**"... there is a sudden rapid passage to
a totally new and more comprehensive type of order or organization,
with quite new emergent properties ..."**

(Huxley & Huxley 1947)

The SM as a low energy effective theory and the role of the Higgs in the early universe

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Lectures at IFJ-PAN, Krakow, October 14-27, 2014

Lecture Series at IFJ-PAN Krakow October 2014:

About the role of the Higgs boson in the evolution of the early universe

Articles:

1) The Standard model as a low-energy effective theory:
   what is triggering the Higgs mechanism?
   arXiv:1304.781 – Acta Phys.Polon. B45 (2014) 1167-1214

2) The hierarchy problem of the electroweak Standard Model revisited
   arXiv:1305.6652

3) Higgs inflation and the cosmological constant
   arXiv:1402.37.38 – Acta Phys.Polon. B45 (2014) 1215-1254

and References therein.

Abstract:

After the discovery of the Higgs particle the most relevant structures of the SM have been verified and for the first time we know all parameters of the SM within remarkable accuracy. Together with recent calculations of the SM renormalization group coefficients up to three loops we can safely extrapolate running couplings high up in energy. Assuming that the SM is a low energy effective theory of a cutoff theory residing at the Planck scale, we are able to calculate the effective bare parameters of the underlying cutoff system. It turns out that the effective bare mass term changes sign not far below the Planck scale, which means that in the early universe the SM was in the symmetric phase. The sign-flip, which is a result of a conspiracy between the SM couplings and their screening/antiscreening behavior, triggers the Higgs mechanism. Above the Higgs phase transition the bare mass term in the Higgs potential must have had a large positive value, enhanced by the quadratic divergence of the bare Higgs mass. Likewise the quartically enhanced positive vacuum energy density is present in the symmetric

phase. The Higgs system thus provides the large dark energy density in the early universe, which triggers slow-roll inflation, i.e. the SM Higgs is the inflaton scalar field. Reheating is dominated by the decay of the heavy Higgses into (in the symmetric phase) massless top/anti-top quark pairs. The new scenario possibly could explain the baryon-asymmetry essentially in terms of SM physics

Within this context a careful renormalization group analysis of the electroweak Standard Model reveals that there is no hierarchy problem in the SM. In the broken phase a light Higgs turns out to be natural as it is self-protected and self-tuned by the Higgs mechanism. It means that the scalar Higgs needs not be protected by any extra symmetry, specifically super symmetry, in order not to be much heavier than the other SM particles which are protected by gauge- or chiral-symmetry. Thus the existence of quadratic cutoff effects in the SM cannot motivate the need for a super symmetric extensions of the SM, but in contrast plays an important role in triggering the electroweak phase transition and in shaping the Higgs potential in the early universe to drive inflation as supported by observation.

A Colloquium
”About the role of the Higgs boson in the early universe”
will be presented at IFJ-PAN, Thursday October 23th 11am.

Program: (lectures 2 x 45 minutes)

Lecture 1:	1.1 Overview and Context,
	1.2 On the construction and properties of low energy effective theories
	Topics:
	- On the path to physics at the Planck scale
	  o No QFT without cutoff
	  o The Planck length
	- Phase Transitions in a Nutshell

Lecture 2:	the SM as a low energy effective theory (LEET)
	Topics:
	- Low energy effective QFT of a cutoff system
	  o The emergence of multipole forces
	  o The emergence of non-Abelian gauge structures
	  o Conspiracies tuning for criticality

Lecture 3:	SM parameters, MS renormalization group and matching conditions
	Topics:
	- The SM renormalization group equations
	  o SM RG running at NNLO
	- Matching MS parameters with physical parameters
	  o Matching relation for the effective Fermi constant
	  o The issue of tadpoles and non-decoupling of heavy particles
	  o Matching and running of the top mass
	  o The top Yukawa coupling from form Mt
	  o The Higgs self-coupling form MH

Lecture 4:	4.1 Summary and main consequences of the LEESM scenario, Topics: - Low energy effective QFT of a cutoff system - Matching conditions - SM RG evolution to the Planck scale - The issue of quadratic divergences in the SM - Remark on the impact on inflation

4.2 what is triggering the Higgs mechanism and inflation

Topics:

- Higgs vacuum stability versus metastability?
- the role of the quadratic ”singularities” for inflation
- Higgs mass ”self-tuning”, dynamical Higgs mass
- the ”no hierarchy problem” issue

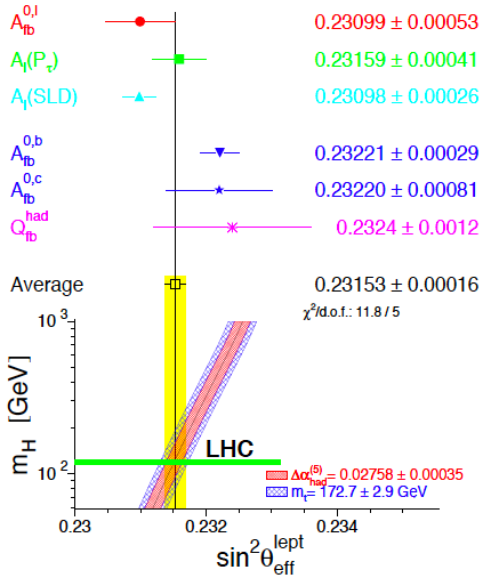
Lecture 5:	Basics of Cosmology and Inflation Topics: - Cosmological solutions of Friedmann's equations - Energy momentum tensor and matter in the universe - Messages from the Cosmological Microwave Background - Why do we need inflation? - Finite temperature effects in the early universe
Lecture 6:	Analysis of SM Higgs Inflation Topics: - running couplings impact on cosmological solutions - Higgs field at the Planck scale - how to get enough inflation - slow-roll inflation criteria - evolution of the various energy components

Lecture 7:	The Cosmological Constant Problem Topics: - what provides a large cosmological constant needed for inflation - how to get rid of the huge cosmological constant - reheating and Baryogenesis
Lecture 8:	Summary and outline Topics: - a new view on naturalness and physics beyond the SM - an alternate path to new physics? - open problems - how to scrutinize SM Higgs inflation - cold dark matter, what could it be? - quo vadis particle physics?

Introduction

✌ LHC ATLAS&CMS Higgs discovered ⇒ the SM completion

Higgs mass found by ATLAS and CMS agrees perfectly with the indirect bounds



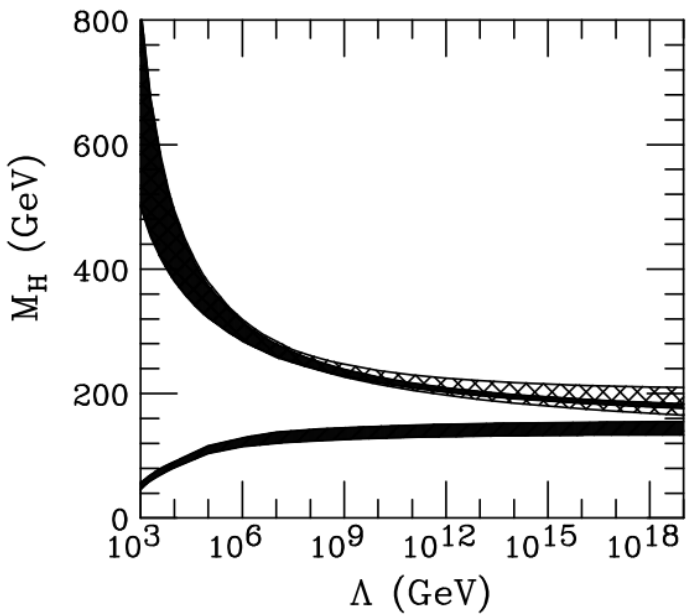
LEP 2005 +++ LHC 2012

Englert&Higgs Nobel Prize 2013

Higgs mass found in very special mass range 125.9 ± 0.4 **GeV**

Common Folklore: hierarchy problem requires supersymmetry (SUSY) extension of the SM (no quadratic/quartic divergences) SUSY = infinity killer!

Do we need new physics? Stability bound of Higgs potential in SM:



$$V = \frac{m^2}{2} H^2 + \frac{\lambda}{24} H^4$$

Riesselmann, Hambye 1996

$M_H < 180$ GeV

– first 2-loop analysis, knowing M_t –

SM Higgs remains perturbative up to scale Λ if it is light enough (upper bound=avoiding Landau pole) and Higgs potential remains stable ($\lambda > 0$) if Higgs mass is not too light [parameters used: $m_t = 175[150 - 200]$ GeV ; $\alpha_s = 0.118$]

The Higgs is special!

- the Higgs boson was invented 1964 by Higgs, Englert and others to formulate a renormalizable theory of weak interactions

