$\mathcal{L}_{\mathrm{Yukawa}}$ in the electroweak SM

Fermion masses and mixing structure

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Outline of Talk:

① **Basic ingredients of the Standard Model: the SM fermions**

② **WEYL, DIRAC and MAJORANA spinors**

③ **The Yukawa sector and quark flavor mixing**

④ **The Yukawa sector and neutrino flavor mixing**

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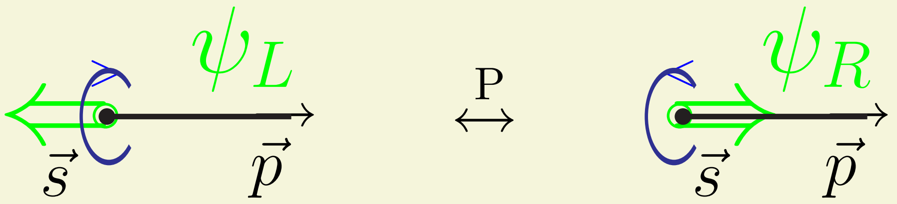
① Basic ingredients of the Standard Model

Fundamental constituents of matter: QUARKS & LEPTONS

Dirac particles (spin 1/2, Pauli principle), "in first place" massless

left or right handed: (helicity, chirality)



Forces are exchange forces of a particular type: gauge forces

which derive from an equivalence principle

Weyl 1918,1929, Yang & Mills 1954

Constituents of matter only communicate via spin 1 gauge bosons

The SM fermions

SM gauge group

$$G_{loc} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$$

The weak quantum numbers and multiplets are summarized in the following Tables.

	Doublets				Singlets			
	$(\nu_\ell)_L$	$(\ell^-)_L$	$(u,c,t)_L$	$(\tilde{d},\tilde{s},\tilde{b})_L$	$(\nu_\ell)_R$	$(\ell^-)_R$	$(u,c,t)_R$	$(d,s,b)_R$
Q	0	-1	$2/3$	$-1/3$	0	-1	$2/3$	$-1/3$
T_3	$1/2$	$-1/2$	$1/2$	$-1/2$	0	0	0	0
Y	-1	-1	$1/3$	$1/3$	0	-2	$4/3$	$-2/3$

$\Uparrow$

sterile

neutrinos

GROUP	MULTIPLET	REPRESENTATION	
$SU(3)_c$:	LEPTONS	1	color singlet
	QUARKS	3	color triplet
	ANTIQUARKS	3^*	anticolor triplet
$SU(2)_L$:	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \begin{pmatrix} u \\ \tilde{d} \end{pmatrix}_L$		
	$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \begin{pmatrix} c \\ \tilde{s} \end{pmatrix}_L$	$2=2^*$	weak iso-spin doublets
	$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L , \begin{pmatrix} t \\ \tilde{b} \end{pmatrix}_L$		
	$\nu_{eR}, e^-_R, u_R, d_R$		
	$\nu_{\mu R}, \mu^-_R, c_R, s_R$	1	weak iso-spin singlets
	$\nu_{\tau R}, \tau^-_R, t_R, b_R$		
$U(1)_Y$:	$Y = 2(Q - T_3)$	abelian	weak hypercharge

By $\tilde{q}$ we denoted the Cabibbo-Kobayashi-Maskawa (CKM) rotated quarks $\tilde{q}_d = U_{CKM} q_d$ where $q_d = (d, s, b)$ is a horizontal vector (in family space) of quark mass eigen-fields.

② WEYL, DIRAC and MAJORANA spinors

1) Lorentz transformations and spinors

Classical special relativity:

Lorentz group $L_+^\uparrow$ rotations and velocity transformations (boosts)

Relativistic quantum mechanics:

$SL(2,C)$, the group of complex unimodular 2 $\times$ 2 matrices

Relationship: via four-vector of hermitean matrices $\sigma_\mu = (\mathbf{1}, \vec{\sigma}\,)$

(σ_i, $i = 1,2,3$ the Pauli matrices)

