha_{\text{had}}^{(5)}(s) = -\frac{\alpha s}{3\pi} \left( \int_{4m_{\pi}^2}^{E_{\text{cut}}^2} \frac{ds'}{s'(s' - s)} \int_{0}^{\infty} ds' \frac{R_{\gamma}^{\text{data}}(s')}{s'(s' - s)} \right)$$

where

$$R_{\gamma}(s) \equiv \frac{\sigma(0)(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})}{4\pi \alpha^2 \frac{3s}{3s}}$$

hadronic vacuum polarization

$$\alpha(s) = \frac{\alpha}{1 - \Delta\alpha(s)} ; \quad \Delta\alpha(s) = \Delta\alpha_{\text{lep}}(s) + \Delta\alpha_{\text{had}}^{(5)}(s) + \Delta\alpha_{\text{top}}(s)$$
Present situation: (after KLOE, BaBar and first BESIII results)

\[
\Delta \alpha_{\text{hadrons}}^{(5)}(M_Z^2) = 0.027648 \pm 0.000176 \\
0.027504 \pm 0.000118 \quad \text{Adler}
\]

\[
\alpha^{-1}(M_Z^2) = 128.942 \pm 0.021 \\
128.961 \pm 0.011 \quad \text{Adler}
\]

Scan: CMD-2, SND (NSK); ISR: KLOE (pioneered the method), BaBar, BESIII

New experimental input for HVP: BESIII-ISR, VEPP-2000

a) Initial state radiation (ISR),  b) Standard energy scan.
Still: most precise ISR measurements in conflict. BESIII steps to resolve this

Recent/preliminary results:

- $e^+ e^- \rightarrow \pi^+ \pi^-$ from CMD-3
- $e^+ e^- \rightarrow \pi^+ \pi^- \pi^0$ from Belle
- $e^+ e^- \rightarrow K^+ K^-$ from CMD-3
- $e^+ e^- \rightarrow K^+ K^-$ from SND
- $e^+ e^- \rightarrow \omega \pi^0 \rightarrow \pi^0 \pi^0 \gamma$ from SND
- $e^+ e^- \rightarrow \pi^+ \pi^-$ from BESIII recent, most important
- $R_{uds}$ and $R$ from 3.21 GeV to 3.72 GeV from KEDR
New from BESIII

BESIII vs BaBar and KLOE

BESIII Collab. arXiv:1507.08188v3: 1.9 $\sigma$ below BaBar in agreement with KLOE
\( \Delta \alpha_{\text{had}}(M_Z^2) \) results from ranges:

for \( M_Z = 91.1876 \) GeV in units \( 10^{-4} \). 2015 update in terms of \( e^+ e^- \)-data and pQCD. 46% data, 54% perturbative QCD. pQCD is used between 5.2 GeV and 9.5 GeV and above 13.0 GeV.

<table>
<thead>
<tr>
<th>final state</th>
<th>range (GeV)</th>
<th>( \Delta \alpha_{\text{had}}^{(5)} \times 10^4 ) (stat) (syst) [tot]</th>
<th>rel</th>
<th>abs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>(0.28, 1.05)</td>
<td>33.91 (0.05) (0.18) [0.18]</td>
<td>0.5%</td>
<td>1.1%</td>
</tr>
<tr>
<td>( \omega )</td>
<td>(0.42, 0.81)</td>
<td>3.10 (0.04) (0.08) [0.09]</td>
<td>3.0%</td>
<td>0.3%</td>
</tr>
<tr>
<td>( \phi )</td>
<td>(1.00, 1.04)</td>
<td>4.76 (0.07) (0.11) [0.13]</td>
<td>2.7%</td>
<td>0.5%</td>
</tr>
<tr>
<td>( J/\psi )</td>
<td></td>
<td>12.38 (0.60) (0.67) [0.90]</td>
<td>7.2%</td>
<td>25.5%</td>
</tr>
<tr>
<td>( \Upsilon )</td>
<td></td>
<td>1.30 (0.05) (0.07) [0.09]</td>
<td>6.9%</td>
<td>0.3%</td>
</tr>
<tr>
<td>had</td>
<td>(1.05, 2.00)</td>
<td>16.22 (0.07) (0.89) [0.89]</td>
<td>5.5%</td>
<td>25.1%</td>
</tr>
<tr>
<td>had</td>
<td>(2.00, 3.10)</td>
<td>15.34 (0.08) (0.61) [0.62]</td>
<td>4.0%</td>
<td>12.1%</td>
</tr>
<tr>
<td>had</td>
<td>(3.10, 3.60)</td>
<td>4.93 (0.03) (0.13) [0.14]</td>
<td>2.8%</td>
<td>0.6%</td>
</tr>
<tr>
<td>had</td>
<td>(3.60, 5.20)</td>
<td>16.62 (0.11) (0.05) [0.12]</td>
<td>0.3%</td>
<td>0.6%</td>
</tr>
<tr>
<td>pQCD</td>
<td>(5.20, 9.46)</td>
<td>33.84 (0.00) (0.03) [0.03]</td>
<td>0.0%</td>
<td>0.1%</td>
</tr>
<tr>
<td>had</td>
<td>(9.46, 13.00)</td>
<td>18.32 (0.24) (1.01) [1.04]</td>
<td>5.7%</td>
<td>34.0%</td>
</tr>
<tr>
<td>pQCD</td>
<td>(13.0, ( \infty ))</td>
<td>115.73 (0.00) (0.04) [0.04]</td>
<td>0.0%</td>
<td>0.1%</td>
</tr>
<tr>
<td>data</td>
<td>(0.28, 13.00)</td>
<td>126.86 (0.67) (1.64) [1.78]</td>
<td>1.4%</td>
<td>100.0%</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>276.43 (0.67) (1.64) [1.78]</td>
<td>0.6%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>
Correlation between different contributions to $\alpha_\mu^{\text{had}}$ and $\Delta\alpha^{\text{had}}(5)$. 

Contributions from $e^+e^-$ data ranges and form pQCD to $\alpha_\mu^{\text{had}}$ and $\Delta\alpha^{\text{had}}(5)$. 