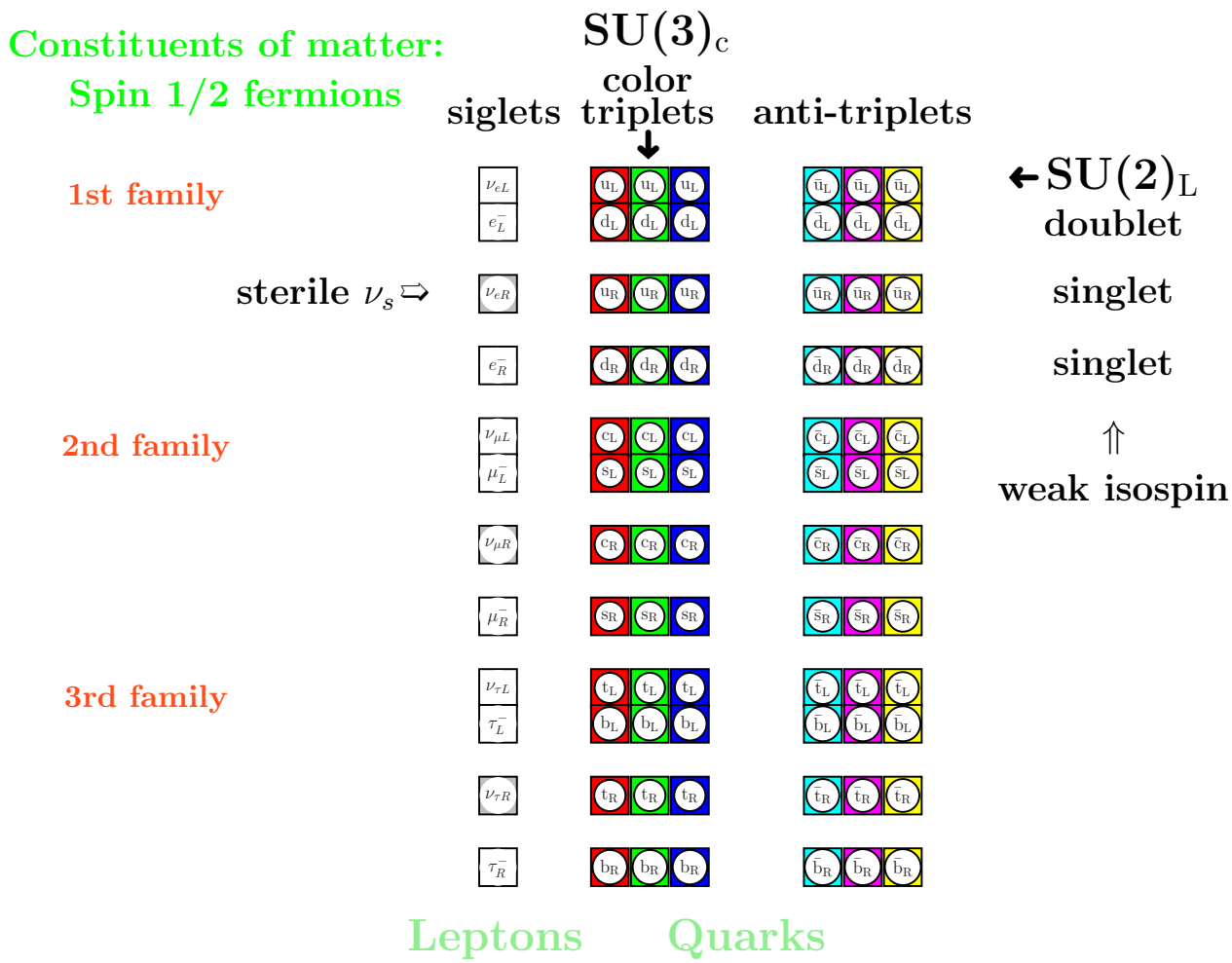
- the secret behind: all particle masses are dynamically generated by spontaneous symmetry breaking (SSB) [Higgs mechanism]

- one needs one scalar field $\equiv$ Higgs field, which develops a finite constant vacuum expectation value (VEV) v [vacuum condensate]

➜ originally massless particles must move in a ground state filled with Bose condensate such that particles appear to have a mass.

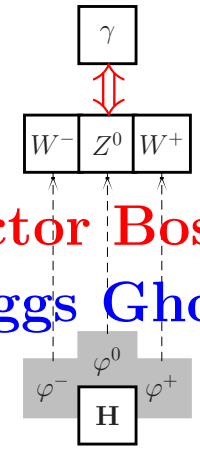
- at least as important: Higgs particle and associated Higgs exchange force needed to tame bad high energy behavior (renormalizability)

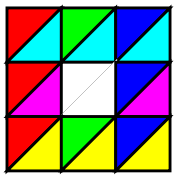
The Standard Model completed



Carrier of Forces:
Spin 1 Gauge Bosons

Photon





Vector Bosons

Higgs Ghosts

Octet of Gluons

"Cooper Pairs":
Spin 0 Higgs Boson

Higgs Particle

Higgs mechanism: breaking weak isospin spontaneously

weak bosons W, Z and all fermions get masses

$m_i \propto y_i\, v$; $v = \langle H \rangle >\simeq 246.21$ **GeV**; $G_{\text{Fermi}} = \frac{1}{\sqrt{2}v^2}$ **(radioactivity)**

Before Higgs mechanism [symmetric phase]: $W^\pm, Z$ and all fermions massless
Higgs "ghosts" $\phi^\pm, \phi^0$ heavy physical degenerate with the Higgs!

Basic parameters: gauge couplings $g' = g_1$, $g = g_2$, g_3, top quark Yukawa coupling y_t, Higgs self-coupling λ and Higgs VEV v

mass $\propto$ interaction strength $\times$ Higgs VEV v

$$
\begin{aligned}
m_W^2(\mu) &= \frac{1}{4}\, g^2(\mu)\, v^2(\mu)\ ; \quad m_Z^2(\mu) = \frac{1}{4}\,(g^2(\mu) + g'^2(\mu))\, v^2(\mu)\ ; \\
m_f^2(\mu) &= \frac{1}{2}\, y_f^2(\mu)\, v^2(\mu)\ ; \quad m_H^2(\mu) = \frac{1}{3}\, \lambda(\mu)\, v^2(\mu)\ .
\end{aligned}
$$

Effective parameters depend on renormalization scale μ [normalization reference energy!], scale at which ultraviolet (UV) singularities are subtracted

energy scale $\leftrightarrow$ center of mass energy of a physical process

e.g. at Large Electron Positron Collider [LEP] (pre LHC e^+e^- storage ring)

Prologue

Maybe the most remarkable thing we learned from the emergence of the Standard Model (SM) of the weak, electromagnetic and strong interaction is it high intrinsic symmetry, intrinsically the SM is massless (except from the Higgses). The SM is constructed in such a way that particle masses do not spoil the symmetries of the Lagrangian as it is possible if masses are generated by spontaneous symmetry breaking. The symmetry is broken by the ground state via a vacuum condensate of a single scalar field. If we consider the SM as a low energy effective theory of some cutoff system exhibiting a very large cutoff, many of the SM structures are generated 'automatic', as we can learn from the theory of critical phenomena in condensed matter physics.

For the SM the relevant cutoff we assume to be the Planck scale. The energy scale, which we identify with the $\overline{\mathrm{MS}}$ scale μ, in general, is to be associated with the temperature as $E = k_{\mathrm{B}}T$ in the evolution of the universe.

This series of Lectures should provide background materiel supporting my main theses on the SM, considered as a low energy effective theory, after Higgs discovery:

- There is no hierarchy problem of the SM in the Higgs phase (in which we live)
- A super symmetric or any other extension of the SM cannot be motivated by the (non-existing) hierarchy problem
- SM running couplings trigger the Higgs mechanism at about 10^{17} GeV as the universe cools down; in the broken phase the Higgs is naturally as light as other SM particles which are generated by the Higgs mechanism
- in the early symmetric phase the hierarchy problem is back but not a problem; the quadratically enhanced bare mass term in the Higgs potential triggers inflation, if the Higgs is to be the inflaton this enhancement is crucial and would be missing in supersymmetric extensions of the SM

Lecture 1:

Low energy effective theories in Critical Phenomena of statistical mechanics systems. The path to the quantitative construction of LEETs.

Outline of Lecture 1:

① **On the path to physics at the Planck scale**

- ❑ **No QFT without cutoff**
- ❑ **The Planck length**

② **Phase Transitions in a Nutshell**

① On the path to physics at the Planck scale

String Paradigm: the closer we look the more symmetric the world looks

$$\text{M–THEORY} \sim \text{STRINGS} \leftarrow \text{SUGRA} \leftarrow \text{SUSY} \leftarrow \text{SM},$$

Experience from condensed matter physics and a number of known facts suggest that a completely different picture could be behind what we observe as elementary particle interactions at low energies. It might well be that many known features and symmetries we observe result as a consequence of "blindness for details" at long distances of some unknown kind of medium exhibiting a fundamental cutoff, which we expect to be the Planck length. The symmetry pattern thus could look like:

$$\text{ETHER} \rightarrow \text{QFT} \overset{???}{\simeq} \text{SM (symmetric)} \underset{symmetry\ breaking}{\overset{spontaneous}{\rightarrow}} \text{SM (broken phase).}$$

Emergence Paradigm: symmetries emerge as you only see the long distance tail

the closer you look the more there is to see

In nature many apparent symmetries are broken if you look closer!

- **the only unbroken symmetries we know are the local Abelian $U(1)_{\rm em}$ of QED and the non-Abelian $SU(3)_c$ of QCD**
- **even Lorentz- and Poincaré-invariance are not exact symmetries of nature (broken by gravity in any case), as all others: chiral symmetry, the weak $SU(2)_L$, P, C, T, CP, $U(1)_\ell$ $(\ell = e, \mu, \tau)$, $U(1)_B$, flavor $SU(2)$, $SU(3)$ $\cdots$ etc**

Unlike in renormalized QFT, here the relationship between bare and renormalized parameters has a physical meaning.

Such ideas are quite old, among many others

Landau 55, Wilson 71, FJ 78, 't Hooft 80
and in some aspects are now commonly accepted among particle physicists (naturalness arguments etc.).
Key question:

Simplicity at long distances or simplicity at high energies?

Physics at the Planck scale cannot be described by local quantum field theory. The curvature of space-time is relevant and special relativity is modified by gravitational effects. One expects a world which exhibits an intrinsic cutoff corresponding to the fundamental length $a_{\rm Pl} \simeq 10^{-33}$ cm. But not only Lorentz and Poincaré invariance are expected to break down, also the laws of quantum mechanics need not hold any longer at Λ_P .

The "microscopic" theory at distances $a_{\rm Pl}$ is unknown, but we know it belongs to the "universality class" of possible theories, which exhibit as a universal low energy effective asymptote: the known electroweak and strong interactions as well as classical gravity.

Long distance universality is a well known phenomenon from condensed matter physics, where we know that a ferromagnetic, a liquid-gas system and a superconductor may exhibit identical long range properties (phase diagram, critical exponents etc.). Our hypothesis could be that there exist some kind of a "Planck solid", for example.

We should mention right here that there is a principal difference between a normal solid and a "Planck solid"; of the latter we only can observe its long range properties, the critical or quasi–critical behavior. Its true short range properties will never be observable, since we will never be able to built a "Planck microscope", which would allow us to perform experiments at Λ_P .

Also the observations which tell us about the properties of the early universe will never suffice to pin down in detail the structure at distances $a_{\rm Pl}$.

What we call “ether” can stand only for the universality class, the totality of possible systems which exhibit identical critical behavior. It is a non–trivial task to specify possible candidate models belonging to the universality class which manifests itself as the SM at low energies.