To any real four-vector x^μ we can associate a Hermitean 2×2 matrix given by

$$x_\mu \to X = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \quad .$$

Conversely, every Hermitean 2×2 matrix X determines a real four-vector

$$X \to x^\mu = \frac{1}{2}\text{Tr}\,(X\sigma^\mu) \quad .$$

Thus we have a linear one-to-one correspondence between real four-vectors and Hermitean 2×2 matrices. Furthermore we note that

$$\det X = x^2 = x^\mu x_\mu \quad .$$

This establishes the following

Theorem 1: The mapping

$$x^\mu = \frac{1}{2}\text{Tr}\,(X\sigma^\mu) \;\in \mathbf{M_4} \leftrightarrow X = x^\mu \sigma_\mu \;\in \mathbf{H(2)}$$

establishes a linear isomorphism (one-to-one mapping) between the Minkowski space M_4 and the space of two-dimensional Hermitean matrices $\mathbf{H(2)}$.

Definition 1: The complex 2×2 matrices U with determinant $\det U = 1$ with matrix multiplication as a composition law form a group: the special linear group or unimodular linear group in two complex dimensions $SL(2,C)$.

An element $U \in SL(2,C)$ provides a mapping

$$X \to X' = UXU^+ \quad \text{i.e.} \quad x'^{\mu}\sigma_\mu = x^\nu U \sigma_\nu U^+$$

between Hermitean matrices, which preserves the determinant

$$\det X' = \det U \det X \det U^+ = \det X \quad .$$

The transformation

$$X \to X' = UXU^+ \;\; \text{on } \mathbf{H(2)}$$

corresponds to the real linear transformation

$$x^\mu \to x'^{\mu} = \Lambda^\mu{}_\nu \, x^\nu \;\; \text{on } \mathbf{M_4}$$

which satisfies $x'^{\mu}x'_\mu = x^\mu x_\mu$ and therefore is a Lorentz transformation.

The Lie algebra of $SL(2,C)$, the group of complex unimodular 2×2 matrices, is identical to the one of Lorentz group $L_+^\uparrow$. The 6 generators are given by the generators of the rotations $\vec{J}$ plus the generators of the Lorentz boosts $\vec{K}$. The Lie algebra given by

$J_i = \sigma_i/2 \;\; K_i = \sigma_i/2$ **with** $\sigma_i \;\; (i=1,2,3)$ **the Pauli matrices (basis of hermitian traceless 2** $\times$ **2 matrices)**

$$[J_i, J_k] = i\epsilon_{ikl} J_l \;,\;\; [J_i, K_k] = i\epsilon_{ikl} K_l \;,\;\; [K_i, K_k] = -i\epsilon_{ikl} J_l \quad .$$

is a coupled algebra of the J_i**'s and** K_i**'s. Since these generators are Hermitean** $\vec{J} = \vec{J}^+$ **and** $\vec{K} = \vec{K}^+$ **the group elements** $e^{-i\vec{\omega}\vec{J}}$ **and** $e^{i\vec{\beta}\vec{K}}$ **are unitary. This algebra can be decoupled by the linear transformation**

$$\vec{A} = \frac{1}{2}\left(\vec{J} + i\vec{K}\right) \;,\;\; \vec{B} = \frac{1}{2}\left(\vec{J} - i\vec{K}\right)$$

under which the Lie algebra takes the form

$$\vec{A} \times \vec{A} = i\vec{A} \;,\;\; \vec{B} \times \vec{B} = i\vec{B} \;,\;\; [A_i, B_j] = 0$$

of two decoupled angular momentum algebras. Since $\vec{A}^+ = \vec{B}$ **and** $\vec{B}^+ = \vec{A}$**, the new generators are not Hermitean any more and hence give raise to non–unitary irreducible representations. These are finite dimensional and evidently characterized by a pair** (A,B)**, with** $2A$ **and** $2B$ **integers. The dimension of the representation** (A,B) **is** $(2A+1)\cdot(2B+1)$**. The rotation group** $SO(3) = \mathrm{diag}\{SU(2) \otimes SU(2)\} \sim SU(2)$ $\Rightarrow$ **generators** $(I_{1i} + I_{2i})/2 \Rightarrow$ **Angular momentum of representation** (A,B)**:**

$J = A+B, A+B-1, \cdots |A-B|$. Note that this classification exhausts all finite dimensional irreducible representations which are necessarily non-unitary. The unitary representations, under which physical states transform, on the other hand are necessarily infinite dimensional.