F. Jegerlehner – INFN LNF - Laboratori Nazionali di Frascati, –  
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4. Reducing uncertainties via the Euclidean split trick: Adler function controlled pQCD

- experiment side: new more precise measurements of $R(s)$
- future direct measurements Patrick Janot
- theory side: $\alpha_{em}(M_Z^2)$ by the "Adler function controlled" approach

\[
\alpha(M_Z^2) = \alpha^{\text{data}}(-s_0) + \left[ \alpha(-M_Z^2) - \alpha(-s_0) \right]^{\text{pQCD}} + \left[ \alpha(M_Z^2) - \alpha(-M_Z^2) \right]^{\text{pQCD}}
\]

where the space-like $-s_0$ is chosen such that pQCD is well under control for $-s < -s_0$. The monitor to control the applicability of pQCD is the Adler function

\[
D(Q^2 = -s) = \frac{3\pi}{\alpha} s \frac{d}{ds} \Delta\alpha_{\text{had}}(s) = -(12\pi^2) s \frac{d\Pi'_\gamma(s)}{ds} = \frac{Q^2}{s + Q^2} = \int_{4m^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} 
\]

which also is determined by $R(s)$ and can be evaluated in terms of experimental $e^+e^-$—data. Perturbative QCD tail: $D(Q^2) \to N_c \sum_f Q_f^2 (1 + O(\alpha_s))$ as $Q^2 \to \infty$.

\[ \Delta \alpha^\text{had} \text{ Adler function controlled} \]

✓ use old idea: Adler function: Monitor for comparing theory and data

\[
D(-s) = \frac{3\pi}{\alpha} s \frac{d}{ds} \Delta \alpha^\text{had}(s) = -(12\pi^2) s \frac{d\Pi'_\gamma(s)}{ds}
\]

\[
\Rightarrow D(Q^2) = Q^2 \left( \int_{4m_i^2}^{E_{\text{cut}}^2} d s \frac{R(s)_{\text{data}}}{(s + Q^2)^2} + \int_{E_{\text{cut}}^2}^{\infty} d s \frac{R_{\text{pQCD}}(s)}{(s + Q^2)^2} \right).
\]

<table>
<thead>
<tr>
<th>pQCD ↔ ( R(s) )</th>
<th>pQCD ↔ ( D(Q^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>very difficult to obtain in theory</td>
<td>smooth simple function in Euclidean region</td>
</tr>
</tbody>
</table>

Conclusion:

❄ time-like approach: pQCD works well in “perturbative windows”

3.00 - 3.73 GeV, 5.00 - 10.52 GeV and 11.50 - \( \infty \) Kühn, Steinhauser

❄ space-like approach: pQCD works well for \( \sqrt{Q^2} = -q^2 > 2.0 \) GeV (see plot)
“Experimental” Adler–function versus theory (pQCD + NP)

Error includes statistical + systematic here (in contrast to most $R$-plots showing statistical errors only)!

(Eidelman, F. J., Kataev, Veretin 98, FJ 08/15 updates)
theory based on results by Chetyrkin, Kühn et al.
⇒ pQCD works well controlled to predict $D(Q^2)$ down to $s_0 = (2.0 \text{ GeV})^2$; use this to calculate

$$\Delta \alpha_{\text{had}}(-Q^2) \sim \frac{\alpha}{3\pi} \int dQ'^2 \frac{D(Q'^2)}{Q'^2}$$

and obtain, for $s_0 = (2.0 \text{ GeV})^2$:

- $\Delta \alpha_{\text{had}}^{(5)}(-s_0)_{\text{data}} = 0.006392 \pm 0.000064$
- $\Delta \alpha_{\text{had}}^{(5)}(-M_Z^2) = 0.027466 \pm 0.000118$
- $\Delta \alpha_{\text{had}}^{(5)}(M_Z^2) = 0.027504 \pm 0.000119$

- shift $+0.000008$ from the 5-loop contribution
- error $\pm 0.000100$ added in quadrature form perturbative part

QCD parameters: $\bullet \alpha_s(M_Z) = 0.1189(20)$,
- $m_c(m_c) = 1.286(13) \ [M_c = 1.666(17)] \ \text{GeV}$, $\bullet m_b(m_c) = 4.164(25) \ [M_b = 4.800(29)] \ \text{GeV}$

based on a complete 3–loop massive QCD analysis Kühn et al 2007

\[ \Delta \alpha_{\text{had}}(-M_0^2) \] results from ranges:

for \( M_0 = 2 \) GeV in units \( 10^{-4} \). 2015 update in terms of \( e^+e^- \)-data and pQCD. 95% data, 5% perturbative QCD. pQCD is used between 5.2 GeV and 9.5 GeV and above 13.0 GeV.

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<td>( \rho )</td>
<td>(0.28, 1.05)</td>
<td>29.77 (0.04) (0.15)[0.16]</td>
<td>0.5%</td>
<td>5.8%</td>
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<td>1.5%</td>
</tr>
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<td>3.78 (0.05) (0.09)[0.10]</td>
<td>2.7%</td>
<td>2.4%</td>
</tr>
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<td>( J/\psi )</td>
<td></td>
<td>3.21 (0.15) (0.15)[0.21]</td>
<td>6.7%</td>
<td>10.6%</td>
</tr>
<tr>
<td>( \Upsilon )</td>
<td></td>
<td>0.05 (0.00) (0.00)[0.00]</td>
<td>6.8%</td>
<td>0.0%</td>
</tr>
<tr>
<td>had</td>
<td>(1.05, 2.50)</td>
<td>13.76 (0.05) (0.56)[0.56]</td>
<td>4.1%</td>
<td>72.0%</td>
</tr>
<tr>
<td>had</td>
<td>(2.50, 3.10)</td>
<td>2.49 (0.01) (0.18)[0.18]</td>
<td>7.2%</td>
<td>7.4%</td>
</tr>
<tr>
<td>had</td>
<td>(3.10, 3.60)</td>
<td>1.30 (0.01) (0.03)[0.04]</td>
<td>2.8%</td>
<td>0.3%</td>
</tr>
<tr>
<td>had</td>
<td>(3.60, 5.20)</td>
<td>2.87 (0.02) (0.00)[0.02]</td>
<td>0.0%</td>
<td>0.1%</td>
</tr>
<tr>
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<td>(5.20, 9.46)</td>
<td>2.66 (0.00) (0.00)[0.00]</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>had</td>
<td>(9.46, 13.00)</td>
<td>0.57 (0.01) (0.03)[0.03]</td>
<td>5.5%</td>
<td>0.2%</td>
</tr>
<tr>
<td>pQCD</td>
<td>(13.00, 0.00)</td>
<td>0.70 (0.00) (0.00)[0.00]</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>data</td>
<td>(0.28, 13.00)</td>
<td>60.49 (0.18) (0.63)[0.66]</td>
<td>1.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>63.85 (0.18) (0.63)[0.66]</td>
<td>1.0%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>
Of $\Delta \alpha_{\text{had}}^{(5)}(M_Z^2)$ 22% data, 78% pQCD!