The approach discussed here is based on experimentally well established physics and on observations from the theory of critical phenomena which revealed the emergence of local Euclidean renormalizable QFT as a low energy effective structure of complex condensed matter systems
see e.g. Wilson, Kogut 74, FJ 75/98, Kogut 79.

Ken Wilson NP 1982
solved a problem which persisted for over 70 years!

The question:

Has nature imposed highly symmetric Planck physics, which then gets broken down who knows how, or is it the other way round that the SM is a low energy effective theory that looks "simple" because we only see what we can see from far away?

I advocate to have a closer look at the second alternative.

Some misapprehensions:

- ❑ **Renormalizability is a fundamental law of nature**
- ❑ **Symmetries have to be imposed by hand to be there**
- ❑ **Cutoffs are an auxiliary tool to overcome technicalities at intermediate steps of perturbative calculations**
- ❑ **the hierarchy problem of the SM requires a supersymmetric extension of the latter or some alternative extensions**

Paths to Physics at the Planck Scale

M–THEORY(BRAIN WORLD)	Energy scale	E–THEORY(REAL WORLD)
candidate TOE		"chaotic" system
exhibits intrinsic cut-off	Planck scale	with intrinsic cut–off
↓	$\parallel$	
STRINGS	10^{19} GeV	
↓	↓	↑ QFT
SUGRA		↑
↓		
SUSY–GUT	10^{16} GeV	↑
↓	↓	
SUSY	1 TeV	"??SM??"
↘	↓	↗
	SM	

top-down approach

bottom-up approach

symmetry high $\rightarrow$ $\rightarrow$ $\rightarrow$ symmetry low

?? symmetry $\equiv$ blindness for details ??

❑ No QFT without cutoff

In order to define any continuum QFT in $d > 2$ dimensions one has to introduce a cutoff because of ultraviolet (UV) divergences. In fact the only non-perturbative definition of a field theory known requires some kind of a lattice e.g. lattice QCD. UV divergences are not just a mathematical complication, they very likely have a physical origin!

Dimensional regularization, which mimics that there is no need to introduce a cutoff, is a purely mathematical (non-physical) finite part prescription, which only works in perturbation theory. The related $\overline{\mathrm{MS}}$ quantities are not observables. Nevertheless, they have a physical meaning: they properly describe the scale dependence in the high energy regime where mass effects can be neglected.

Suppose, we have no infrared problem (no massless states, mass gap m) then the logarithms, which are at the heart of the problem

$$I_{\text{UV singular}} = \int_m^\infty \frac{dx}{x} \to I_{\text{UV reg}} = \int_m^{\text{some UV cutoff}} \frac{dx}{x}$$

where either some UV cutoff$= \Lambda$ is a true cutoff and $I_{\text{UV reg}} = \log(\Lambda/m)$ or formally, as in $\overline{\text{MS}}$ prescription, some UV cutoff$= \mu$ such that $I_{\text{UV reg}} = \log(\mu/m)$ Also the $\overline{\text{MS}}$ renormalization group equations only reflect the UV properties of the theory!

In condensed matter systems, which exhibit a physical cutoff π/a (e.g. a the lattice size), critical long range fluctuations imply an IR problem

$$I_{\text{IR singular}} = \int_0^{\pi/a} \frac{dx}{x} \to I_{IR\ reg} = \int_{\text{some IR regulator}}^{\pi/a} \frac{dx}{x}$$

where as some IR regulator$= 1/\xi$, ξ the correlation length.

Ken Wilson first discovered the intimate relation of the two problems, which allowed him to solve the long standing puzzle of critical behavior and scaling in condensed matter physics.

❑ The Planck length

When applying condensed matter analogies to particle physics we have to ask: What is the fundamental physical cutoff in Nature

Gravitation, relativity (electromagnetism) and quantum theory each are characterized by a typical fundamental constant $\Rightarrow$

Basic constants: G, c, $\hbar$ **(Newton 1687, Einstein 1905, Planck 1913)**

They may be combined into

"smallest" distance: Planck length:

$$\ell_{\rm Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.616252(81) \times 10^{-33}\ {\rm cm}$$

"biggest" mass: Planck mass:

$$m_{\rm Pl} = \sqrt{\frac{\hbar/c}{G}} = 2.2 \times 10^{-5}\ {\rm gr}$$

shortest time: Planck time:

$$t_{\rm Pl} = \ell_{\rm Pl}/c = 5.4 \times 10^{-44}\ {\rm sec}$$

Using $E = mc^2$ we obtain:

highest energy $m_{\rm Pl} = \sqrt{\frac{c\hbar}{G}} = 1.22 \times 10^{19}$ **GeV**

Planck scale

Using $E_{\rm thermal} = k_{\rm B} T$ we obtain:

highest temperature $T_{\rm Pl} = \frac{m_{\rm Pl} c^2}{k_{\rm B}} = \sqrt{\frac{\hbar c^5}{G k_{\rm B}^2}} = 1.416786(71) \times 10^{32}\ {}^\circ{\rm K}$

(Temperature of the Big Bang)

Classical cosmology based on GRT stops to be valid for $t \sim t_{\rm Pl}$!

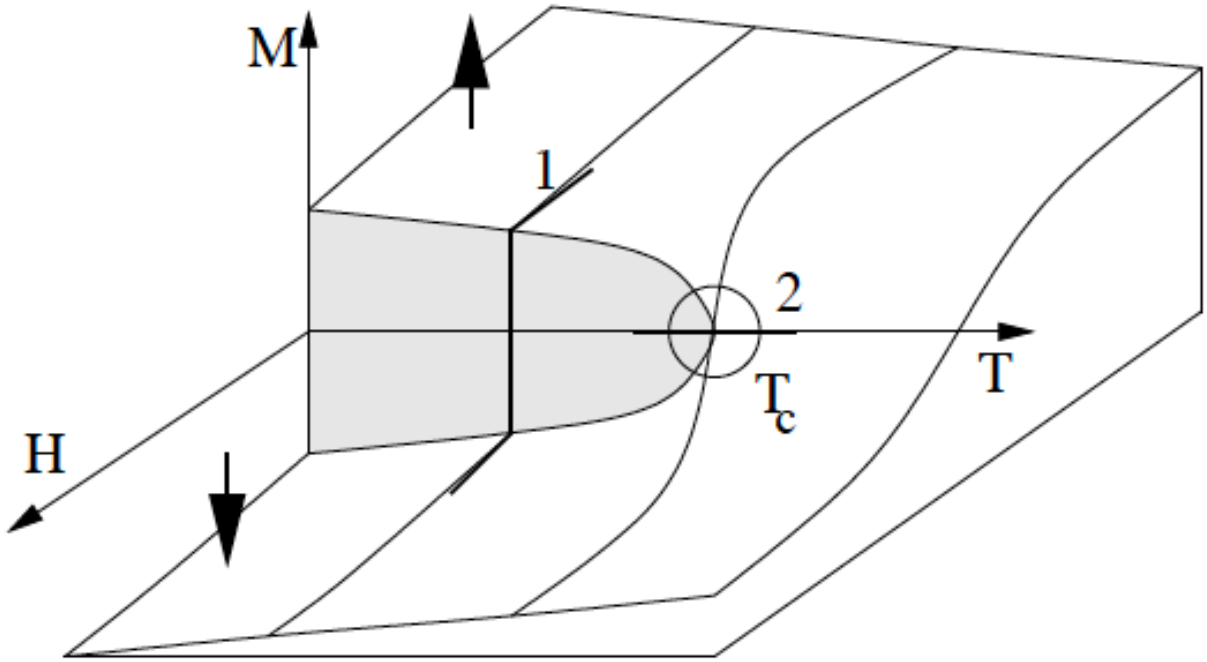
Big Bang singularity $R(0) = 0$ artifact of classical Big Bang Cosmology model.

At the Planck scale: unification of gravity with the other forces quantum gravity expected. No compelling answers, no convincing theory.

Back to the "Ether"

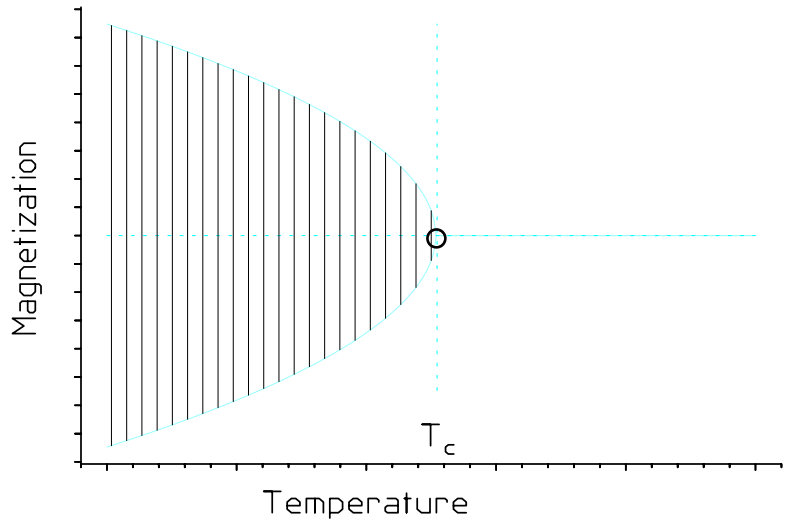
② Phase Transitions in a Nutshell

Example: Ferromagnetic spin system



Spontaneous magnetization M is occurs at $H = 0$ and $T < T_c$:
M is the order parameter and T_c the critical temperature

Cut $H = 0$:



Critical point $T = T_c$, $H = 0$: **phase transition of second order**

Cold phase $T < T_c$, $H = 0$: **spontaneous magnetization $M \neq 0$; spins spontaneously align, ordered phase, M is the order parameter**

sensitive to small perturbations $\delta H \to -\delta H$: Jump $M \to -M$ phase transition of first order