Summary: in vicinity of the unit matrix (e.g. for the Lie algebras) we have the equivalence

$$\boxed{L_+^\uparrow \sim SO(1,3) \sim SL(2,C) \sim SU(2) \otimes SU(2)}$$

The simplest irreducible representations are of type $(j,0)$, $(0,j)$ and (j,j) :

$$(j,0) \quad : \qquad \left.\begin{array}{l} A \to J^{(j)} \\ B \to 0 \end{array}\right\} \Rightarrow \quad \begin{array}{ll} J \to & J^{(j)} \\ K \to & -iJ^{(j)} \end{array}$$

$$\Rightarrow \ \text{Representation :} \quad D^{(j)}(\Lambda) = e^{(\vec{\beta}-i\vec{\omega})\,\vec{J}^{\,(j)}}$$

$$(0,j) \quad : \qquad \left.\begin{array}{l} A \to 0 \\ B \to J^{(j)} \end{array}\right\} \Rightarrow \quad \begin{array}{ll} J \to & J^{(j)} \\ K \to & iJ^{(j)} \end{array}$$

$$\Rightarrow \ \text{Representation :} \quad \bar{D}^{(j)}(\Lambda) = e^{(-\vec{\beta}-i\vec{\omega})\,\vec{J}^{\,(j)}}$$

Note that $\bar{D}^{(j)}(\Lambda)$ **is the adjoint of** $D^{(j)}(\Lambda)$ $:\bar{D}^{(j)}(\Lambda) = D^{(j)+}(\Lambda^{-1})$.

We are now ready to classify the fields in terms of the simplest finite dimensional (non-unitary) representations. We denote group element as follows: $A \in SL(2,C)$ **and**

$\Lambda \in \mathcal{P}_+^\uparrow$. **The simplest representations the are:**

1	$D(0,0)$	**scalar field**
A	$D(1/2,0)$	**Weyl field (neutrino field)**
$\bar{A}$	$D(0,1/2)$	**Weyl field (antineutrino field)**
$\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$	$D(1/2,0) \oplus D(0,1/2)$	**Dirac field**
$\begin{pmatrix} \mathrm{Re}A & \mathrm{Im}A \\ -\mathrm{Im}A & \mathrm{Re}A \end{pmatrix}$	$D(1/2,0) \oplus D(0,1/2)$	**Majorana field**
Λ	$D(1/2,1/2)$	**vector field (contravariant)**
$(\Lambda^{-1})^T$	$D(1/2,1/2)$	**vector field (covariant)**

Spinors

The advantage to work directly with $SL(2,C)$ becomes obvious once we deal with spinors: A transformation $U \in SL(2,C)$ may be understood as a mapping

$$U: \quad V \to V$$

of a two-dimensional complex vector space V onto itself. The "complex two-vectors"

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in V$$

are called spinors. The spinors

$$u(\uparrow) = \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \;\text{and}\; u(\downarrow) = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}$$

are eigenvectors to the eigenvalues $\pm 1/2$ of the 3rd component of the spin matrix $\sigma_3/2$:

$$\frac{\sigma_3}{2} u(\uparrow) = +\frac{1}{2} u(\uparrow) \;\text{and}\; \frac{\sigma_3}{2} u(\downarrow) = -\frac{1}{2} u(\downarrow) \quad .$$

Since there are two types of inequivalent representations we have to distinguish two types of spinors, called undotted and dotted spinors.

Definition 1a: A covariant undotted spinor is a spinor which transforms according to the representation $(\frac{1}{2},0)$, i.e., it is characterized by the transformation law

$$u_a \in V \to u'_a = {U_\Lambda}_a^{\ b}\, u_b \in V \quad ; \quad a,b = 1,2 \quad .$$

Definition 1b: A contravariant dotted spinor is a spinor which transforms according to the representation $(0,\frac{1}{2})$, i.e., it is characterized by the transformation law

$$u^{\dot a} \in V \to u'^{\dot a} = \bar{U}_{\Lambda\ \dot b}^{\ \dot a}\, u^{\dot b} \in V \quad ; \quad \dot a, \dot b = 1,2 \quad .$$

