The muon g-2: where we are?

F. Jegerlehner
Humboldt-Universität zu Berlin & DESY Zeuthen

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(g – 2)_μ

Outline of Talk:

1. The Anomalous Magnetic Moment of the Muon
2. Standard Model Prediction for a_μ
3. Hadronic Light-by-Light Scattering Contribution to g – 2
4. Outlook
\( (g - 2)_\mu \)

#### The Anomalous Magnetic Moment of the Muon

\[
\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c} \vec{s} ; \quad g_\mu = 2 (1 + a_\mu)
\]

**Dirac:** \( g_\mu = 2 \), \( a_\mu \) muon anomaly

Stern, Gerlach 22: \( g_e = 2 \); Kusch, Foley 48: \( g_e = 2 (1.00119 \pm 0.00005) \)

\[
\gamma(q) \quad \mu(p')
\]

\[
(-ie) \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m_\mu} F_2(q^2) \right] u(p)
\]

\[
F_1(0) = 1 ; \quad F_2(0) = a_\mu
\]

\( a_\mu \) responsible for the Larmor precession

directly proportional at magic energy \( \sim 3.1 \text{ GeV} \)

CERN, BNL g-2 experiments

\[
\vec{\omega}_a = \frac{e}{m} \left[ a_\mu \vec{B} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \vec{\beta} \times \vec{E} \right]^{E \sim 3.1 \text{ GeV}}
\]

at "magic \( \gamma \)" \( \simeq \frac{e}{m} \left[ a_\mu \vec{B} \right] \)
The BNL muon storage ring
Measured: ratio $\bar{R} = \omega_s / \omega_c$, $\omega_s =$ precession frequency, $\omega_c =$ cyclotron frequency; also need ratio $\mu_{\mu} / \mu_p$ between muon magnetic moment and proton magnetic moment

$$a_{\mu} = \frac{\bar{R}}{\mu_{\mu} \mu_p}$$

Distribution of counts versus time for the 3.6 billion decays in the 2001 negative muon data-taking period
(g – 2)μ

Experiment Theory

\[ a_μ = 116592080(63) \times 10^{-11} \]

2.9 σ (e^+ e^-) [ 1.4 σ (τ) ]

BNL - E821
Muon (g-2) Collaboration

hep-ex/0401008/0602035

Uncertainties:

- Experiment: 0.5ppm
  = 6 × 10^{-10} [ 5 × 10^{-10} \text{ stat, } 4 × 10^{-10} \text{ syst } ]

- Theory: 0.6ppm
  = 7×10^{-10} (e^+ e^-) [0.6ppm = 7 × 10^{-10} (τ) ]

New BNL 969 prop.: 0.2ppm
  = 2.4 × 10^{-10}

New physics sensitivity: (example)

\[ \Delta a_μ^{\text{SUSY}}/a_μ \simeq 1.25 \text{ppm} \left( \frac{100\text{GeV}}{\tilde m} \right)^2 \tan \beta \]

\( \tilde m \) lightest SUSY particle; SUSY requires two Higgs doublets

\[ \tan \beta = \frac{v_1}{v_2}, v_i = < H_i > ; \ i = 1, 2 \]

\[ \tan \beta \sim m_t/m_b \sim 40 \ [4 – 40] \]

most NP models yield

\[ |a_μ(\text{New Physics})| \simeq m_μ^2/M^2 \frac{\alpha}{\pi} \Rightarrow M \simeq 1 – 2 \text{ TeV} \]
The QED contribution to $a_\mu$ has been computed (or estimated) through 5 loops. Growing coefficients in the $\alpha/\pi$ expansion reflect the presence of large $\ln \frac{m_\mu}{m_e} \approx 5.3$ terms coming from electron loops.