Non-critical system: spin correlations

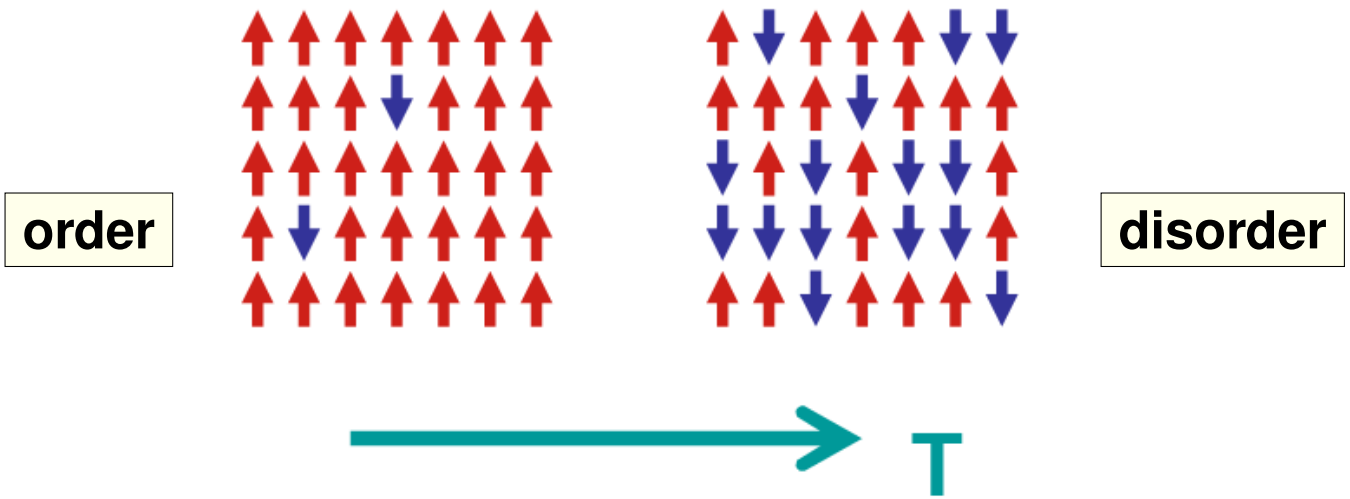
$\langle \sigma_{\vec{n}} \sigma_0 \rangle^c \simeq \frac{e^{-|\vec{x}|/\xi}}{|\vec{x}|}$; $\vec{x} = \vec{n} \cdot a \to \infty$; ξ finite

only a finite number of degrees of freedom are relevant: those in box of size ξ

in terms of Ising model up/down spins: spin $\sigma = \pm 1$ in z direction in which external field H is applied when non-zero

Suppose magnetization is in up direction; majority of spins aligned up – temperature fluctuations –: flipping a spin requires energy, keeps system stable, some of the spins flip, clusters of down spins of size of correlation length ξ

Hot phase $T > T_c$, $H = 0$: **system disordered, no long range order**



Approaching critical point from below $T \overset{<}{\to} T_c$, $H = 0$: **clusters of misaligned spins grows with ξ. As $\xi \to \infty$ magnetization disappears $M \to 0$ and with it the energy needed to flip a spin ➜ fluctuations of the local magnetization over large distances,**

∞ many degrees of freedom relevant

Characteristics of critical point:

❑ **divergence of correlation length**

$\xi \propto t^{-\nu}$; $H = 0,\ t \to 0$; $t \doteq \frac{T-T_c}{T_c}$

❑ **weak decay of correlations of** relevant (dim < 4) fields (operators)

$\langle \sigma_{\vec{n}}\, \sigma_0 \rangle^c \propto \frac{1}{|\vec{x}|^{2\mathrm{d}_\sigma}}$; **field correlation** ➜ **dyn. dim.** d_σ **of field** σ_x

$|\vec{x}| \to \infty$

$\langle E_{\vec{n}}\, E_0 \rangle^c \propto \frac{1}{|\vec{x}|^{2\mathrm{d}_E}}$; **energy correlation** ➜ **dyn. dim.** d_E **of field** σ_x^2

As a consequence of slow decay:

Susceptibility : $\chi = \sum_{\vec{n}} \langle \sigma_{\vec{n}}\, \sigma_0 \rangle^c \to \infty$

Specific heat : $C = \sum_{\vec{n}} \langle E_{\vec{n}}\, E_0 \rangle^c \to \infty$

singular behavior! some quantities diverge at critical point!

Basics of statistical lattice system, $\vec{x} = a\vec{n}$, $\vec{n}$ lattice vector, a lattice spacing
Relevant fluctuation fields

$$\sigma_x \text{ (site spin)} \ , \ E_x \doteq \sum_{y,|x-y|=a} \sigma_x \sigma_y \text{ (site energy density)}$$

Hamiltonian $\mathbf{H}$: K attractive n.n.coupling (interaction energy), H external magnetic field

$$\mathbf{H} = \beta \mathcal{H} \ , \ \ \mathcal{H}(\sigma) = -K \sum_x E_x - H \sum_x \sigma_x \ ; \ \ \beta = \frac{1}{k_B T}$$

Partition function: finite system of N spin lattice sites

$$Z_N = \sum_{\text{conf}\,\sigma} \exp -H_N$$

Correlation functions:

$$\langle \sigma_x \cdots \rangle = Z_N^{-1} \sum_{\text{conf}\,\sigma} \sigma_x \cdots \mathrm{e}^{-H_N}$$

Free energy density:

$$f(k,h) = \frac{F_N}{N}\ ; \quad F_N = -\ln Z_N\ ; \quad k = K/\beta\ , \quad h = H/\beta$$

Magnetization	$M \doteq \frac{\partial f}{\partial h}$	=	$\langle\sigma_x\rangle = \langle\sigma_0\rangle$
Energy density	$E \doteq \frac{\partial f}{\partial k}$	=	$\langle E_x\rangle = \langle E_0\rangle$
Susceptibility	$\chi \doteq -\frac{\partial^2 f}{\partial h^2}$	=	$\sum_x \{\langle\sigma_x\sigma_0\rangle - \langle\sigma_x\rangle\langle\sigma_0\rangle\}$
Specific heat	$C \doteq -\frac{\partial^2 f}{\partial k^2}$	=	$\sum_x \{\langle E_x E_0\rangle - \langle E_x\rangle\langle E_0\rangle\}$

Parametrization of the critical singularities: Widom, Domb, Hunter 1965

scaling hypothesis

$$
\begin{array}{llll}
\textbf{Magnetization:} & M|_H & \simeq & \left\{ \begin{array}{l} 0 \\ (-t)^{\beta} \end{array} \right. \\
\textbf{Susceptibility:} & \chi|_H & \simeq & \left\{ \begin{array}{l} t^{-\gamma} \\ (-t)^{-\gamma'} \end{array} \right. \qquad t \to \left\{ \begin{array}{l} +0 \\ -0 \end{array} \right. \\
\textbf{Specific Heat:} & C|_H & \simeq & \left\{ \begin{array}{l} t^{-\alpha} \\ (-t)^{-\alpha'} \end{array} \right. \\
\textbf{Magnetization:} & M|_{t=0} & \simeq & \pm|H|^{1/\delta}\ ; \qquad H \to \pm 0
\end{array}
$$

Results:

i) At criticality: scaling (homogeneous functions)

ii) All critical indices are given in terms of two independent ones

$x = \mathrm{d} - \mathrm{d}_\sigma$ **and** $y = \mathrm{d} - \mathrm{d}_E$

iii) scaling relations: ($t \to -0$ unprimed, $t \to +0$ primed)

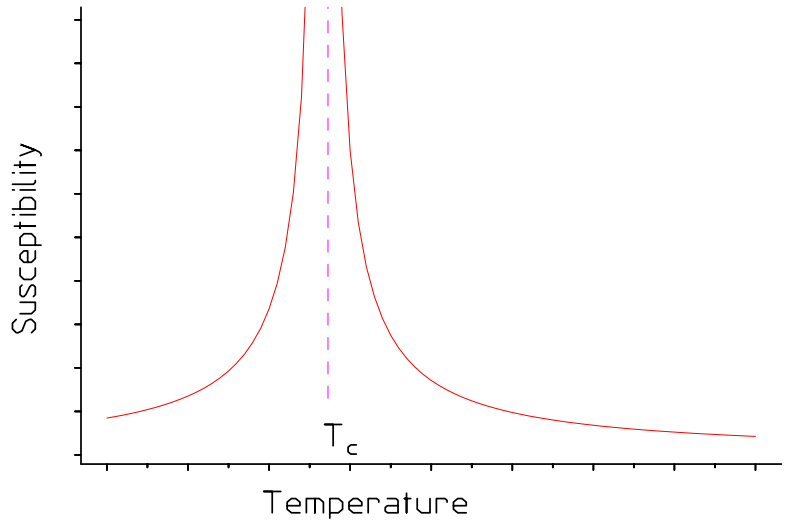
$$\alpha' = \alpha \; ; \quad \gamma' = \gamma \; ; \quad \nu' = \nu \; ; \quad \beta\,(\delta+1) = 2\beta + \gamma = 2 - \alpha$$

$$\begin{aligned} \mathrm{d}_\sigma &= \tfrac{d-2}{2} + \tfrac{\eta_\sigma}{2} = d - x \quad ; \quad \eta_\sigma = d + 2 - 2x\,, \quad \text{anomalous dim of } \sigma_x \\ \mathrm{d}_E &= 2\,\tfrac{d-2}{2} + \tfrac{\eta_E}{2} = d - y \quad ; \quad \eta_E = 4 - 2y\,, \quad \text{anomalous dim of } E_x \sim \sigma_x^2 \end{aligned}$$

$$\begin{aligned} \beta &= \tfrac{d-x}{y} \quad ; \quad \delta^{-1} = \tfrac{d-x}{y} \\ \gamma = \gamma' &= \tfrac{2x-d}{y} \quad ; \quad \alpha = \alpha' = \tfrac{2y-d}{y} \\ \nu = \nu' &= \tfrac{1}{y} \end{aligned}$$