The matrix

$$g^{ab} = (i\sigma_2)^{ab} = \varepsilon^{ab} = -\varepsilon_{ab} = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$$

plays the role of the metric and is numerically invariant. It is the only numerically invariant tensor for the group $SL(2,C)$, besides the Kronecker symbol δ_a^b. In the

dotted space we similarly have

$$g^{\dot{a}\dot{b}} = (i\sigma_2)^{\dot{a}\dot{b}} = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Parity and reflexion of spinors

The parity P, as we know, acts on four–vectors like $Px = (x^0, -\vec{x}\,)$ and satisfies $P^2 = 1$. With respect to the rotation group O_3 P is just a rotation by the angle 2π and thus in the context of the rotation group P has no special meaning. This is different for the Lorentz group. While

$$U_P \vec{J} = \vec{J} U_P$$

commutes

$$U_P \vec{K} = -\vec{K} U_P$$

does not commute. As a consequence, we learn that

$$U_P U(\vec{\chi}, \vec{n}\,\theta) = U(-\vec{\chi}, \vec{n}\,\theta) U_P$$

and hence

$$U_P U_\Lambda = \bar{U}_\Lambda U_P \quad . \tag{1}$$

We thus have the

Theorem 1: Under parity a undotted spinor is transformed into a dotted spinor and vice versa.

The position of the indices of U_P may be read off from the position of the indices of the transformation matrices U_Λ and $\bar{U}_\Lambda$. Since U_P commutes with $\vec{J}$, and in particular with the z–component of the spin J_3, each component transforms individually, which means that U_P acts diagonal. Furthermore, $P^2 = 1$ implies $U_P^2 = \pm 1$ or

$$\boxed{U_P = u_P = \pm 1 \quad \text{or} \quad \pm\, i} \quad .$$

This follows from the correspondences $x^\mu \to X$, $Px^\mu \to U_P X U_P$ and $P^2 x^\mu = x^\mu \to U_P^2 X U_P^2 = X$ and U_P diagonal, which implies that we may write it as a c–number phase u_P, simply.

Bispinors (Dirac spinor in helicity representation)

Parity can only be a symmetry if undotted and dotted spinors enter a theory in a symmetric fashion. This means that P–invariance requires undotted and dotted spinors to be grouped in pairs into a single object which is called bispinor.

Definition 2: A bispinor is a four–spinor

$$u_\alpha \doteq \begin{pmatrix} u_a \\ v^{\dot{b}} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ v^{\dot{1}} \\ v^{\dot{2}} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

which transforms according to the reducible representation $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$

$$u_\alpha \to u'_\alpha = \begin{pmatrix} U_\Lambda & 0 \\ 0 & \bar{U}_\Lambda \end{pmatrix}_{\alpha\beta} u_\beta = (S_\Lambda u)_\alpha$$

and under parity

$$u_\alpha = \begin{pmatrix} u_a \\ v^{\dot{b}} \end{pmatrix} \to u_P \begin{pmatrix} v^{\dot{a}} \\ u_b \end{pmatrix} = u_P \begin{pmatrix} v^{\dot{1}} \\ v^{\dot{2}} \\ u_1 \\ u_2 \end{pmatrix} = u_P \begin{pmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{pmatrix}$$

We may rewrite the parity operation acting on a bispinor as follows:

Definition 3: γ^0 denotes the 4×4 matrix

$$\gamma^0 \doteq \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad .$$

It acts on bispinors as

$$\gamma^0 \left(\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right) = \left(\begin{array}{c} u_3 \\ u_4 \\ u_1 \\ u_2 \end{array} \right) .$$

With the help of γ^0 a parity transformation may be written in the compact form

$$\boxed{u_\alpha \to u_P\, \gamma^0_{\alpha\beta} u_\beta} \; .$$

Properties of γ^0 are

$$(\gamma^0)^2 = \mathbf{1} \;\; , \quad \gamma^0 = \gamma^{0\,+} = \gamma^{0\,-1} \;\; .$$

Definition 4: γ_5 denotes the 4×4 matrix

$$\gamma_5 \doteq \left(\begin{array}{cc} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{array} \right) \; .$$

It acts on bispinors as

$$\gamma_5 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ -u_3 \\ -u_4 \end{pmatrix} .$$