**New:** $a_e = 0.00115965218085(76)$ Gabrielse et al. 2006

$$\alpha^{-1}(a_e) = 137.035999710(96) \ [0.70 \ ppb]$$

based on work of Kinoshita, Nio 04

$$a_\mu^{QED} = 116584718.20 \ (0.03) \ (1.15) \ (0.08) \times 10^{-11}$$

The current uncertainty is well below the $\pm 60 \times 10^{-11}$ experimental error from E821.
<table>
<thead>
<tr>
<th># n of loops</th>
<th>( C_i \left[ (\alpha/\pi)^n \right] )</th>
<th>( a_\mu^{\text{QED}} \times 10^{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+0.5</td>
<td>116140972.87 (0.44)</td>
</tr>
<tr>
<td>2</td>
<td>+0.765 857 376(27)</td>
<td>413217.60 (0.02)</td>
</tr>
<tr>
<td>3</td>
<td>+24.050 508 98(44)</td>
<td>30141.90 (0.00)</td>
</tr>
<tr>
<td>4</td>
<td>+126.07(41)</td>
<td>367.01 (1.19)</td>
</tr>
<tr>
<td>5</td>
<td>+930.0(170)</td>
<td>6.29 (1.15)</td>
</tr>
<tr>
<td>tot</td>
<td></td>
<td>116584705.66 (2.80)</td>
</tr>
</tbody>
</table>

1 diagram

![Diagram 1](image1)

Schwinger 1948

2 7 diagrams

![Diagrams 2-7](image2)

Peterman 1957, Sommerfield 1957

3 72 diagrams

Lautrup, Peterman, de Rafael 1974, Laporta, Remiddi 1996

4 about 1000 diagrams

Kinoshita 1999, Kinoshita, Nio 2004

5 estimate of leading terms

Karshenboim 93, Czarnecki, Marciano 00, Kinoshita, Nio 05
\( (g - 2)_\mu \)

**Weak Contributions**

\[
a_\mu^{\text{weak}(1)} = (195 \pm 0) \times 10^{-11}
\]

Brodsky, Sullivan 67, ...,

Bardeen, Gastmans, Lautrup 72

Higgs contribution tiny!

\[
a_\mu^{\text{weak}(2)} = -(44 \pm 4) \times 10^{-11}
\]

Kukhto et al 92

potentially large terms \( \sim G_F m_\mu^2 \alpha / \pi \ln \frac{M_Z}{m_\mu} \)

Peris, Perrottet, de Rafael 95

quark-lepton (triangle anomaly) partial

Czarnecki, Krause, Marciano 96

Heinemeyer, Stöckinger, Weiglein 04, Gribouk, Czarnecki 05 final full 2–loop result known

**Most recent evaluations: improved hadronic part (beyond QPM)**

\[
a_\mu^{\text{weak}} = (152 \pm 1[\text{had}] \pm ?) \times 10^{-11} \quad (\text{Knecht, Peris, Perrottet, de Rafael 02})
\]

\[
a_\mu^{\text{weak}} = (154 \pm 1[\text{had}] \pm 2[m_\text{H}, m_\text{t}, 3 - \text{loop}]) \times 10^{-11} \quad (\text{Czarnecki, Marciano, Vainshtein 02})
\]
Hadronic Contributions

General problem in electroweak precision physics:
contributions from hadrons (quark loops) at low energy scales

(a) Hadronic vacuum polarization $O(\alpha^2), O(\alpha^3)$
(b) Hadronic light-by-light scattering $O(\alpha^3)$
(c) Hadronic effects in 2-loop EWRC $O(\alpha G_F m^2_{\mu})$
Evaluation of $\alpha_{\mu}^{\text{had}}$

Leading non-perturbative hadronic contributions $\alpha_{\mu}^{\text{had}}$ can be obtained in terms of

$$R_{\gamma}(s) \equiv \frac{\sigma^{(0)}(e^+e^- \to \gamma^* \to \text{hadrons})}{4\pi \alpha^2} \frac{4\pi \alpha^2}{3s} \text{data via dispersion integral:}$$

$$\alpha_{\mu}^{\text{had}} = \left( \frac{\alpha m_\mu}{3\pi} \right)^2 \left( \int_{4m_\pi^2}^{E_{\text{cut}}^2} ds \frac{R_{\gamma}^{\text{data}}(s)}{s^2} \hat{K}(s) \right)$$

$$+ \int_{E_{\text{cut}}^2}^{\infty} ds \frac{R_{\gamma}^{\text{pQCD}}(s)}{s^2} \hat{K}(s)$$

- Experimental error implies theoretical uncertainty!
- Low energy contributions enhanced: $\sim 67\%$ of error on $\alpha_{\mu}^{\text{had}}$ comes from region $4m_\pi^2 < m_\pi^2 < M_\Phi^2$

$$\alpha_{\mu}^{\text{had}(1)} = (692.0 \pm 6.0) \times 10^{-10}$$

e$^+e^- - data based

Data:..., CMD2, KLOE, SND
Key role: VEPP-2M/Novosibirsk, DAFNE/Frascati

F. Jegerlehner, Final EURIDICE Meeting, Kazimierz, Poland – August 24-27, 2006 –

\[ (g - 2)_\mu \]
Novosibirsk – Frascati battle fields

after CMD-2 $e^+e^-$ (direct) contra ALEPH $\tau$ (indirect); KLOE (new technique: radiative return) clearly is on CMD-2 side (3% discrepancy vs 10 to 20%), SND joined, at the moment $\tau$–data not included since indirect and probably corrections not fully understood or experimental problem (Belle preliminary)