$$\boxed{y = \nu^{-1}\,, \;\; x = \nu^{-1}\beta\delta}$$

Example: Quantum field theory taught by the Ising model



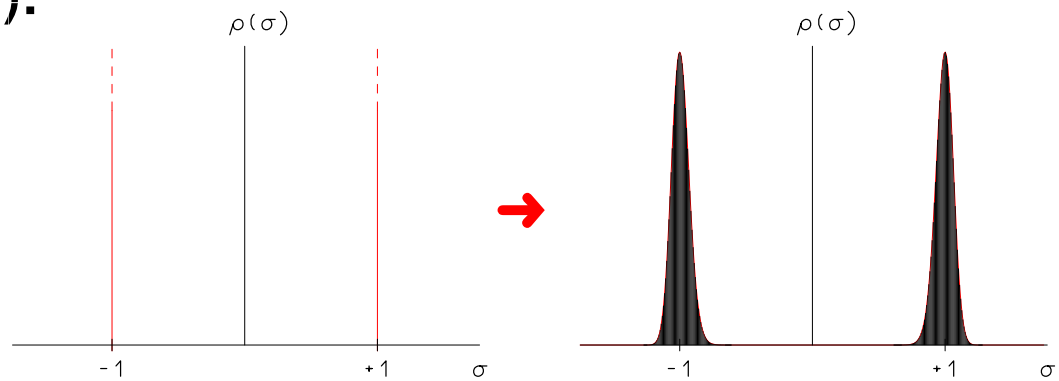
Ising Ferromagnetic model: N **classical spins** $\sigma_x = \pm 1$ **on a lattice of spacing** a **and box size** L**, with nearest neighbor interaction, two parallel spins attract each other with energy** $-K$ **and a spin parallel to an external field has energy** $-H$**:**

$$\mathcal{H}[\sigma] = -K \sum_{n.n.} \sigma_x \sigma_y - H \sum \sigma_x$$

$$Z = Z_0^{-1} \int \Pi_{\vec{n}}\, \mathrm{d}\sigma_{\vec{n}}\, \rho(\sigma_{\vec{n}})\, \mathrm{e}^{-H_N} \;;\quad H = \beta\mathcal{H} \;;\quad \beta = \frac{1}{k_B T}$$

Phase transition require thermodynamic limit $N \to \infty$ **i.e.** $\boxed{La \gg \xi, x, a}$

Approximation 1):



$$\rho(\sigma_{\vec{n}}) = \delta(\sigma_{\vec{n}}^2 - 1) \simeq \sqrt{\frac{u_0}{\pi}}\, \mathrm{e}^{-u_0\,(\sigma_{\vec{n}}^2-1)^2} \;;\quad u_0 \gg 1$$

$$Z \simeq \tilde{Z} = \tilde{Z}_0^{-1} \int \Pi_{\vec{n}}\, \mathrm{d}\sigma_{\vec{n}}\, \mathrm{e}^{-\mathcal{H}[\sigma]}$$

$$\mathcal{H}_0 = \frac{1}{2} \int_{-\pi/a}^{+\pi/a}{}_q \tilde{\sigma}(\vec{q})\tilde{\sigma}(-\vec{q})\, G_0^{-1}(\vec{q})$$

$$\mathcal{H}_i = u_0 \int_{-\pi/a}^{+\pi/a}{}_{q_1\cdots q_4} \tilde{\sigma}(\vec{q}_1)\cdots\tilde{\sigma}(\vec{q}_4)\, \delta(\Sigma\vec{q})$$

Approximation 2): expand propagator for small $\vec{q}$ and replace cutoff box by sphere of radius $\Lambda = \pi/a$ ➜ emerging rotational invariance !!! long range properties of original system unaffected!

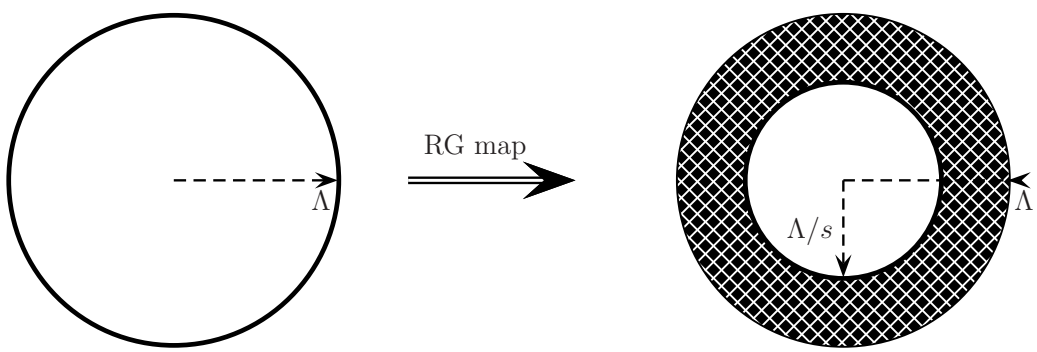
- $$G_0^{-1}(\vec{q}) = m_0^2 + 4a^{-2} \sum_{i=1}^{d} \sin^2 \tfrac{aq_i}{2} \;\rightarrow\; m_0^2 + q^2 + \Lambda^{-2}q^4 \;;\quad q^2 = \vec{q}^{\,2}$$

- $\int_{-\pi/a}^{+\pi/a} \mathrm{d}q \cdots \to \int_{|\vec{q}|\leq\Lambda} \mathrm{d}q \cdots \; ; \quad \sigma_{\vec{n}} \to \varphi(\vec{x})$ **up to field renormalization**

and looking at behavior for Λ large, resulting correlation functions are identical with those of an Euclidean QFT with a cutoff! **cutoff φ^4 theory**

$$Z = \int_{p\leq\Lambda} \mathcal{D}\varphi \, \exp[-S_\Lambda[\varphi]]$$

Definition of Wilson's RG: integrate out momenta in shell $\Lambda/s < |q_i| \leq \Lambda$
information not needed to study correlations at $|q_i| < \Lambda/s$



$$\exp[-S^{\rm eff}_{\Lambda'}[\varphi]]] \stackrel{\rm def}{=} \int_{\Lambda' \le p \le \Lambda} \mathcal{D}\varphi \, \exp[-S_\Lambda[\varphi]]$$

and obviously

$$Z = \int_{p \le \Lambda'} \mathcal{D}\varphi \, \exp[-S^{\rm eff}_{\Lambda'}[\varphi]]$$

The RG map $S_\Lambda[\varphi] \stackrel{\rm RG}{\to} S^{\rm eff}_{\Lambda'}[\varphi]$ is transitive, the Wilson exact RG is a semi-group (transitivity).

Effective Hamiltonian very complicated, the crux is that most of the generated effective couplings are suppressed by the cutoff!

Effective Landau-Ginsburg Hamiltonian:

$$
\begin{aligned}
H^{\rm eff}_{\Lambda'}[\varphi'] &= \sum_{n=0}^{\infty} \int_{q_1\cdots q_n} (2\pi)^d\, \delta^{(d)}\left(\sum_i q_i\right) \tilde\varphi'_{q_1}\cdots\tilde\varphi'_{q_n} \times u'_n(q_1\cdots q_n; u) \\
u_n &= u_n^{\rm SD} + u_n^{\rm LD}\ ; \qquad \text{LD = long distance}\ ,\quad \text{SD = short distance} \\
u_n^{\rm SD} &= \sum_{n\ell\alpha} P(q_1^{\ell_1}\cdots q_n^{\ell_n})\, g_{n\ell\alpha}\ ; \quad \textbf{analytic in q} \\
u_n^{\rm LD} &= \textbf{see below}
\end{aligned}
$$

Generating functional:

$$
Z\{J\} = \frac{1}{Z_0}\int_{p\le\Lambda} \mathcal{D}\varphi\, \exp[-S_\Lambda[\varphi] + (J\varphi)]
$$

Correlations at $|q| \ll \Lambda$ via Wilson's RG: $H^{\rm eff}_{\Lambda'}[\varphi'] = R_s\, H_\Lambda[\varphi]$ or $g' = R_s\, g$

Linearization: $g'_{n\ell\alpha} \propto s^{\omega_{n.\ell}} g_{n\ell\alpha}$ **up to logs**

$\omega_{n,\ell} > 0$	:	**relevant**	=	**super-renormalizable**
$\omega_{n,\ell} = 0$	:	**marginal**	=	**renormalizable**
$\omega_{n,\ell} < 0$	:	**irrelevant**	=	**non-renormalizable**

Universality:

$$H^{\text{eff}}(\varphi)_\Lambda = \sum_{\substack{n\ell\alpha \\ \omega_{n,\ell} \geq 0}} \int_{q_1 \cdots q_n} g_{n\ell\alpha} \tilde{O}_{n\ell\alpha}(\varphi)$$

- **only a few effective couplings survive ➜ renormalizability**
- **effective system exhibits more symmetries ➜ dynamical generation of symmetries (like rotation symmetry)**

- **in $d = 4$ Euclidean $O(4)$ ➜ $d = 3 + 1$ Minkowski $O(3,1)$ = special relativity (Lorentz-, Poincaré invariance)** **Osterwalder-Schrader**

Landau-Wilson:

$H_{\rm eff}(\varphi)_\Lambda$ = **free** H_0 **in** $d > 4$ | $\lesssim 4$: $H_{\rm eff}(\varphi)_\Lambda$ = **perturbative**

- **d=4 space-time "compactifies" itself;** $d - 4$ **complement dies out at long distances (what Wilson's RG says)**

Wilson-Fisher's ε**-expansion** ➜ **canonical power counting:**

$$g'(s) \quad \propto \quad s^\omega \; ; \quad \omega = d - 2nd_\varphi - \ell$$
$$d_\varphi \quad = \quad d - x \; ; \quad x : \alpha_s = s^x$$

such that, including fermionic terms, leading q **- dependence is** $O(1)$
[normalization to kinetic term, which rules propagation]

$$u^{\rm LD}_{2\,\alpha\beta}(q) = r_{\alpha\beta} + {\rm i}\, a_{\alpha\beta,\gamma} q^\gamma + b_{\alpha\beta,\mu\nu} q^\mu q^\nu + O(q^3)$$

- **form non-degenerate if all fluctuation variables have nearest neighbor (n.n.) interactions besides others**

Wilson-model RG in $d = 4 - \varepsilon$ dimensions:

- **in $d = 4$ Gaussian fixed point expected (free field); triviality of φ^4 theory in low energy limit**

Important note: φ^4 in $d = 4$ is not itself trivial (as claimed by some experts) only its long range behavior is trivial!