Contributions from $e^+e^-$ data ranges and form pQCD to $\Delta \alpha_{\text{had}}^{(5)}(-M_0^2)$ vs. $\Delta \alpha_{\text{had}}^{(5)}(M_Z^2)$. 
### How much pQCD?

<table>
<thead>
<tr>
<th>Method</th>
<th>range [GeV]</th>
<th>pQCD</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard approach:</td>
<td>5.2 - 9.5</td>
<td>33.84(0.03)</td>
<td></td>
</tr>
<tr>
<td>My choice</td>
<td>13.0 - ∞</td>
<td>115.73(0.04)</td>
<td>→ 149.57 (0.05)</td>
</tr>
<tr>
<td>Standard approach:</td>
<td>2.0 - 9.5</td>
<td>72.09(0.07)</td>
<td></td>
</tr>
<tr>
<td>Davier et al.</td>
<td>11.5 - ∞</td>
<td>123.24(0.05)</td>
<td>→ 195.33 (0.09)</td>
</tr>
<tr>
<td>Adler function controlled:</td>
<td>5.2 - 9.5</td>
<td>2.66(0.00)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13.0 - ∞</td>
<td>0.70(0.00)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>−∞ − −2.0</td>
<td>210.68(0.99)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>−$M_Z \rightarrow M_Z$</td>
<td>0.38(0.00)</td>
<td>→ 214.42 (0.99)</td>
</tr>
</tbody>
</table>

Note: the Adler function monitored Euclidean data vs pQCD split approach is only moderately more pQCD-driven, than the time-like approach adopted by Davier et al. and others.
The time-like vs space-like effective charge

Note that the smooth space-like effective charge agrees rather well with the non-resonant “background” above the $\Phi$ (kind of duality)

No proof that this cannot produce non-negligible shifts!

Time-like VP-subtraction cannot be implemented locally near OZI suppressed resonances: $J/\psi, \psi'$ and $\Upsilon_1, \Upsilon_2, \Upsilon_3$
\( \alpha_{\text{QED},\text{eff}} \): time-like vs. space-like

\[ 10^{-3} \]

\begin{center}
\begin{tikzpicture}
\begin{axis}[
width=\textwidth,
height=\textwidth,
xlabel = \( \sqrt{s} \) (GeV),
ylabel = \( \alpha_{\text{QED},\text{eff}}(s) \),

\addplot[red, thick, solid] coordinates {
(10^{-2}, 7.40) (10^{-1}, 7.44) (10^{0}, 7.40) (10^{1}, 7.44) (10^{2}, 7.48) (10^{3}, 7.52)
};
\addlegendentry{time-like}
\addplot[blue, thick, dashed] coordinates {
(10^{-2}, 7.40) (10^{-1}, 7.44) (10^{0}, 7.40) (10^{1}, 7.44) (10^{2}, 7.48) (10^{3}, 7.52)
};
\addlegendentry{space-like}
\end{axis}
\end{tikzpicture}
\end{center}

\( \alpha_{\text{QED},\text{eff}} \) duality: \( \alpha_{\text{QED},\text{eff}}(s) \) is varying dramatically near resonances, but agrees quite well in average with space-like version
An alternative method: measure space-like $\alpha_{\text{QED, eff}}(t)$

Newly proposed recently: [arXiv:1504.02228]
“A new approach to evaluate the leading hadronic corrections to the muon g-2”
Carloni Calame, Passera, Trentadue, Venenzoni 2015

- measuring directly low energy running $\alpha_{\text{QED}}(s)$ in space-like region via

**Bhabha scattering**
\[ e^+(p_+) \ e^-(p_-) \rightarrow e^+(p'_+) \ e^-(p'_-) \]

VP dressed tree level Bhabha scattering in QED

has two tree level diagrams the $t$– and the $s$–channel. With the positive c.m. energy
square \( s = (p^+ + p^-)^2 \) and the negative momentum transfer square

\[
t = (p_+ - p'_-)^2 = -\frac{1}{2} (s - 4m_e^2) (1 - \cos \theta),
\]

\( \theta \) the \( e^- \) scattering angle, there are two very different scales involved

\[\Box \text{ space-like } \alpha_{QED,\text{eff}}(-Q^2) \text{ determines } a_{\mu}^{\text{had}} \text{ via} \]

\[
a_{\mu}^{\text{had}} = \frac{\alpha}{\pi} \int_0^1 \text{d}x \; (1 - x) \; \Delta\alpha_{\text{had}}(-Q^2(x))
\]

where \( Q^2(x) \equiv \frac{x^2}{1-x}m_\mu^2 \) is the space–like square momentum–transfer or

\[
x = \frac{Q^2}{2m_\mu^2} \left( \sqrt{1 + \frac{4m_\mu^2}{Q^2}} - 1 \right).
\]

Also in the Euclidean region the integrand is highly peaked, now around half of the \( \rho \) meson mass scale.
The integrand of $a_{\mu}^{\text{had}}$ integral as functions of $x$ and $Q$. Strongly peaked at about 330 MeV.

The VP dressed lowest order cross–section is

$$\frac{d\sigma}{d \cos \Theta} = \frac{s}{48\pi} \sum_{ik} |A_{ik}|^2$$

where $A_{ik}$ tree level helicity amplitudes, $i, k = \text{L,R}$ left– and right–handed electrons.
Dressed transition amplitudes: \((m_e \approx 0)\)

\[
|A_{\mathrm{LL},\mathrm{RR}}|^2 = \frac{3}{8} (1 + \cos \theta)^2 \left| \frac{e^2(s)}{s} + \frac{e^2(t)}{t} \right|^2
\]

\[
|A_{\mathrm{LR},\mathrm{RL}}|^2 = \frac{3}{8} (1 - \cos \theta)^2 \left| \frac{e^2(s)}{s} + \frac{e^2(t)}{t} \right|^2
\]

Preferably one uses small angle Bhabha scattering (small \(|t|\)) as a normalizing process which is dominated by the \(t\)-channel \(\sim 1/t\), however, detecting electrons and positrons along the beam axis often has its technical limitations.