Soon: BABAR and Belle will join the battle new KLOE result awaited!
$(g - 2)_\mu$
Recent evaluations of $a^\text{had}_\mu$

<table>
<thead>
<tr>
<th>data</th>
<th>$a^\text{had}(1) \times 10^{10}$</th>
<th>Ref.</th>
<th>hep-ph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^+e^-$</td>
<td>694.8[8.6]</td>
<td>FJ,GJ 03</td>
<td>0310181</td>
</tr>
<tr>
<td>$e^+e^-$</td>
<td>692.0[6.0]</td>
<td>FJ 06</td>
<td>0608xxx**</td>
</tr>
<tr>
<td>$e^+e^-$</td>
<td>696.3<a href="6.2">7.2</a><em>{\text{exp}}(3.6)</em>{\text{rad}}</td>
<td>DEHZ 03</td>
<td>0308213</td>
</tr>
<tr>
<td>$e^+e^-$</td>
<td>690.9<a href="3.9">4.4</a><em>{\text{exp}}(1.9)</em>{\text{rad}}(0.7)_{\text{QCD}}</td>
<td>DEHZ 06</td>
<td>ICHEP06**</td>
</tr>
<tr>
<td>$e^+e^-$</td>
<td>692.4<a href="5.9">6.4</a><em>{\text{exp}}(2.4)</em>{\text{rad}}</td>
<td>HMNT 03</td>
<td>0312250</td>
</tr>
<tr>
<td>$e^+e^-$</td>
<td>699.6<a href="8.5">8.9</a><em>{\text{exp}}(1.9)</em>{\text{rad}}(2.0)_{\text{proc}}</td>
<td>ELZ 03</td>
<td>0312114</td>
</tr>
<tr>
<td>$\tau$</td>
<td>711.0<a href="5.0">5.8</a><em>{\text{exp}}(0.8)</em>{\text{rad}}(2.8)_{SU(2)}</td>
<td>DEHZ 03</td>
<td>0308213</td>
</tr>
<tr>
<td>$e^+e^- + \tau$</td>
<td>693.5<a href="5.0">5.9</a><em>{\text{exp}}(1.0)</em>{\text{rad}}(3.0)_{\ell\times\ell}</td>
<td>TY 04</td>
<td>0402285</td>
</tr>
<tr>
<td>$e^+e^- + \tau$</td>
<td>701.8<a href="4.9">5.8</a><em>{\text{exp}}(1.0)</em>{\text{rad}}(3.0)_{\ell\times\ell}</td>
<td>TY 04</td>
<td>0402285</td>
</tr>
</tbody>
</table>

Differences in errors mainly by utilizing more or less theory: pQCD, SR, low energy QCD methods

** most recent data SND, CMD-2, BaBar included
Contributions to $a^\text{had}_\mu \times 10^{10}$ with relative (rel) and absolute (abs) error in percent.

<table>
<thead>
<tr>
<th>Energy range</th>
<th>$a^\text{had}_\mu$ <a href="error">%</a> $\times 10^{10}$</th>
<th>rel [%]</th>
<th>abs [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho, \omega \ (E &lt; 2M_K)$</td>
<td>538.58 <a href="3.84"> 77.8</a></td>
<td>0.7</td>
<td>37.9</td>
</tr>
<tr>
<td>$2M_K &lt; E &lt; 2 \text{ GeV}$</td>
<td>102.33 <a href="4.72"> 14.8</a></td>
<td>4.6</td>
<td>57.3</td>
</tr>
<tr>
<td>$2 \text{ GeV} &lt; E &lt; M_{J/\psi}$</td>
<td>22.13 <a href="1.23"> 3.2</a></td>
<td>5.6</td>
<td>3.9</td>
</tr>
<tr>
<td>$M_{J/\psi} &lt; E &lt; M_Y$</td>
<td>26.39 <a href="0.59"> 3.8</a></td>
<td>2.2</td>
<td>0.9</td>
</tr>
<tr>
<td>$M_Y &lt; E &lt; E_{\text{cut}}$</td>
<td>1.40 <a href="0.09"> 0.2</a></td>
<td>6.2</td>
<td>0.0</td>
</tr>
<tr>
<td>$E_{\text{cut}} &lt; E$ pQCD</td>
<td>1.53 <a href="0.00"> 0.2</a></td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>$E &lt; E_{\text{cut}}$ data</td>
<td>690.83 <a href="6.23"> 99.8</a></td>
<td>0.9</td>
<td>100.0</td>
</tr>
<tr>
<td>total</td>
<td>692.36 <a href="6.23">100.0</a></td>
<td>0.9</td>
<td>100.0</td>
</tr>
</tbody>
</table>
Comparison of error profiles between