Perturbation about $d = 4$: $\kappa \doteq \Lambda/\Lambda'$
1) perturbation expansion for small u i.e. $\exp -\mathcal{H}_i = 1 - \mathcal{H}_i + \frac{1}{2}\mathcal{H}_i^2 - \cdots$

with $\text{—}^{-1} = \kappa^2\,\vec{p}^{\,2} + m^2$ **and** $\times = u_0$ **with** $\varphi' = \alpha_\kappa\,\varphi$ **we find**

$$u_2' = \frac{1}{2}\alpha_\kappa^2\kappa^d \left\{ \text{—}^{-1} + 12 \bigcirc + \cdots \right\}$$

$$u_4' = \alpha_\kappa^4\kappa^{3d} \left\{ \times - 12 \bigcirc + \cdots \right\}$$

2) evaluation for small $\vec{p}^{\,2}$**: using** $\alpha_\kappa = \kappa^x\,;\quad d_\varphi = d - x = \frac{d-2+\eta}{2}\,;\quad \eta_{\rm LO} = 0$

$$m'^2 = \kappa^{-2}\left\{ m_0^2 + \frac{u_0}{8\pi^2}\left[\frac{1}{2}\Lambda^2(1-\kappa^2) + m_0^2\,\ln\kappa + \cdots\right]\right\}$$

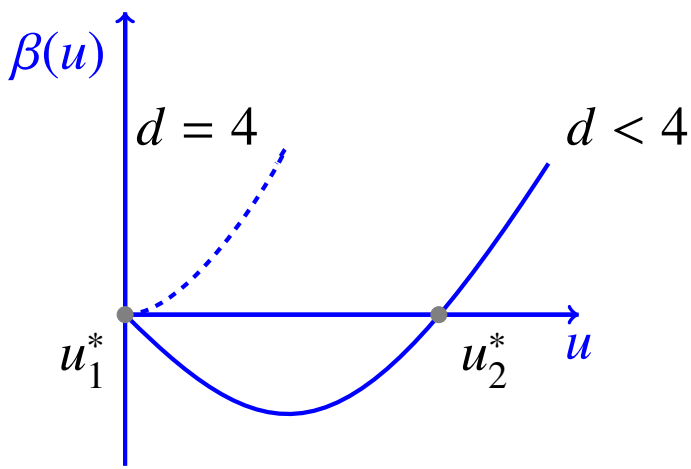
$$u' = \kappa^{-\varepsilon}\left\{ u_0 + \frac{9}{4}\frac{u_0^2}{8\pi^2}\,\ln\kappa\cdots\right\}$$

$\boxed{d > 4\,;\quad \varepsilon < 0}$ **trivial IR fixed point (Gaussian, mean field)**

$m^* = 0$ **must be tuned**
$u^* = 0$ **automatic**

$\boxed{d < 4\,; \quad \varepsilon > 0}$ $\exists$ **non-trivial IR fixed point**

$$m^* = -\frac{1}{6}\,\varepsilon\Lambda^2 + O(\varepsilon)$$
$$u^* = \frac{16}{9}\,\pi^2\,\varepsilon + O(\varepsilon^2)$$



Expand RG about (m^*, u^*):

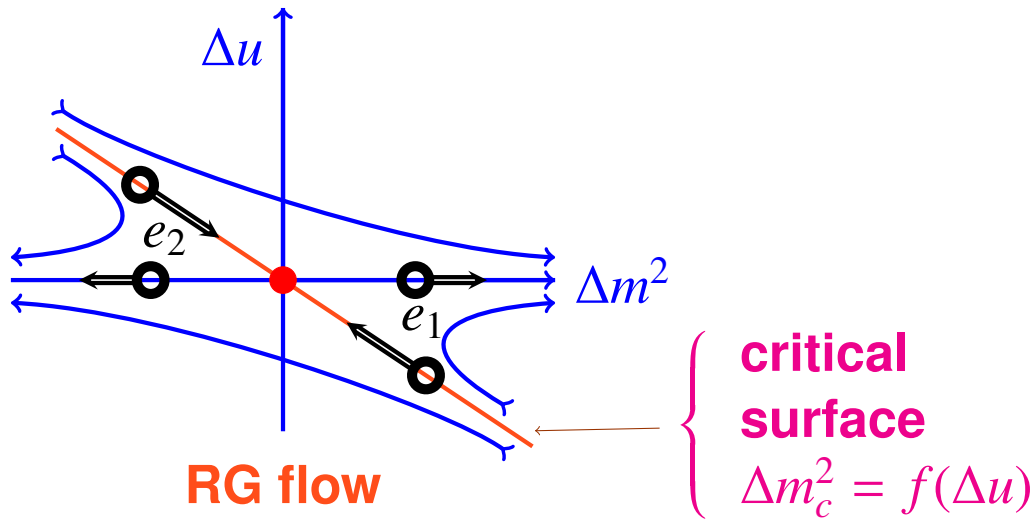
$$R_\kappa^{\rm lin} = \begin{pmatrix} \kappa^{-y_1} & c \\ 0 & \kappa^{-y_2} \end{pmatrix}$$

Eigenvalues:

$$\begin{array}{lll} E_1 = \kappa^{-y_1} & ; & y_1 \sim 2 - \frac{\varepsilon}{3} \\ E_2 = \kappa^{-y_2} & ; & y_2 \sim -\varepsilon \end{array}$$

Eigenvectors:

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; e_2 = \begin{pmatrix} a \\ 1 \end{pmatrix} \quad ; \quad a = -\frac{3}{4}\left(\frac{\Lambda}{\pi}\right)^2$$



$$R\begin{pmatrix}1\\0\end{pmatrix} = \kappa^{-y_1}\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\infty\\0\end{pmatrix}; \quad y_2 > 0$$

$$R\begin{pmatrix}a\\a\end{pmatrix} = \kappa^{-y_2}\begin{pmatrix}a\\1\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}; \quad y_2 < 0$$

Herewith we have determined two critical exponents:

$$\nu = \frac{1}{2} + \frac{\varepsilon}{12} + O(\varepsilon^2)$$
$$\eta = 0 + O(\varepsilon^2)$$

Higher approximations for $d = 3$ ($\varepsilon = 1$)

Exponent	Exp	ε	ε^2	ε^3	MF
α	~0.1	0.125	0.077	0.196	0
γ	~ 1.36	1.25	1.244	1.195	1

If the IR fixed point is non-degenerate i.e. the β–function has a simple zero $\beta(g^*) = 0\ ;\ \ \beta'(g^*) > 0)$, the critical correlation functions at the critical point are homogeneous functions under dilatations which implies that they define a conformally invariant Euclidean QFT.

❒ **the realization that effective critical theories are QFTs finally lead Ken Wilson to solve the very long standing problem of understanding scaling laws and critical exponents (anomalous dimensions) in phase transitions**

❒ **the key point after all was that QFT methods (Feynman graph expansions**

etc) near 2nd order phase transition points where applicable to classical statistical mechanics systems
Results relevant in our context:

- **emergence of (Euclidean) QFT as a low energy effective structure**
- **emergence of symmetries like rotational invariance**
- **universality, universality classes (blindness to specific short distance structure)**

Examples of very pronounced transition phenomena

Approaching Critical behavior: the finite system "early" knows about

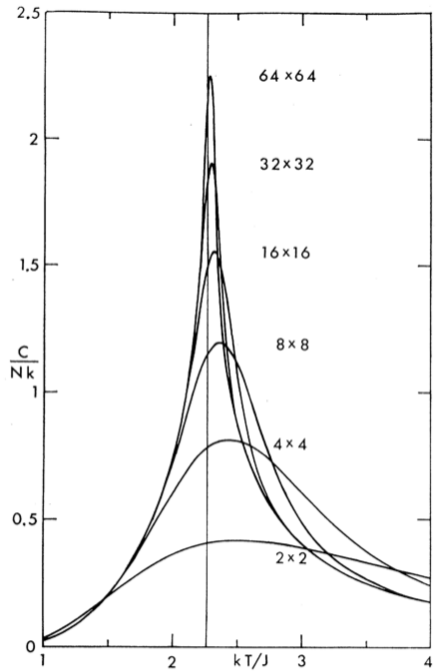


FIG. 1. The specific heat per spin for small Ising lattices; exact results for the $m \times n$ square lattice with periodic boundary conditions are displayed for $m=n=2, 4, 8, 16, 32$, and 64 ($N=mn$). The limiting critical point is marked by a vertical line.

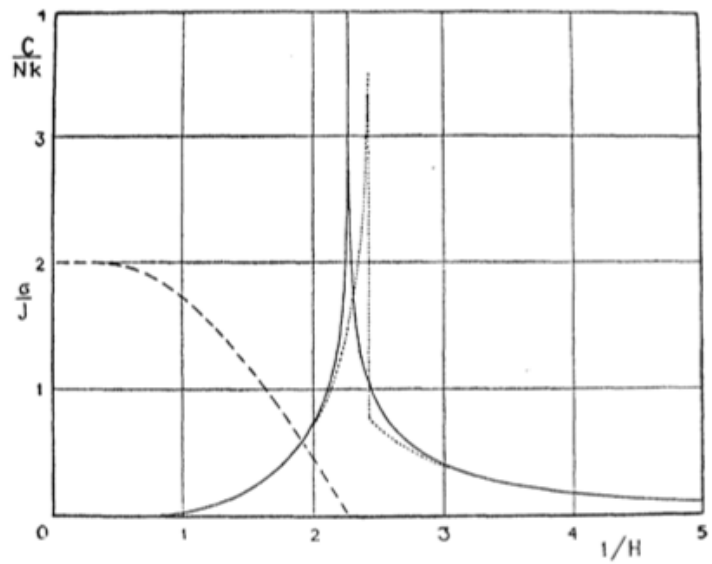


FIG. 6. Properties of quadratic crystal. — — — — — — Boundary tension σ between regions of opposite order. ———— Specific heat C. - - - - - - - - - - - - Approximate computation of C by Kramers and Wannier.