Care also is needed concerning the ISR corrections because cuts for the Bhabha process \((e^+e^- \rightarrow e^+e^-)\) typically are different from the ones applied to \(e^+e^- \rightarrow \text{hadrons}\). Usually, experiments have included corresponding uncertainties in their systematic errors, if they not have explicitly accounted for all appropriate radiative corrections.
The primary goal of [arXiv:1504.02228]: determining $d_\mu^{\text{had}}$ in an alternative way

**My proposal here:** determine very accurately

\[ \Delta \alpha_{\text{had}}(-Q^2) \text{ at } Q \approx 2 \text{ GeV} \]

by this method (one single number!) as the non-perturbative part of $\Delta \alpha_{\text{had}}(M_Z^2)$ as in “Adler function” approach.

- extremely important for the FCC-ee/ILC physics program

❌ $t$–channel dominance requires $|t|/s \ll 1 \implies$ better for high energy machine?

Best: can be done at future $e^+e^-$ machine at $Z$ peak:

\[ \cos \theta \approx 1 - 2|t|/s \]

\[ \sqrt{|t|} = 2 \text{ GeV}; \quad \sqrt{s} = M_Z \implies \theta \sim 2.5^\circ \]
Offset problem: need very accurate Luminosity \( \Rightarrow \) actually one obtains
\[ \Delta \alpha_{\text{had}}(t) - \Delta \alpha_{\text{had}}(t_0) \] where \( t_0 \) corresponds to the minimum reference angle at which the Luminosity is determined. How to get \( \Delta \alpha_{\text{had}}(t_0) \)? In any case we would require \(|t_0| \ll |t|\)

Note: different for \( a^\mu_{\text{had}} \) integrand peaks at 0.33 GeV i.e. dominant contribution from \(|t|/s \approx 0.1\) at Dafne, corresponding to an angle 37°, however, \( s \) over \( t \) channel at 10% level only.

How to get the OFFSET?
\[ \Delta \alpha_{\text{had}}^{(5)}(-M_0^2) = 63.86(65) \]
\[ \Delta \alpha_{\text{had}}^{(5)}(-M_0^2) \]
\[ M_0 = 1 \text{ GeV} \]

\[ \Delta \alpha_{\text{had}}(-(1.0 \text{ GeV})^2) = 36.67(30)[20] \]
\[ \Delta \alpha_{\text{had}}(-M_0^2) = 15.93(10)[7] \times 10^{-4} \]

Question: can we reduce uncertainty by changing

\[ \Delta \alpha_{\text{had}}(-M_0^2)^{\text{direct}} \]

from

\[ \Delta \alpha_{\text{had}}(-M_0^2)^{\text{direct}} \]

to

\[ \left\{ \Delta \alpha_{\text{had}}(-M_0^2) - \Delta \alpha_{\text{had}}(-M_{\text{lumi}}^2) \right\}^{\text{Bhabha}} + \Delta \alpha_{\text{had}}(-M_{\text{lumi}}^2)^{\text{direct}} \]

What will be the offset scale \( M_{\text{lumi}} \)?
5. Prospects for future improvements

Mandatory pQCD improvements required are:

• 4–loop massive pQCD calculation of Adler function;
  required are a number of terms in the low and high momentum series expansions
  which allow for the appropriate Padé improvements
  [essentially equivalent to a massive 4–loop calculation of $R(s)$];
• $m_c$, $m_b$ improvements by sum rule and/or lattice QCD evaluations;
• improved $\alpha_s$ in low $Q^2$ region.

Theory: (QCD parameters) has to improve by factor 10 ! $\rightarrow \pm 0.20$

Requirement may be realistic:

❖ pin down experimental errors to 1% level in all non-perturbative regions up to 2.5 GeV
❖ switch to Euclidean approach, monitored by the Adler function
❖ improve on QCD parameters, mainly on $m_c$ and $m_b$

Settling the HVP issue for $a_\mu$ settles it largely for $\Delta \alpha(-M_0^2)$
Error profiles (standard approach):

Contributions to the total error from different energy regions to the hadronic lowest order vacuum polarization contribution to $a_\mu$, $\Delta \alpha(M_Z^2)$ and $\Delta \alpha(-M_0^2)$ for $M_0 = 2\ GeV$ in percent. These errors are to be added in quadrature to get the total uncertainty. The graph illustrates where experimental effort is needed in order to get a better precision.

The virtues of Adler function approach are obvious:

- no problems with physical threshold and resonances
- pQCD is used only where we can check it to work (Euclidean, $Q^2 \gtrsim 2.0\ GeV$).
- no manipulation of data, no assumptions about global or local duality.
- non–perturbative “remainder” $\Delta \alpha_{\text{had}}^{(5)}(-s_0)$ is mainly sensitive to low energy data !!!
\( \Delta \alpha(\lambda) \) directly accessible in Bhabha Scattering! modulo offset!

- To be analyzed in detail
- Maybe most promising option
Future: ILC/FCC-ee requirement: improve by factor 10 in accuracy

- direct integration of data: 46% from data 54% p-QCD
  \[ \Delta \alpha^{(5)} \text{data}_{\text{had}} \times 10^4 = 126.86 \pm 1.78 \text{ (1.4\%)} \]
  1% overall accuracy ±1.27
  1% accuracy for each region (divided up as in table)
  added in quadrature: ±0.40
  Data: [1.78] vs. [0.40] ⇒ improvement factor 4.5
  \[ \Delta \alpha^{(5)} \text{pQCD}_{\text{had}} \times 10^4 = 149.57 \pm 0.05 \text{ (0.0\%)} \]
  Theory: no improvement needed!

- integration via Adler function: 22% from data 78% p-QCD
  \[ \Delta \alpha^{(5)} \text{data}_{\text{had}} \times 10^4 = 063.87 \pm 0.66 \text{ (1.1\%)} \]
  1% overall accuracy ±0.60
  1% accuracy in region 1.0 to 2.5 GeV
  added in quadrature: ±0.28
  Data: [1.19] vs. [1.03,0.57,0.37] ⇒ improvement factor 2.1-3.2 (Adler vs Adler)
  [1.78] vs. [1.03,0.57,0.37] ⇒ improvement factor 3.1-4.8 (Standard vs Adler)
  \[ \Delta \alpha^{(5)} \text{pQCD}_{\text{had}} \times 10^4 = 214.48 \pm 1.00 \text{ (0.05\%)} \]
  Theory: massive 4-loop needed and more accurate \( m_c, m_b \) and \( \alpha_s \)!