\[ \Delta \alpha^{(5)}_{\text{had}}(M_Z^2) \]
\[ \Delta \alpha^{(5)}_{\text{had}}(-s_0) \]

and

\[ a_\mu \]

Note high improvement potential for

VEPP-2000 and DANAE/KLOE-2
“Experimental” Adler-function versus theory (pQCD + NP) in the low energy region ((EJKV98)). Note that the error includes both statistical and systematic ones, in contrast to R-data plots where only statistical errors are shown !!!.

\[ D_{\gamma}(Q^2) = Q^2 \int_{4m_{\pi}^2}^{\infty} ds \frac{R_{\gamma}(s)}{s+Q^2} \]
Higher order hadronic contributions

\[ a_{\mu}^{\text{had}(2)} = -(100 \pm 6) \times 10^{-11} \quad \text{(Krause 96)} \]

\[ a_{\mu}^{\text{had}(2)} = -(98 \pm 1) \times 10^{-11} \quad \text{(Hagiwara et al. 03)} \]
3 Hadronic Light-by-Light Scattering Contribution to \( g - 2 \)

- Hadronic light–by–light scattering \( a_{\mu}^{\text{lbl}} = (80 \pm 40) \times 10^{-11} \) (Knecht & Nyffeler 02)
- \( a_{\mu}^{\text{lbl}} = (136 \pm 25) \times 10^{-11} \) (Melnikov & Vainshtein 03)

Low energy effective theory: e.g. ENJL
- Kinoshita-Nizic-Okamoto 85, Hayakawa-Kinoshita-Sanda 95, Bijnens-Pallante-Prades 95

\( q = (u, d, s, \ldots) \)
Hadrons in $< 0 | T \{ A^\mu(x_1) A^\nu(x_2) A^\rho(x_3) A^\sigma(x_4) \} | 0 >$

Key object full rank-four hadronic vacuum polarization tensor

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 \, d^4x_2 \, d^4x_3 \, e^{i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \times \langle 0 | T \{ j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\rho(0) \} | 0 \rangle .$$

- non-perturbative physics
- general covariant decomposition involves 138 Lorentz structures of which
- 32 can contribute to $g - 2$
fortunately, dominated by the pseudoscalar exchanges $\pi^0, \eta, \eta', \ldots$ described by the effective Wess-Zumino Lagrangian

generally, pQCD useful to evaluate the short distance (S.D.) tail

dominated long distance (L.D.) part must be evaluated using some low energy effective model which includes the pseudoscalar Goldstone bosons as well as the vector mesons play key role

Hadronic light–by–light scattering is dominated by $\pi^0$ exchange in the odd parity channel, pion loops etc. at long distances (L.D.) and quark loops incl. hard gluonic corrections at short distances (S.D.)
The spectrum of invariant $\gamma\gamma$ masses obtained with the Crystal Ball detector. The three rather pronounced spikes seen are the $\gamma\gamma \rightarrow$ pseudoscalar (PS) $\rightarrow \gamma\gamma$ excitations: PS=$\pi^0$, $\eta$, $\eta'$.
Pion-pole contribution dominating hadronic contributions = neutral pion exchange

Leading hadronic light–by–light scattering diagrams

The key object here is the $\pi^0\gamma\gamma$ form factor $F_{\pi^0\gamma^*\gamma^*}(m_{\pi}^2, q_1^2, q_2^2)$ which is defined by the matrix element

$$i \int d^4x e^{iq\cdot x} \langle 0|T\{j_\mu(x)j_\nu(0)\}|\pi^0(p)\rangle = \varepsilon_{\mu\nu\alpha\beta} q^\alpha p^\beta F_{\pi^0\gamma^*\gamma^*}(m_{\pi}^2, q^2, (p-q)^2).$$