D=2 Ising system developing critical behavior (singularity)

Range of a transition region:

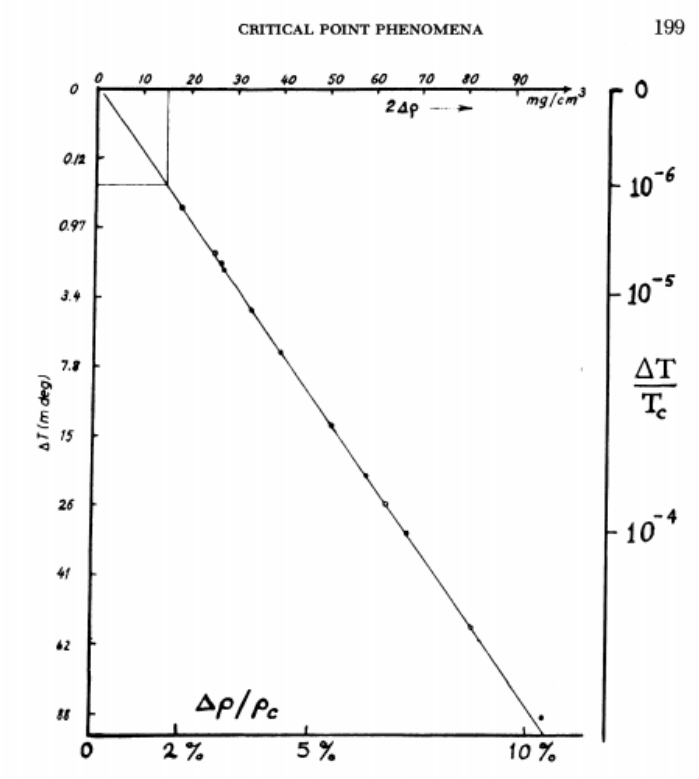
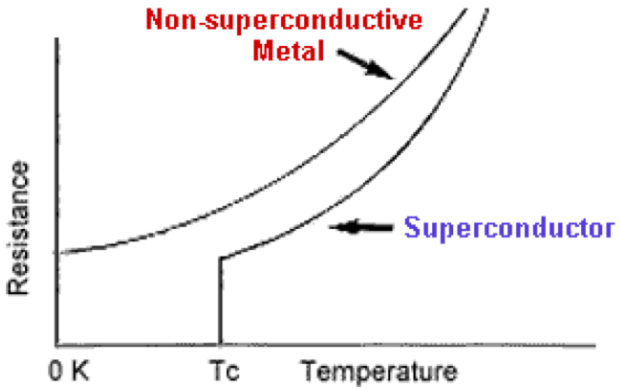


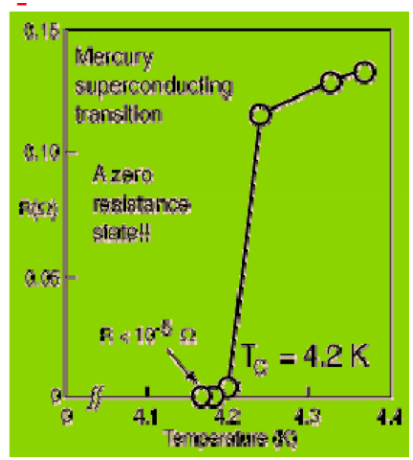
Figure 1. The coexistence curve of carbon dioxide close to the critical point, showing the variation of the density discontinuity $\Delta\rho(T)$ with temperature deviation $\Delta T = |T - T_c|$. The temperature scale is nonlinear, the ordinate of the graph being proportional to $(\Delta T/T_c)^{1/3}$. (After Lorentzen [10].)

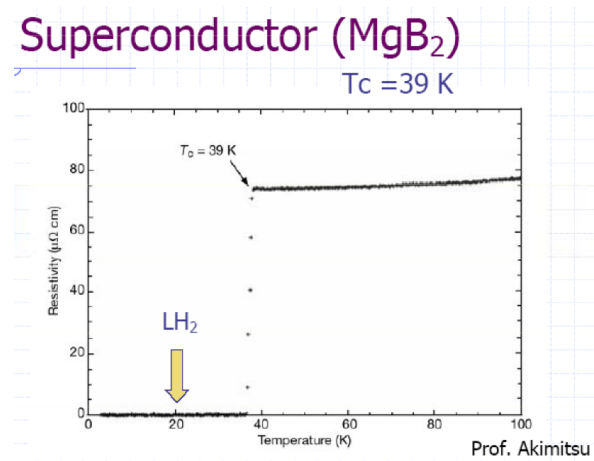
Real world CO_2 critical scaling

Sharpness of a transition I:

Superconductivity: Perfect conductor
(Electrical resistivity of superconductor Less than 10^{-20} ohm.cm)
Kamerlingh Onnes 1911

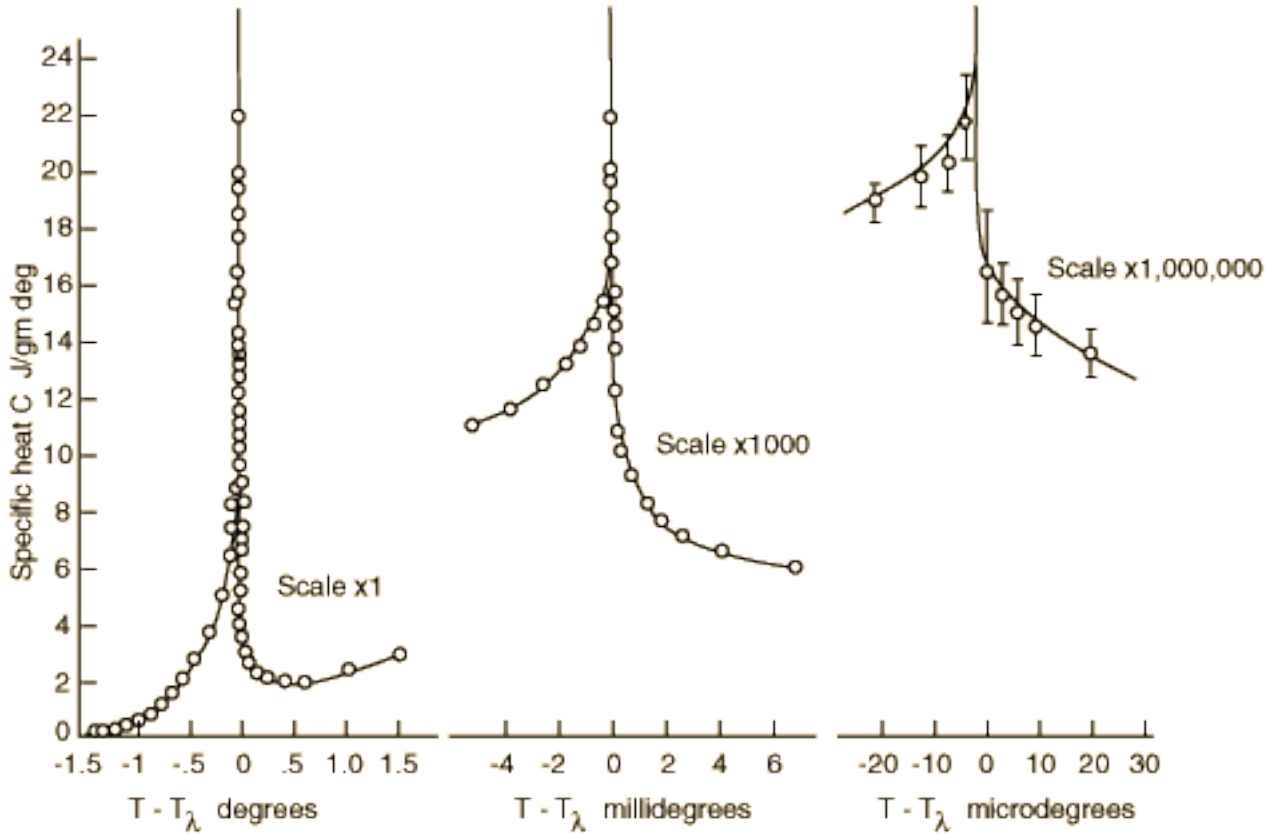






Dramatic drop in resistivity in transition to superconducting phase

Sharpness of a transition II



Specific heat of liquid Helium (Fairbank et al. 1957)

History:

- **1873 van der Waals: first model of a phase transition – the van der Waals liquid-gas system**

- **1900 Verschaffelt: investigating CO_2 liquid-vapor transition van der Waals model fails:** $\rho_l - \rho_v = 0.243\ (1 - T/T_c)^{0.367}$ **not mean field i.e. real world** $\beta = 0.367$ **not** $\beta = 0.5$**; takes over 70 years to solve the problem by Ken Wilson 1971 [NP 1980] (Wilson's RG, QFT gives right answer!)**

- ...

- **1937 Landau theory of Phase Transitions: Landau mean field approach to phase transitions, claims long distance behavior in any case is mean field (i.e. free fields in QFT language)**

- ...

- **1971 Wilson: critical behavior non-trivial, anomalous dimensions etc.**
  ➜ critical behavior governed by QFT!!!

Mean field, Gaussian, free field:

$$\alpha = 0 \; ; \;\; \beta = \tfrac{1}{2} \; ; \;\; \gamma = 1 \; ; \;\; \delta = 3$$

Dynamical, non-classical:

$$\alpha = 0.12 \; ; \;\; \beta = 0.313 \; ; \;\; \gamma = 1.25 \; ; \;\; \delta = 5.0$$

As far as the values of the non-classical critical exponents obtained before 1900 by Van der Waals and Verschaffelt are concerned, they have been fully confirmed by modern research (Levelt Sengers 1975).

Example: Ising like systems (universality class)

D	2	3	4
α	0	0.110(1)	0
β	1/8	0.3265(3)	1/2
γ	7/4	1.2372(5)	1
δ	15	4.789(2)	3
η	1/4	0.0364(5)	
ν	1	0.6301(4)	1/2
ω	2	0.84(4)	

- $D = 2$ **Onsagers 1943 exact solution**
- $D = 3$ **Wilson 1971, Wilson-Fisher 1972 via ε-expansion**
- $D = 4$ **Landau theory 1937, free field (triviality of D=4 critical theories)**

Universality in the real world: liquid-gas, ferromagnetic and superconducting system may have identical critical (long distance) behavior! only if one looks closer one sees the difference.