Thus the undotted part of a bispinor is an eigenvector to the eigenvalue $+1$, the dotted part is eigenvector to the eigenvalue -1:

$$\gamma_5 \begin{pmatrix} u_a \\ 0 \end{pmatrix} = \begin{pmatrix} u_a \\ 0 \end{pmatrix} \quad ; \quad \gamma_5 \begin{pmatrix} 0 \\ v^{\dot{b}} \end{pmatrix} = - \begin{pmatrix} 0 \\ v^{\dot{b}} \end{pmatrix} .$$

Other properties of γ_5 are

$$(\gamma_5)^2 = \mathbf{1} \ , \quad \gamma_5 = \gamma_5^+ = \gamma_5^{-1} \ .$$

Definition 5: The 4×4 matrices

$$\Pi_\pm \doteq \frac{1}{2} (1 \pm \gamma_5)$$

are projection operators

$$\Pi_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad \Pi_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

for the upper and lower components of a bispinor, respectively; they are called chiral projectors. Thus

$$\Pi_+ \begin{pmatrix} u_a \\ v^{\dot{b}} \end{pmatrix} = \begin{pmatrix} u_a \\ 0 \end{pmatrix} \quad ; \quad \Pi_- \begin{pmatrix} u_a \\ v^{\dot{b}} \end{pmatrix} = \begin{pmatrix} 0 \\ v^{\dot{b}} \end{pmatrix}$$

are projections to the irreducible blocks of the bispinors.

Right–handed and left–handed bispinors

With the help of the chiral projectors acting on bispinors we may reformulate the notion of undotted and dotted spinors as right–handed and left–handed bispinors, respectively.

Definition 6: Given a bispinor u_α, we define

$$u_{R\alpha} \doteq \Pi_+ u_\alpha \sim u_a \text{ and } u_{L\alpha} \doteq \Pi_- u_\alpha \sim v^{\dot{b}} \quad .$$

We call u_R right-handed (bi)spinor and u_L left–handed (bi)spinor.

Later, when discussing spinor fields, we will see that the “handedness” is directly related to the helicity of a particle. Right–handed refers to positive helicity and left–handed refers to negative helicity.

We note that there is a perfect equivalence between right–handed (bi)spinors and undotted spinors and left–handed (bi)spinors and dotted spinors. In the physics literature commonly the bispinor language together with the handedness is used instead of the undotted and dotted two–spinor terminology.

Properties of the representation S_Λ

The reducible representation

$$S_\Lambda \equiv \left(\begin{array}{cc} U_\Lambda & 0 \\ 0 & \bar{U}_\Lambda \end{array} \right) = e^{i\left(\vec{\chi}\,\vec{\mathcal{K}}+\vec{\omega}\,\vec{\mathcal{J}}\,\right)}$$

has the generators

$$1) \quad \text{Rotations} \quad : \quad \vec{\mathcal{J}} \quad \equiv \quad \begin{pmatrix} \vec{J} & 0 \\ 0 & \vec{J} \end{pmatrix}$$

$$2) \quad \text{Boosts} \quad : \quad \vec{\mathcal{K}} \quad \equiv \quad i \begin{pmatrix} \vec{J} & 0 \\ 0 & -\vec{J} \end{pmatrix}$$

with $\vec{J} = \vec{\sigma}/2$. With the help of γ_5 we may write

$$\vec{\mathcal{K}} = i\,\gamma_5\,\vec{\mathcal{J}} \quad .$$

A crucial property of the representation S_Λ is the following:

Theorem 2: S_Λ^{+-1} is equivalent to S_Λ .

Indeed,

$$S_\Lambda^{+-1} = S_{\Lambda^{-1}}^{+} = \gamma^0 S_\Lambda \gamma^0 \quad ,$$

which is easily verified. Thus, there exist only one such representation . One consequence is that we may obtain an invariant Hermitean form from one single bispinor. It is convenient to first introduce an adjoint bispinor:

Definition 7: to a bispinor u_α we associate the adjoint bispinor

$$\bar{u}_\alpha \doteq \left(u^+\gamma^0\right)_\alpha = (u_3^*, u_4^*, u_1^*, u_2^*) \quad .$$

We then have the

Theorem 3: the bilinear form

$$\bar{u}_\alpha u_\alpha = u_\alpha^+ {\gamma^0}_{\alpha\beta} u_\beta \quad ,$$

where repeated spinor indices are summed over, is *Hermitean and Lorentz invariant*.