Properties:

- bose symmetric $F_{\pi^0\gamma^*\gamma^*}(s, q_1^2, q_2^2) = F_{\pi^0\gamma^*\gamma^*}(s, q_2^2, q_1^2)$
- need it off-shell in integrals; if $(s, q_1^2, q_2^2) \neq (m_{\pi}^2, 0, q^2)$ in fact not known
- pion pole approximation in mind (pion pole dominance) $s = m_{\pi}^2$
- for generality approximation use vertex function (not matrix element)

$$i \int d^4x e^{iq\cdot x} \langle 0|T\{j_\mu(x)j_\nu(0)\tilde{\varphi}_{\pi^0}(p)\}|0\rangle = \varepsilon_{\mu\nu\alpha\beta} q^\alpha p^\beta F_{\pi^0\gamma^*\gamma^*}(p^2, q^2, (p-q)^2) \times \frac{i}{p^2 - m_{\pi}^2}.$$
with \( \tilde{\varphi}(p) = \int d^4 y \, e^{ipx} \varphi(y) \) the Fourier transformed \( \pi^0 \)-field. \( \Rightarrow \) representation in terms of form factors

To compute \( a^{l-b; \pi^0}_\mu \equiv F_M(0) \)\text{\_pole}, we need \( i \frac{\partial}{\partial k^\nu} \Pi^{(\pi^0)}_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) \) at \( k = 0 \) where \( p_3 = -(p_1 + p_2) \). Computing the Dirac traces yields

\[
a^{l-b; \pi^0}_\mu = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m^2][(p - q_2)^2 - m^2]} \times \left[ \frac{\mathcal{F}_{\pi^0*\gamma*\gamma^*}(q_2^2, q_1^2, q_3^2)}{q_2^2 - m_{\pi}^2} \mathcal{F}_{\pi^0*\gamma^*\gamma}(q_2^2, q_2^2, 0) T_1(q_1, q_2; p) \right. \\
+ \left. \frac{\mathcal{F}_{\pi^0*\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2)}{q_3^2 - m_{\pi}^2} \mathcal{F}_{\pi^0*\gamma^*\gamma}(q_3^2, q_3^2, 0) T_2(q_1, q_2; p) \right],
\]

where \( T_1(q_1, q_2; p) \) and \( T_2(q_1, q_2; p) \) are scalar kinematics factors; two terms unified by bose symmetry.
The pion-photon-photon transition form factor

- Form factor function $\mathcal{F}_{\pi^0*\gamma^*\gamma^*}(s, s_1, s_2)$ is largely unknown
- Fortunately some experimental data is available:
  - The constant $\mathcal{F}_{\pi^0\gamma\gamma}(m_{\pi}^2, 0, 0)$ is well determined by the $\pi^0 \rightarrow \gamma\gamma$ decay rate

The invariant matrix element reads (follows from Wess-Zumino Lagrangian) ($f_\pi \sim F_0$)

$$M_{\pi^0\gamma\gamma} = e^2 \mathcal{F}_{\pi^0\gamma\gamma}(0, 0, 0) = \frac{e^2 N_c}{12\pi^2 f_\pi} = \frac{\alpha}{\pi f_\pi} \approx 0.025 \text{ GeV}^{-1},$$

Information on $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_{\pi}^2, -Q^2, 0)$ comes from experiments $e^+e^- \rightarrow e^+e^-\pi^0$ where the electron (positron) gets tagged to high $Q^2$ values.

Measurement of the $\pi^0$ form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_{\pi}^2, -Q^2, 0)$ at high space–like $Q^2$
Data for $F_{\pi^0 \gamma^* \gamma}(m^2_\pi, -Q^2, 0)$ is available from CELLO and CLEO. Brodsky–Lepage interpolating formula

$$F_{\pi^0 \gamma^* \gamma}(m^2_\pi, -Q^2, 0) \approx \frac{1}{4\pi^2 f_\pi} \frac{1}{1 + (Q^2/8\pi^2 f^2_\pi)}$$

gives an acceptable fit to the data.

Assuming the pole approximation this FF has been used by all authors (HKS,BPP,KN) in the past, but has been criticized recently (MV).

In fact in $g - 2$ we are at zero momentum such that only the FF

$$F_{\pi^0 \gamma^* \gamma}(-Q^2, -Q^2, 0) \neq F_{\pi^0 \gamma^* \gamma}(m^2_\pi, -Q^2, 0)$$

is consistent with kinematics. Unfortunately, this off–shell form factor is not known and in fact not measurable.