❒ **Simplicity of long distance physics.**

Epilogue

Scaling and self-similarity are among the most interesting phenomena observed all over in nature. In spite of the dramatic nature of phase transition phenomena, it took theoretical physics from Van der Waals, who to my knowledge invented the first model of a phase transition in 1873[1] to Ken Wilson 1971[2], who for the first time understood critical behavior, the emergence of physics patterns at long distances, in a quantitative way. The key was anomalous dimensions and what prevented the earlier solution of the problem in condensed matter systems was the fact that these systems exhibit a natural cut-off, some atomic size in a gas or liquid or a related lattice spacing in a solid. What was hard to understand at that time was the fact that in iterated maps fixed scales do not exist, because the effective cutoff gets rescaled dynamically (Kadanoff's block-spin iteration, Wilson's renormalization semi-group)[3].

[1] The Van der Waals equation is a modification of the ideal gas law, and Van der Waals received the Nobel prize in 1910 for "his work on the equation of state for gases and liquids".

[2] He got the Nobel prize in 1982 for "for his theory for critical phenomena in connection with phase transitions".

[3] For almost 70 years experts thought that a study of the long range behavior consists in studying the limit $p \ll \Lambda =$

Typical is the appearance of logarithms integrals

$$\int_{\lambda}^{\Lambda} \frac{dx}{x}$$

which pick equal contributions from all scales, it diverges logarithmically in the infrared (IR) as well as in the ultraviolet (UV). Iterated map resum logarithms as we know and usually lead to non-trivial (non-Gaussian) critical exponents and thus to power law corrections with anomalous dimensions. Anomalous dimensions are related to fractals.

✦✦✦✦✦✦✦✦✦✦✦

$1/a$ for fixed Λ and hence critical behavior was not supposed to have something to do with UV divergences. At the end critical behavior turned out can be understood only if the $\Lambda \to \infty$ is taken into account. The reason is that the critical effective theory is a renormalizable QFT, with its UV properties playing an important role.

Exercise:

Free energy density: (here H the Hamiltonian, β the inverse temperature times the Boltzmann factor)

$$f = -\frac{1}{N}\ln Z_N\ ;\quad Z_N = \sum_{conf\sigma} \exp -H_N\ ,\quad H = \beta\mathcal{H}\ ,\quad \beta = \frac{1}{k_B T}$$

the singular part under spatial dilatations $x \to \kappa x$ transforms as (here H the magnetic field and β the critical exponent)

$$f(t,h) = \kappa^d\, f(t\,\kappa^{-y_E}, h\,\kappa^{-y_\sigma})\ ;\quad t \doteq \frac{T-T_c}{T_c}\ ,\quad h \doteq \beta H$$

The equation of state is given by

$$M = \frac{\partial f}{\partial h} = \kappa^{d-y_\sigma}\frac{\partial f}{\partial h}\,(t\kappa^{-y_E}, h_0)_{h=h_0}$$

and the coexistence curve $h = 0$ reads

$$M(t, h = 0) = |t|^{\frac{d-y}{y_E}} M(\pm, 0)$$

Derive the formulas

a) free energy density (singular part)
$f(t, h) = |t|^{2-\alpha} f(\pm 1, \frac{h}{|t|^{\Delta}})$
$\Delta = \beta + \gamma$ gap exponent

b) equation of state scales like
$\frac{h(t,M)}{M^{\delta}} = h(\frac{t}{M^{1/\beta}})$

An early experimental test shows the equation of states in the form $\frac{h}{M|t|^{\gamma}} = w_{\pm}\left(\frac{M}{|t|^{\beta}}\right)$, i.e. $w_{\pm}$ is a function of $M|t|^{-\beta}$ only and not on M and t separately:

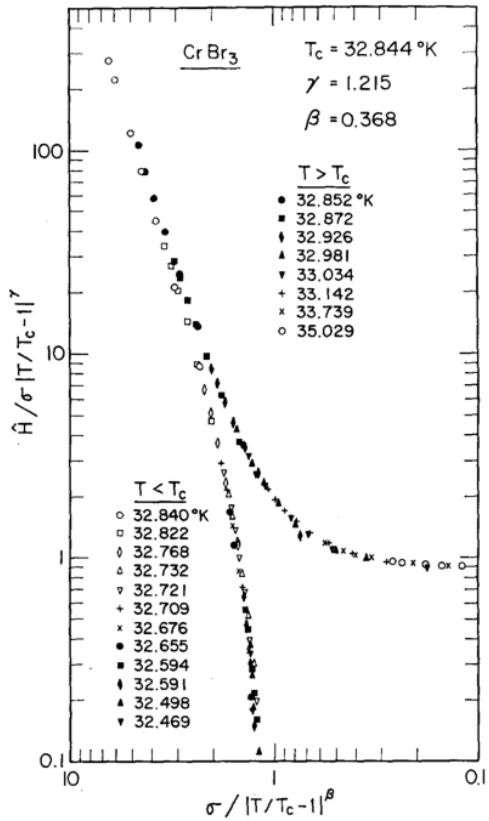


FIG. 2. This plot of $H/(\sigma\,|\,t\,|^{\gamma}) = \Phi(1, m^{1/\beta})$ against $\sigma\,|\,t\,|^{-\beta} = m$ confirms the scaling hypothesis for $CrBr_3$. The two branches are for $T > T_c$ and $T < T_c$.

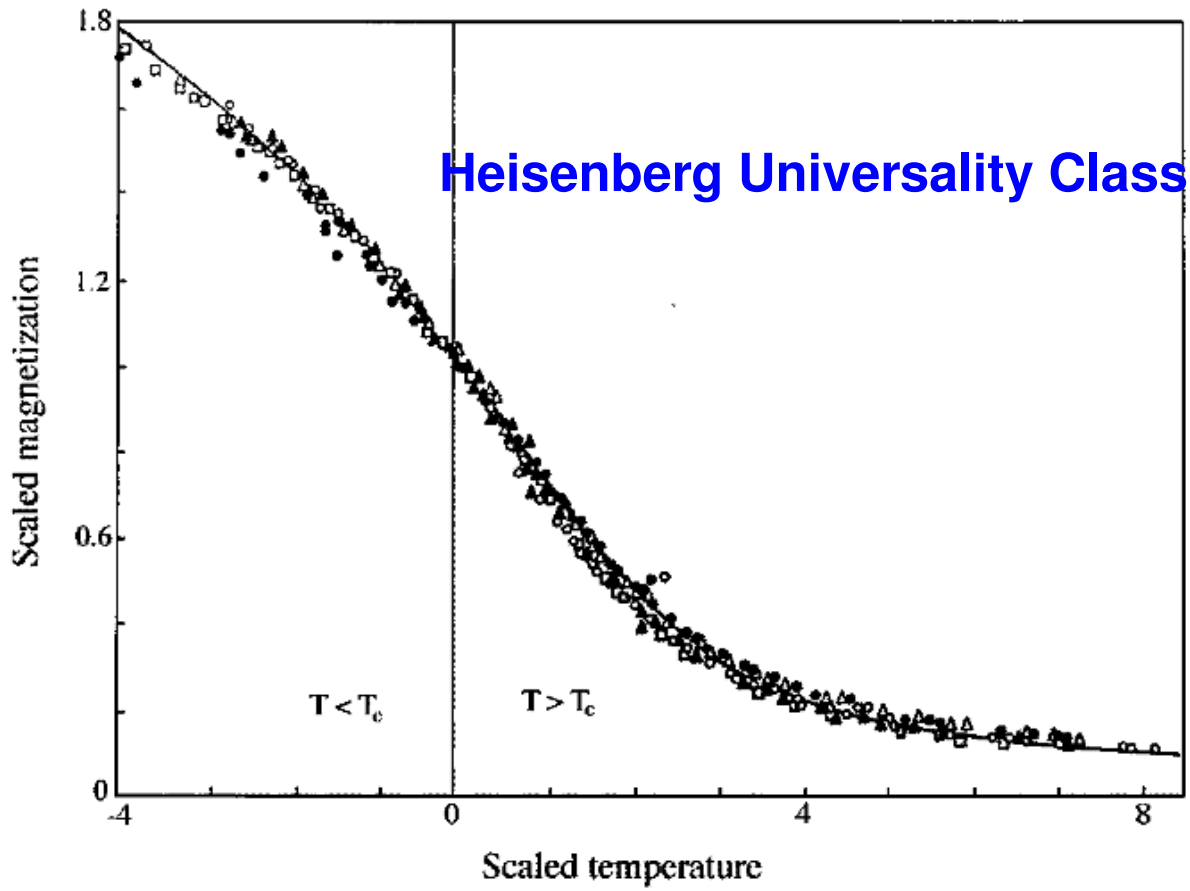
Divergences of the Magnetic Properties of CrBr3 near the Critical Point
Plot of $h/(M|t|^{\gamma})$ against $M/|t|^{\beta}$ confirms the scaling hypothesis for $CrBr_3$. The two branches are for $T > T_c$ and $T < T_c$.
(Ho, Litster, J. Appl. Phys. 40 (1969) 1270)

The physical universality can be understood to some extent (at least intuitively) from our general discussion and from model calculations.

System	α	β	γ	δ	ν	η
CO_2		0.352 ± 0.008	1.22 ± 0.01	4.47 ± 0.12		
X_e	$0.04 - 0.06$	0.35 ± 0.7	1.232 ± 0.02	4.6 ± 0.1		
H_e^4	$0.04 - 0.11$	0.355 ± 0.009	1.22 ± 0.002	4.44 ± 0.16		
N_i		0.374 ± 0.016	1.35 ± 0.02	4.45 ± 0.10		
Gd		0.370 ± 0.01	1.33	4.39 ± 0.1		
$CrBr_3$		0.368	1.215	4.3		
MnF_2		0.335 ± 0.01	1.27		0.63	0.05
fluids		0.35 ± 0.01	1.21 ± 0.05			
ferrom.		0.37 ± 0.01				
binary fluids		0.342 ± 0.016				
Ising (d=3)	0.110(1)	0.3265(3)	1.2372(5)	4.789(2)	0.6301(4)	0.0364(5)
Heisenberg (d=3)	−0.1336(15)	0.3689(3)	1.3960(9)	4.783(3)	0.7112(5)	0.0375(5)

What definitely looks different if we look close at its substructure often has identical emergent behavior when seen from far away!

Data Collapse: Scaled plot of magnetization vs temperature for five different materials



Experimental MHT data on five different magnetic materials (CrBr3, EuO, Ni, YIG, Pd3Fe) plotted in scaled form.

Next ⊳⋙ lecture.