- direct measurement (near/off \( Z \) peak)
<table>
<thead>
<tr>
<th></th>
<th>$s_0$</th>
<th>$M_Z^2$</th>
<th>$-(2.5 \text{ GeV})^2$</th>
<th>$-(2.0 \text{ GeV})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>data</strong></td>
<td>$126.86 \pm 1.78 \ [1.4%]$</td>
<td>$73.72 \pm 0.79 \ [1.1%]$</td>
<td>$63.87 \pm 0.66 \ [1.1%]$</td>
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<tr>
<td><strong>pQCD</strong></td>
<td>$149.57 \pm 0.05 \ [0.0%]$</td>
<td>$201.23 \pm 0.99 \ [0.5%]$</td>
<td>$210.74 \pm 1.04 \ [0.5%]$</td>
<td></td>
</tr>
<tr>
<td>$\Delta \alpha_{\text{had}}(M_Z^2)$</td>
<td>$0.027643 \pm 0.000178$</td>
<td>$0.027535 \pm 0.000127$</td>
<td>$0.027501 \pm 0.000124$</td>
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</tr>
<tr>
<td>$\alpha^{-1}(M_Z^2)$</td>
<td>$128.953 \pm 0.024$</td>
<td>$128.968 \pm 0.017$</td>
<td>$128.972 \pm 0.017$</td>
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</table>

**accuracy in $10^{-5}$**

<p>| | | | |</p>
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<td><strong>18.92</strong></td>
<td><strong>13.52</strong></td>
<td><strong>13.15</strong></td>
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**improvement**

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<td><strong>1.0</strong></td>
<td><strong>1.0</strong></td>
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</tbody>
</table>
\( \Delta \alpha_{\text{had}}(s_0) \times 10^4 \). Future I: improving data only a) direct \( \delta \sigma \lesssim 1\% \) above \( \phi \) to 11.5 GeV, b) low energy space-like cut at \( \sqrt{s_0} \), \( \delta \sigma \lesssim 1\% \) above \( \phi \) to 2.5 GeV

<table>
<thead>
<tr>
<th>( s_0 )</th>
<th>( M_Z^2 )</th>
<th>( -(2.5 \text{ GeV})^2 )</th>
<th>( -(2.0 \text{ GeV})^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>data</td>
<td>126.86 ± 0.41 [0.3%]</td>
<td>73.72 ± 0.33 [0.4%]</td>
<td>63.87 ± 0.28 [0.4%]</td>
</tr>
<tr>
<td>pQCD</td>
<td>149.57 ± 0.05 [0.0%]</td>
<td>201.23 ± 1.03 [0.5%]</td>
<td>210.74 ± 1.04 [0.5%]</td>
</tr>
<tr>
<td>( \Delta \alpha_{\text{had}}(M_Z^2) )</td>
<td>0.027643 ± 0.000041</td>
<td>0.027535 ± 0.000105</td>
<td>0.027501 ± 0.000112</td>
</tr>
<tr>
<td>( \alpha^{-1}(M_Z^2) )</td>
<td>128.953 ± 0.006</td>
<td>128.968 ± 0.014</td>
<td>128.972 ± 0.015</td>
</tr>
<tr>
<td>accuracy in ( 10^{-5} )</td>
<td>4.39</td>
<td>11.16</td>
<td>11.95</td>
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<tr>
<td>improvement</td>
<td>4.3</td>
<td>1.2</td>
<td>1.1</td>
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$\Delta \alpha_{\text{had}}(s_0) \times 10^4$. Future II: improving data as above plus pQCD to 0.2% in case b)

<table>
<thead>
<tr>
<th>$s_0$</th>
<th>$M_Z^2$</th>
<th>$-(2.5 \text{ GeV})^2$</th>
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<td>data</td>
<td>126.86 ± 0.41 [0.3%]</td>
<td>73.72 ± 0.33 [0.4%]</td>
<td>63.87 ± 0.28 [0.4%]</td>
</tr>
<tr>
<td>pQCD</td>
<td>149.57 ± 0.05 [0.0%]</td>
<td>201.23 ± 0.40 [0.2%]</td>
<td>210.74 ± 0.42 [0.2%]</td>
</tr>
<tr>
<td>$\Delta \alpha_{\text{had}}(M_Z^2)$</td>
<td>0.027643 ± 0.000041</td>
<td>0.027535 ± 0.000053</td>
<td>0.027501 ± 0.00060</td>
</tr>
<tr>
<td>$\alpha^{-1}(M_Z^2)$</td>
<td>128.953 ± 0.006</td>
<td>128.968 ± 0.007</td>
<td>128.972 ± 0.008</td>
</tr>
<tr>
<td>accuracy in $10^{-5}$</td>
<td>4.39</td>
<td>5.66</td>
<td>6.37</td>
</tr>
<tr>
<td>improvement</td>
<td>4.3</td>
<td>2.4</td>
<td>2.1</td>
</tr>
</tbody>
</table>

$\Delta \alpha_{\text{had}}(s_0) \times 10^4$. Future III: improving data as above plus pQCD to 0.1% in case b)

<table>
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<tr>
<th>$s_0$</th>
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<th>$-(2.0 \text{ GeV})^2$</th>
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</thead>
<tbody>
<tr>
<td>data</td>
<td>126.86 ± 0.41 [0.3%]</td>
<td>73.72 ± 0.33 [0.4%]</td>
<td>63.87 ± 0.28 [0.4%]</td>
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<tr>
<td>pQCD</td>
<td>149.57 ± 0.05 [0.0%]</td>
<td>201.23 ± 0.40 [0.1%]</td>
<td>210.74 ± 0.21 [0.1%]</td>
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<tr>
<td>$\Delta \alpha_{\text{had}}(M_Z^2)$</td>
<td>0.027643 ± 0.000041</td>
<td>0.027535 ± 0.000040</td>
<td>0.027501 ± 0.00048</td>
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<tr>
<td>$\alpha^{-1}(M_Z^2)$</td>
<td>128.953 ± 0.006</td>
<td>128.968 ± 0.006</td>
<td>128.972 ± 0.007</td>
</tr>
<tr>
<td>accuracy in $10^{-5}$</td>
<td>4.39</td>
<td>4.29</td>
<td>5.06</td>
</tr>
<tr>
<td>improvement</td>
<td>4.3</td>
<td>3.1</td>
<td>2.6</td>
</tr>
</tbody>
</table>
Other issues:

$$\Delta\alpha_{\text{lep}}(M_Z^2) \simeq 0.031419187418$$

A very heavy top decouples like

$$\Delta\alpha_{\text{top}} \simeq -\frac{\alpha}{3\pi} \frac{4}{15} \frac{s}{m_t^2} \to 0$$

when $m_t \gg s$. Thus top quark contribution at $s = M_Z^2$:

$$\Delta\alpha_{\text{top}}(M_Z^2) = -0.76 \times 10^{-4}.$$ 

Remaining problems are the following:

a) contributions to the Adler function up to three–loops all have the same sign and are substantial. Four– and higher–orders could still add up to non-negligible contribution. An
error for missing higher order terms is not included.