An alternative way to look at the problem is to use the anomalous PCAC relation and to relate $\pi^0 \gamma^* \gamma$ to directly the ABJ anomaly.

High energy behavior of $F_{\pi^0 \gamma^* \gamma^*}$ is best investigated by OPE of

$$\langle 0 \mid T\{j_\mu(x_1)j_\nu(x_2)j_\lambda(x_3)\} \mid \gamma(k) \rangle$$

taking into account that the external photon is in a physical state. Looking at first diagram:

- $q_1$ and $q_2$ as independent loop integration momenta
- most important region $q_1^2 \sim q_2^2 \gg q_3^2$, $\Leftrightarrow$ short distance expansion of $T\{j_\mu(x_1)j_\nu(x_2)\}$ for $x_1 \rightarrow x_2$. 
Usual structure of OPE

“perturbative hard short distance coefficient function” \( \times \) “soft long distance matrix element”

Surprisingly, the leading possible term in \( \mathcal{L} \) is just given by the ABJ anomaly diagram known exact to all orders and given by the lowest order (one–loop) result. This requires of course that the leading operator in the short distance expansion must involve the divergence axial current. Indeed,

\[
i \int d^4 x_1 \int d^4 x_2 \, e^{i (q_1 x_1 + q_2 x_2)} \, T \{ j_\mu(x_1) j_\nu(x_2) X \} = \\
\int d^4 z \, e^{i (q_1 + q_2) z} \, \frac{2i}{q^2} \, \varepsilon_{\mu\nu\alpha\beta} \hat{q}^\alpha T \{ j_5^\beta(z) X \} + \cdots
\]

with \( j_5^\mu = \hat{q} \hat{Q}^2 \gamma^\mu \gamma_5 q \) the relevant axial current and \( \hat{q} = (q_1 - q_2)/2 \approx q_1 \approx -q_2 \); the momentum flowing through the axial vertex is \( q_1 + q_2 \) and in the limit \( k \to 0 \) of our interest \( q_1 + q_2 \to -q_3 \) which is assumed to be much smaller than \( \hat{q} (q_1^2 - q_2^2 \sim -2q_3 \hat{q} \sim 0) \).
After the large \( q_1, q_2 \) behavior has been factored out the remaining soft matrix element to be calculated is

\[
T_{\lambda\beta} = i \int d^4z \ e^{i q_3 z} < 0|T\{j_{5\beta}(z)j_{5\lambda}(0)\}|\gamma(k)>,
\]

which is precisely the well known VVA triangle correlator

\[
T^{(a)}_{\lambda\beta} = - \frac{i e N_c}{4\pi^2} \times \\
\omega_L(q_3^2) q_3 \bar{q}_3 \bar{f}_\sigma \lambda + \omega_T(q_3^2) \left(-q_3^2 \bar{f}_\lambda \beta + q_3 q_3 \bar{f}_\sigma \beta - q_3 q_3 \bar{f}_\sigma \lambda\right)
\]

Both amplitudes the longitudinal \( \omega_L \) as well as the transversal \( \omega_T \) are calculable from the triangle fermion one–loop diagram. In the chiral limit they are given by

\[
\omega_L^{(a)}(q^2) = 2\omega_T^{(a)}(q^2) = -2/q^2.
\]

as discovered by (Vainshtein 03). So everything is perturbative with no radiative corrections at all!

At this stage of the consideration it looks like a real mystery what all this has to do with \( \pi^0 \) exchange, as everything looks perfectly controlled by perturbation theory. The clue is that as a low energy object we may evaluate this matrix element at the same time perfectly in terms of hadronic spectral functions by saturating it by a sum over intermediate states.

Melnikov, Vainshtein: vertex with external photon must be non-dressed! i.e. no VDM damping \( \Rightarrow \) result increases by \( \boxed{30\%} \)!
Basic problem: \((s, s_1, s_2)\)–domain of \(\mathcal{F}_{\pi^0, \gamma^*, \gamma^*}(s, s_1, s_2)\); here \((0, s_1, s_2)\)–plane

Compare VP: \(s\)-axis

\(\chi_{\text{PT/ENJL}}\)

\(\chi_{\text{pQCD}}\)

\((g - 2)_\mu\)
Originally VMD model (Kinoshita et al 85):

- \( \rho \) mesons play an important role in the game (see hadronic VP) \( \Rightarrow \) looks natural to apply a vector meson dominance (VDM) model

- Naive VDM

\[
\frac{i}{q^2} \to \frac{i}{q^2} \frac{m^2_\rho}{q^2 - m^2_\rho} = \frac{i}{q^2} - \frac{i}{q^2 - m^2_\rho}.
\]

- provides a damping at high energies, \( \rho \) mass as an effective cut-off (physical version of a Pauli-Villars cut-off) \( \Rightarrow \) photons \( \to \) dressed photons!