The proof is simple and follows from the properties of γ^0 and S_Λ which we have stated above.

Basic (simplest non-trivial) representations: **WEYL spinors**

$$(1/2,0) \equiv \varphi \sim \psi_R \qquad (0,1/2) \equiv \chi \sim \psi_L$$

On a classical level, Weyl spinors w are elements of a vector space V of two complex entries, which transform under $SL(2,C)$ by matrix multiplication: $w' = Uw$, $w \in V$, $U \in SL(2,C)$

$$w = \left(\begin{array}{c} a \\ b \end{array} \right) \quad ; \quad a, b \in C$$

L-transformations: ($\Lambda_R \equiv D^{(\frac{1}{2})}(\Lambda)$; $\Lambda_L \equiv \bar{D}^{(\frac{1}{2})}(\Lambda)$)

$$\psi_{a\ L,R}(x) \rightarrow \psi_{a\ L,R}(x') = (\Lambda_{L,R})_{ab}\, \psi'_{b\ L,R}(\Lambda_{\mu\nu} x^\nu)$$

with

$$(\Lambda_{R,L})_{ab} = e^{-i \frac{\vec{\sigma}}{2} (\vec{\omega} \pm i \vec{\beta})} \quad (\Lambda_R^+ = \Lambda_L^{-1})$$

Using

$$\sigma_2 \sigma_i \sigma_2 = -\sigma_i^*$$

one can show that

$$\sigma_2 \Lambda_L \sigma_2 = \Lambda_R^*$$

Define charge conjugate spinor (up to arbitrary phase)

$$\boxed{\psi_L^c \equiv \sigma_2 \psi_L^*}$$

L-transformation:

$$\boxed{\psi_L^c \sim \psi_R}$$

$$\Lambda_R \psi_L^c = \Lambda_R \sigma_2 \psi_L^* = \sigma_2 \Lambda_L^* \psi_L^* = \sigma_2 \psi_L^{*\prime} = \psi_L^{c\prime}$$

$\Longrightarrow$ **transformation properties**

$$\boxed{\psi_L^c \equiv \sigma_2 \psi_L^* \equiv \varphi \sim \psi_R \qquad \psi_R^c \equiv \sigma_2 \psi_R^* \equiv \chi \sim \psi_L}$$

We observe that the matrix

$$C = e^{i\pi J^{(\frac{1}{2})}} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -C^+ = -C^{-1}$$

provides the transformations

$$U^{-1} = C^{-1} U^T C \quad , \quad \bar{U} = U^{+-1} = C^{-1} U^* C$$

and

$$\hat{\sigma}_\mu = C^{-1} \sigma_\mu^T C \quad .$$

Consequently:

$\psi_L^T \psi_R \sim (\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) \sim (\frac{1}{2}, \frac{1}{2})$ **NOT LORENTZ INVARIANT**

Only

$$\boxed{(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0)=(0,0)+(1,0) \textbf{ and } (0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})=(0,0)+(0,1)}$$

involve singlets. One easily checks that

$$\boxed{\chi_L^T\sigma_2\psi_L \quad \textbf{and} \quad \chi_R^T\sigma_2\psi_R}$$

correspond to the two possible invariants.

Explicitly:

$$\psi_L^T\sigma_2\psi_L=(\psi_{L\,1},\psi_{L\,2})\left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right)\left(\begin{array}{c} \psi_{L\,1} \\ \psi_{L\,2} \end{array}\right)=-i\psi_{L\,1}\psi_{L\,2}+i\psi_{L\,2}\psi_{L\,1}$$

Note: would vanish for regular numbers, however, $\psi_{L\,i}$ are spinor components (field operators or anti–commuting numbers (Grassmann variables).