b) the link between space–like and time-like region is the difference

\[ \Delta = \Delta \alpha^{(5)}_{\text{had}}(M_Z^2) - \Delta \alpha^{(5)}_{\text{had}}(-M_Z^2) = 0.000045 \pm 0.000002 \]

which can be calculated in pQCD. It accounts for the \( i\pi \)–terms from the logs
\[ \ln(-q^2/\mu^2) = \ln(|q^2/\mu^2|) + i\pi. \]

Since the term is small we can get it as well from direct data integration

\[ \Delta \alpha_{\text{had}}(-M_Z^2) = 276.44 \pm 0.64 \pm 1.78 \]
\[ \Delta \alpha_{\text{had}}(M_Z^2) = 276.84 \pm 0.64 \pm 1.90 \]

and taking into account that errors are almost 100% correlated we have

\[ \Delta \alpha_{\text{had}}(M_Z^2) - \Delta \alpha_{\text{had}}(-M_Z^2) = 0.40 \pm 0.12 \]

\[ \ln \alpha^{-1}(M_Z^2) \] we included the latter term as well as the above top contribution.
Comment:

- Since 2009 the error of $\Delta \alpha_{\text{had}}(-s_0)$ went down by 40% (more KLOE, BaBar, BESIII, especially BaBar exclusive channels up to 2.3 GeV)

- The error budget 50% : 50% we had in 2009 now changed to 39% : 61% for $\sqrt{s_0} = 2.5$ GeV and 29% : 71% for $\sqrt{s_0} = 2.0$ GeV

- Therefore the improvement obtained by reducing the experimental error to 1% in the range from $\phi$ to 2.5 GeV gets much less weight, but still is important

- By choosing a higher cut point e.g. for $\sqrt{s_0} = 3.0$ GeV one can balance the importance of data vs pQCD providing an important crosscheck. One gets $\Delta \alpha_{\text{had}}(-s_0) = 82.21 \pm 0.88[0.38]$ in $10^{-4}$. The QCD contribution is then smaller as well as saver, because the mass effects which are responsible for the larger uncertainty of the pQCD prediction also gets substantially reduced.

- In view that a massive 4-loop QCD calculation is very difficult and better cross section in the range up to 2.5 GeV provides the possibility to optimize the cut scale $s_0$. 
Davier et al. 2011: use pQCD above 1.8 GeV  
● no improvement by remeasuring cross sections above 1.8 GeV  
● no proof that pQCD works at 0.04% precision as adopted

My analysis is data driven: pQCD 5.2 – 9.5 and > 11.5 GeV  
× pQCD at 0.2% Adler function: pQCD error = $\frac{1}{2}$ × present error  
× pQCD at 0.1% Adler function: pQCD error = data error ±0.28

Note: theory-driven analyses using pQCD above 1.8 GeV cannot be improved by improved cross-section measurements above 2 GeV !!!
Better data needed in any case for range 1.2 - 2.4 GeV

- one of the main issue in HVP is \( R_\gamma(s) \) from 1.2 GeV to 2.4 GeV
- has been improved dramatically by the exclusive channel measurements by BaBar
- 20 out of more than 30 channels are measured, many known at the 10 to 15% level
- now exclusive channel data much better quality than the very old inclusive data from Frascati
ILC/FCC-ee community should actively support these activities as integral part of $e^+e^-$-collider precision physics!!!

Remember: tremendous progress since middle of 90’s

- Novosibirsk VEPP-2M: MD-1, CMD2, SND, KEDR; VEPP-2000: CMD3, SND
- Frascati DAFNE: KLOE
- Beijing BEPC: BES II, BESIII
- Cornell CESR: CLEO
- Stanford SLAC PEP-II: BaBar; KEK Tsukuba: Belle

Many analyzes exploiting these results: Davier et al., Hagiwara et al., Burkhardt, Pietrzyk, Yndurain et al.:

Indispensable for Muon $g - 2$, indirect vs direct LEP Higgs mass etc. and future precision test at ILC/FCC-ee and new physics signals in precision observables. Impact for cosmology!
big progress in data, CMD-2, SND, BESII, KLOE, BaBar, ... Chetyrkin, Kühn et al. ..., by far more progressive use of pQCD
How much pQCD?
To discover “Physics Behind Precision” at future linear or circular $e^+e^-$–colliders requires improved SM predictions based more precise input parameters: $\Delta \alpha_{\text{had}}(M_Z^2)$ thereby plays a key role.

A factor 5 improvement seem feasible, but is by far not for free.

High precision input parameters crucial to open a new window for precision cosmology of the early universe.