- naive VDM violates electromagnetic WT-identity

Correct implementation in accord with low energy symmetries of QCD:

- vector meson extended CHPT (E\( \chi \)PT) model (Ecker et al 89)
- hidden local symmetry (HLS) model (Bando et al 85)
- extended NJL (ENJL) model (Dhar et al 85, Ebert, Reinhardt 86, Bijnens 96)

to large extend equivalent

Problem: matching L.D. with S.D. \( \Rightarrow \) results depend on matching cut off \( \Lambda \) \( \Rightarrow \) model dependence

(non-renormalizable low energy effective theory vs. renormalizable QCD)

Novel approach: refer to quark–hadron duality of large-\( N_c \) QCD, hadrons spectrum known, infinite series of narrow spin 1 resonances (‘t Hooft 79) \( \Rightarrow \) no matching problem (resonance representation has to match quark
level representation) (De Rafael 94, Knecht, Nyffeler 02)

⇒ lowest meson dominance (LMD) plus one vector state (V) approximation to large-$N_c$ QCD allows for correct matching

⇒ “LMD+V” parametrization of $\pi^0\gamma\gamma$ form–factor (see below)

### Summary of most recent results

<table>
<thead>
<tr>
<th>$10^{11} a_\mu$</th>
<th>BPP</th>
<th>HKS</th>
<th>KN</th>
<th>MV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$</td>
<td>85 ± 13</td>
<td>82.7 ± 6.4</td>
<td>83 ± 12</td>
<td>114 ± 10</td>
</tr>
<tr>
<td>$\pi, K$ loops</td>
<td>−19 ± 13</td>
<td>−4.5 ± 8.1</td>
<td>0 ± 10</td>
<td></td>
</tr>
<tr>
<td>axial vector</td>
<td>2.5 ± 1.0</td>
<td>1.7 ± 0.0</td>
<td>22 ± 5</td>
<td></td>
</tr>
<tr>
<td>scalar</td>
<td>−6.8 ± 2.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>quark loops</td>
<td>21 ± 3</td>
<td>9.7 ± 11.1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>total</td>
<td>83 ± 32</td>
<td>89.6 ± 15.4</td>
<td>80 ± 40</td>
<td>136 ± 25</td>
</tr>
</tbody>
</table>

Note: MV and KN utilize the same model LMD+V form factor:

$$F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{4\pi^2F_\pi^2}{N_c} \frac{q_1^2q_2^2(q_1^2 + q_2^2) - h_2q_1^2q_2^2 + h_5(q_1^2 + q_2^2) + (N_cM_1^4M_2^4/4\pi^2F_\pi^2)}{(q_1^2 + M_1^2)(q_1^2 + M_2^2)(q_2^2 + M_1^2)(q_2^2 + M_2^2)},$$

where $M_1 = 769$ MeV, $M_2 = 1465$ MeV, $h_5 = 6.93$ GeV$^4$. 
with two modifications:

- Form factor $F_{\pi^0\gamma^*\gamma}(q_2^2, q_2^2, 0) = 1$: undressed soft photon (non-renormalization of ABJ) Note: to have anomaly correct doesn not imply that there is no damping! $PVV$ anomaly quark loop is counter example; it has correct $\pi\gamma\gamma$ in chiral limit (anomaly) and goes like $1/q_2^2$ up to logs in all directions

- $F_{\pi^0\gamma^*\gamma^*}(q_2^2, q_1^2, q_3^2) \simeq F_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_3^2) = KN$
  with $h_2 = 0 \pm 20 \text{ GeV}^2$ (KN) vs. $h_2 = -10 \text{ GeV}^2$ (MV) fixed by twist 4 in OPE ($1/q^4$)

- $a_1[f_1, f_1^*]$ different mixing scheme

Criticism: KN Ansatz only covers $(0, q_1^2, q_2^2)$–plane, with consistent kinematics depends on 3 variables $\rightarrow$ 2–dim integral representation no longer valid.