Thanks for your attention!
Extras
<table>
<thead>
<tr>
<th>Authors</th>
<th>data</th>
<th>pQCD</th>
<th>sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jegerlehner 1985:</td>
<td>247.37±7.</td>
<td>38.63±0.37</td>
<td>286.±7.</td>
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<tr>
<td>Martin, Zeppenfeld 1994:</td>
<td>51.5±1.1</td>
<td>221.7±4.1</td>
<td>273.2±4.2</td>
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<td>Swartz 1995:</td>
<td>232.56±4.6</td>
<td>42.64±0.10</td>
<td>275.2±4.6</td>
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<td>Eidelman, Jegerlehner 1995:</td>
<td>237.55±6.43</td>
<td>42.82±0.10</td>
<td>280.37±6.43</td>
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<tr>
<td>Burkhardt, Pietrzyk 1995:</td>
<td>159.±7.</td>
<td>121.±0.2</td>
<td>280.±7.</td>
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<tr>
<td>Adel, Yndurain 1995:</td>
<td>45.99±0.85</td>
<td>226.6±4.0</td>
<td>272.59±4.09</td>
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<tr>
<td>Alemany, Davier, Höcker 1997:</td>
<td>238.01±6.3</td>
<td>42.82±0.10</td>
<td>280.9±6.3</td>
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<tr>
<td>Kühn, Steinhauser 1998:</td>
<td>82.9±1.40</td>
<td>194.45±0.96</td>
<td>277.43±1.70</td>
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<td>Davier, Höcker 1998:</td>
<td>56.53±0.83</td>
<td>219.77±1.40</td>
<td>276.3±1.6</td>
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<td>Erler 1998:</td>
<td>56.9±1.1</td>
<td>220.8±1.5</td>
<td>277.7±1.9</td>
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<td>Burkhardt, Pietrzyk 2001:</td>
<td>155.8±3.6</td>
<td>120.3±0.2</td>
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<td>Hagiwara et al 2004:</td>
<td>150.18±2.3</td>
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<td>Jegerlehner 2006 direct:</td>
<td>106.07±2.24</td>
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<td>276.07±2.25</td>
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<td>Jegerlehner 2006 Adler :</td>
<td>73.69±0.98</td>
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<td>Hagiwara et al 2011:</td>
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<td>Jegerlehner 2016 direct:</td>
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<td>60.49±0.66</td>
<td>214.48±1.00</td>
<td>275.04±1.19</td>
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</tbody>
</table>
Red theory-driven still can be improved by more precise cross sections below 2 GeV
Green data-driven require better cross section up to above the \( \Upsilon \) region
Magenta Adler function based requires 1\% cross section from 1.4 to 2.5 GeV and progress in pQCD [massive PQCD to 4-loops, Euclidean regime]
Advantage of euclidean approach can be supplemented directly by lattice QCD data
Addendum 1: Using $\tau$–decay spectra + isospin breakings

$e^+e^- \rightarrow \pi^+\pi^-$, $\cdots$ vs $\tau^- \rightarrow \bar{\nu}_\tau \pi^0\pi^-$, $\cdots$

$\tau$–spectra: ALEPH, OPAL, CLEO, Belle

d${}^{\text{had}}$ incl. $I = 1$ $\tau \rightarrow \pi\pi\nu_\tau$ in range [0.63-0.96] GeV:

\begin{align*}
    e^+e^- & : 353.82(0.88)(2.17)[2.34] \\
    \tau & : 354.25(1.24)(0.61)[1.38] \\
    e^+e^- + \tau & : 354.14(0.82)(0.86)[1.19]
\end{align*}

The $\tau$ vs $e^+e^-$ puzzle of $\pi^+\pi^-$ data: $\rho^0 - \gamma$ mixing

$-i \Pi_{\gamma\rho}^{\mu\nu}(q) = \ldots + \ldots$.

absent in charged ($\tau$) channel

$\rho^0 - \gamma$ mixing correction to be applied to $\tau$ data [l], ALEPH versus BaBar plot from Davier et al.[r]

$|F_\pi(E)|^2$ in units of $e^+ e^- I=1$ (CMD-2 GS fit): left no mixing correction, right: after mixing correction
\[ \Delta \alpha_{\text{had}}(\rho^0 - \gamma) \times 10^4 \times (0.592 - 0.975) \text{ GeV} \]

\( \Delta \alpha(M_Z) \) contributions form I=1 \( \pi\pi \) channel in range [0.63, 0.96] GeV in units \( 10^{-4} \).

\( \tau \) data corrected for isospin breaking and [left] for missing \( \rho^0 - \gamma \) mixing [right] \( \rho^0 - \gamma \) mixing is not applied.
$\Delta\alpha(M_Z^2)$ contributions form $l=1$ $\pi\pi$ channel in range $[0.63, 0.96]$ GeV in units $10^{-4}$

<table>
<thead>
<tr>
<th></th>
<th>data</th>
<th>incl. $\rho^0 - \gamma$ mixing correction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$ data, IB corrected</td>
<td>26.90 ( 0.06) ( 0.14)</td>
<td>27.48 ( 0.06) ( 0.14)</td>
</tr>
<tr>
<td>$e^+e^-$ data, $l=1$ $\pi\pi$</td>
<td>26.82 ( 0.08) ( 0.18)</td>
<td>26.82 ( 0.08) ( 0.18)</td>
</tr>
<tr>
<td>Combined</td>
<td>26.87 ( 0.05) ( 0.11)</td>
<td>27.23 ( 0.12) ( 0.30)</td>
</tr>
</tbody>
</table>

$\Delta\alpha^{\text{had}}[ee < 2 \text{ GeV}] \times 10^4$

$\Delta\alpha^{\text{had}}[ee + \tau < 2 \text{ GeV}] \times 10^4$

<table>
<thead>
<tr>
<th></th>
<th>data</th>
<th>incl. $\rho^0 - \gamma$ mixing correction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta\alpha^{\text{had}}[ee]$</td>
<td>$(276.43 \pm 0.67(\text{stat}) \pm 1.64(\text{syst})[\pm 1.78(\text{tot})]) \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$\Delta\alpha^{\text{had}}[ee + \tau]$</td>
<td>$(276.48 \pm 0.67(\text{stat}) \pm 1.63(\text{syst})[\pm 1.76(\text{tot})]) \times 10^{-4}$</td>
<td></td>
</tr>
</tbody>
</table>

Central values of pQCD contributions to $\Delta\alpha^{\text{had}}(5)$:

<table>
<thead>
<tr>
<th>range</th>
<th>$\alpha_s = 0.1184$</th>
<th>$\alpha_s = 0.12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2 GeV - 9.46 GeV</td>
<td>33.84</td>
<td>33.91</td>
</tr>
<tr>
<td>13 GeV - $\infty$</td>
<td>115.73</td>
<td>115.82</td>
</tr>
</tbody>
</table>
\( \Delta \alpha(-M_0^2) \) contributions form \( l=1 \) \( \pi \pi \) channel in range \([0.63, 0.96]\) GeV in units \(10^{-4}\)

<table>
<thead>
<tr>
<th>data</th>
<th>incl. ( \rho^0 - \gamma ) mixing correction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau ) data, IB corrected</td>
<td>23.46 ( 0.05) ( 0.12)</td>
</tr>
<tr>
<td>( e^+e^- ) data, ( l=1 ) ( \pi \pi )</td>
<td>23.39 ( 0.07) ( 0.15)</td>
</tr>
<tr>
<td>Combined</td>
<td>23.43 ( 0.04) ( 0.09)</td>
</tr>
<tr>
<td>( \Delta \alpha_{\text{had}}[ee &lt; 2 \text{ GeV}] \times 10^4 )</td>
<td>46.43(0.08)(0.56)[0.57]</td>
</tr>
<tr>
<td>( \Delta \alpha_{\text{had}}[ee + \tau &lt; 2 \text{ GeV}] \times 10^4 )</td>
<td>46.50(0.06)(0.55)[0.56]</td>
</tr>
</tbody>
</table>

In both cases for \( \Delta \alpha_{\text{had}}(-M_0^2) \) and \( \Delta \alpha_{\text{had}}(M_Z^2) \) improvement marginal because \( \pi \pi \) channel alone does not give the dominant contribution!
Addendum 2: The coupling $\alpha_2$, $M_W$ and $\sin^2 \Theta_f$

How to measure $\alpha_2$:

- **charged current channel** $M_W$ ($g \equiv g_2$):

$$M_W^2 = \frac{g^2 v^2}{4} = \frac{\pi \alpha_2}{\sqrt{2} G_\mu}$$

- **neutral current channel** $\sin^2 \Theta_f$

In fact here running $\sin^2 \Theta_f(E)$: LEP scale $\iff$ low energy $\nu_e e$ scattering

$$\sin^2 \Theta_e = \left\{ \frac{1 - \Delta \alpha_2}{1 - \Delta \alpha} + \Delta_{\nu_\mu e,\text{vertex+box}} + \Delta_{\kappa_e,\text{vertex}} \right\} \sin^2 \Theta_{\nu_\mu e}$$

The first correction from the running coupling ratio is largely compensated by the $\nu_\mu$ charge radius which dominates the second term. The ratio $\sin^2 \Theta_{\nu_\mu e} / \sin^2 \Theta_e$ is close to 1.002, independent of top and Higgs mass. Note that errors in the ratio $\frac{1 - \Delta \alpha_2}{1 - \Delta \alpha}$ can be taken to be 100% correlated and thus largely cancel.
Above result allow us to calculate non-perturbative hadronic correction in $\gamma\gamma$, $\gamma Z$, $ZZ$ and $WW$ self energies, as

\[
\begin{align*}
\Pi^{\gamma\gamma} & = e^2 \hat{\Pi}^{\gamma\gamma} \\
\Pi^{Z\gamma} & = \frac{eg}{c_\Theta} \hat{\Pi}_V^{3\gamma} - \frac{e^2 s_\Theta}{c_\Theta} \hat{\Pi}_V^{\gamma\gamma} \\
\Pi^{ZZ} & = \frac{g^2}{c_\Theta^2} \hat{\Pi}_{V-A}^{33\gamma} - 2 \frac{e^2}{c_\Theta^2} \hat{\Pi}_V^{3\gamma} + \frac{e^2 s_\Theta^2}{c_\Theta^2} \hat{\Pi}_V^{\gamma\gamma} \\
\Pi^{WW} & = g^2 \hat{\Pi}_{V-A}^{+-\gamma} \\
\end{align*}
\]

with $\hat{\Pi}(s) = \hat{\Pi}(0) + s\hat{\pi}(s)$.

Leading hadronic contributions:

\[
\begin{align*}
\Delta \alpha_{\text{had}}^{(5)}(s) & = -e^2 \left[ \text{Re} \hat{\pi}^{\gamma\gamma}(s) - \hat{\pi}^{\gamma\gamma}(0) \right] \\
\Delta \alpha_{2\text{had}}^{(5)}(s) & = -\frac{e^2}{s_\Theta^2} \left[ \text{Re} \hat{\pi}^{3\gamma}(s) - \hat{\pi}^{3\gamma}(0) \right] \\
\end{align*}
\]

which exhibit the leading hadronic non-perturbative parts, i.e. the ones involving
the photon field via mixing. $\Delta \alpha_{\text{had}}^{(5)}(s)$ and $\Delta \alpha_{2\text{had}}^{(5)}(s)$ via $e^+e^-$-data and isospin arguments $[(u,d), s$ flavor separation]:

$$\Pi_{ud}^{3\gamma} = \frac{1}{2} \Pi_{ud}^{\gamma\gamma}; \quad \Pi_{s}^{3\gamma} = \frac{3}{4} \Pi_{s}^{\gamma\gamma}$$

$$\Pi_{\gamma\gamma} = \Pi^{(\rho)} + \Pi^{(\omega)} + \Pi^{(\phi)} + \cdots \Rightarrow \Pi_{3\gamma} = \frac{1}{2} \Pi^{(\rho)} + \frac{3}{4} \Pi^{(\phi)} + \cdots$$


Note: gauge boson SE potentially very sensitive to New Physics (oblique corrections)

new physics may be obscured by non-perturbative hadronic effects; need to fix this!
PDG version
\( \Delta \alpha_{\text{em}}(E) \) and \( \Delta \alpha_2(E) \) as functions of energy \( E \) in the time-like and space-like domain. The smooth space-like correction (dashed line) agrees rather well with the non-resonant “background” above the \( \phi \)-resonance (kind of duality). In resonance regions as expected “agreement” is observed in the mean, with huge local deviations.
\( \sin^2 \Theta_W(Q) \) as a function of \( Q \) in the space-like region. Hadronic uncertainties are included but barely visible. Uncertainties from the input parameter \( \sin^2 \theta_W(0) = 0.23822(100) \) or \( \sin^2 \theta_W(M_Z^2) = 0.23153(16) \) are not shown. Future ILC/FCC measurements at 1 TeV would be sensitive to \( Z', H^{--} \) etc.

Except from the LEP and SLD points (which deviate by 1.8 \( \sigma \)), all existing measurements are of rather limited accuracy unfortunately!
$\sin^2 \Theta_W(E)$ as a function of $E$ in the time-like region. Note that $\sin^2 \theta_W(0)/ \sin^2 \theta_W(M_Z^2) = 1.02876$ a 3% correction established at $6.5 \sigma$.

$\sin^2 \Theta_{\text{eff}}$

exhibiting a specific dependence on the gauge boson SEs is an excellent monitor for New Physics