Is this the final answer? How to improve? A limitation to more precise $g - 2$ tests?

Looking for new ideas to get ride of model dependence

In principle lattice QCD could provide an answer [far future (“yellow” region only)]

Ask STRING THEORY? (see below)
Summary of contributions:

\( a_\mu \) in units \( 10^{-6} \), ordered according to their size (L.O. lowest order, H.O. higher order, LBL. light–by–light); \( \alpha^{-1} = 137.0359991100 \)

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>L.O. universal</td>
<td>1161.40973</td>
<td>(0)</td>
</tr>
<tr>
<td>e–loops</td>
<td>6.19457</td>
<td>(0)</td>
</tr>
<tr>
<td>H.O. universal</td>
<td>-1.75755</td>
<td>(0)</td>
</tr>
<tr>
<td>L.O. hadronic</td>
<td>0.06920</td>
<td>(60)</td>
</tr>
<tr>
<td>L.O. weak</td>
<td>0.00195</td>
<td>(0)</td>
</tr>
<tr>
<td>H.O. hadronic</td>
<td>-0.00098</td>
<td>(1)</td>
</tr>
<tr>
<td>LBL. hadronic</td>
<td>0.00110</td>
<td>(40)</td>
</tr>
<tr>
<td>( \tau )–loops</td>
<td>0.00043</td>
<td>(0)</td>
</tr>
<tr>
<td>H.O. weak</td>
<td>-0.00041</td>
<td>(2)</td>
</tr>
<tr>
<td>( e+\tau )–loops</td>
<td>0.00001</td>
<td>(0)</td>
</tr>
<tr>
<td>&quot;New Physics&quot; ???</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a_\mu^{\text{exp}} - a_\mu^{\text{the}} = 277 \pm 94 \cdot 10^{-11} )</td>
<td>[2.9 ( \sigma )]</td>
<td></td>
</tr>
<tr>
<td>theory</td>
<td>1165.91803</td>
<td>(72)</td>
</tr>
<tr>
<td>experiment</td>
<td>1165.92080</td>
<td>(60)</td>
</tr>
</tbody>
</table>
$a_\mu$ Summary: experiment vs. theory

Given theory results only differ by $a_\mu^{\text{had}(1)}$!
All kind of physics meets!
Almost all EURODAFNE & EURIDICE nodes involved in one or the other problem !!!
BNL $g - 2$ experiment 14-fold improvement (vs CERN 1979) to 0.54 ppm

$$a_\mu = 116592080(54)(33) \times 10^{-11}$$

SM Test:
- substantial improvement of CPT test
- confirms for the first time weak contribution: $2\sigma - 3\sigma$
- very sensitive to precise value of the hadronic contribution: limits theoretical precision $2\sigma \rightarrow 1\sigma$, now $\sim \delta_{LBL}$
- Light-by-Light (model-dependent) not far from being as important as the weak contribution, may become the limiting factor for future progress (theory?)

New Physics ??? 2.9 [3.4] $\sigma$

$$|a_\mu^{\text{exp}} - a_\mu^{\text{SM}}| = (28 \pm 9)[(27 \pm 8)] \times 10^{-10}$$

Small discrepancy persists, however, established deviation from the SM, in contrary: very strong constraint on many NP scenarios
• On “prediction” side:
  – big experimental challenge: attempt cross-section measurements at 1% level up to $J/\psi[\Upsilon]$ !!! crucial for $g - 2$ and $\alpha_{\text{QED}}(M_Z)$ at GigaZ
  – new ideas on how to get model–independent hadronic LbL contribution
  – theory of pion FF below 1 GeV allows for further improvement (Colangelo, Gasser, Leutwyler); constraints from $\pi\pi$ scattering phase shifts

• Plans for new $g - 2$ experiment!
  Present: BNL E821 0.5 ppm
  Future: BNL E969 0.2 ppm
  J-PARC 0.1 ppm

Hunting for precision goes on!!!
History of sensitivity to various contributions

- 4th
- QED 6th
- 8th
- hadronic VP
- hadronic LBL
- weak
- New Physics
- SM precision

(a - 2)_μ

BNL CERN III CERN II CERN I

log scale

Uncertainty [ppm]

10^{-1} 1 10 10^2 10^3 10^4

F. Jegerlehner Final EURIDICE Meeting, Kazimierz, Poland – August 24-27, 2006 –