Using $\chi_L = \sigma_2 \psi_R^*$ we get

$$\chi^T \sigma_2 \psi_L = (\sigma_2 \psi_R^*)^T \sigma_2 \psi_L = \psi_R^+ \sigma_2^* \sigma_2 \psi_L = -\psi_R^+ \sigma_2 \sigma_2 \psi_L = -\psi_R^+ \psi_L$$

which we can make hermitian:

$$\psi_R^+ \psi_L + \mathrm{h.c.} = \psi_R^+ \psi_L + \psi_L^+ \psi_R$$

One can check also that

$$\psi_L^+ \sigma^\mu \psi_L \equiv (\psi_L^+ \psi_L, \quad \psi_L^+ \vec{\sigma} \psi_L) \sim \text{LH } \textbf{4-vector}$$
$$\psi_R^+ \hat{\sigma}^\mu \psi_R \equiv (\psi_R^+ \psi_R, -\psi_R^+ \vec{\sigma} \psi_R) \sim \text{RH } \textbf{4-vector}$$

The two–spinor fields satisfy the following linear coupled system of first order differential equations:

$$\boxed{\begin{array}{rcl} i(\hat{\sigma}^\mu)^{\dot{a}a} \partial_\mu \varphi_a(x) & = & m \chi^{\dot{a}}(x) \\ i(\sigma^\mu)_{a\dot{a}} \partial_\mu \chi^{\dot{a}}(x) & = & m \varphi_a(x) \end{array}}$$

As we shall see below this is just an unusual form of the Dirac equation. Only if $m = 0$ the two equetions are decoupled! For $m = 0$ the L–fields and the R–fields are each separetely invariant under L–transformations (for particles travelling at the speed of light it is not possible to flip the spin by a change of the inertial frame).

2) DIRAC spinors

The Dirac field: parity doubled spin $\frac{1}{2}$ field

The Dirac field is the bispinor field obtained by combining the irreducible fields $\varphi_a(x)$ and $\chi^{\dot{a}}(x)$ into one reducible field $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. It is the natural field to be used to describe fermions participating parity conserving interactions like QED and QCD.

Explicitly, the Dirac field is given by

$$\psi_\alpha(x) = \left(\begin{array}{c} \varphi_a \\ \chi^{\dot{a}} \end{array}\right)(x) = \sum_r \int d\mu(p) \left\{ u_\alpha(p,r)\, a(\vec{p},r)\, e^{-ipx} + v_\alpha(p,r)\, b^+(\vec{p},r)\, e^{ipx} \right\} \quad .$$

where

$$u_\alpha = \left(\begin{array}{c} u_a \\ \hat{u}^{\dot{a}} \end{array}\right) \quad ; \quad v_\alpha = \left(\begin{array}{c} v_a \\ \hat{v}^{\dot{a}} \end{array}\right) \quad .$$

$\psi_\alpha(x)$ satisfies the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)_{\alpha\beta}\,\psi_\beta(x) = 0$$

where

$$\gamma^\mu \doteq \begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix}$$

are the Dirac matrices in the helicity representation (Weyl basis). The Dirac equation is nothing but the Weyl equations written in terms of the bispinor ψ. The Dirac spinor combines a LH Weyl spinor with a LH Weyl spinor of the anti-particle (not with RH Weyl spinor of the particle itself).

Action of Dirac field:

$$\boxed{\mathcal{L}_{\rm Dirac} = \frac{1}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi = \mathcal{L}_L + \mathcal{L}_R}$$

(4 degrees of freedom)

Dirac mass term:

$$\boxed{m\bar{\psi}\psi = m\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right) = m\left(\varphi_R^+\varphi_L + \varphi_L^+\varphi_r\right) = m\left(\left(\varphi_L^c\right)^T\left(-i\sigma_2\right)\varphi_L\right) + \textbf{h.c.}}$$

3) **MAJORANA spinors**

This is a spin 1/2 field for which the antiparticle is identical to the particle ($b = a$). In order that the two Weyl equations remain consistent we must have a further relationship, a reality condition, between $\chi^{\dot{a}}$ and φ_a. A solution is given by

$$\chi^{\dot{a}} = (i\sigma_2)^{\dot{a}a}\varphi_a{}^* \quad .$$

This may be checked by using the complex conjugation matrix $C = i\sigma_2$ discussed earlier. Since we are more familiar with the four–spinor notation, we write the Majorana field like a Dirac field

$$\psi_\alpha(x) = \begin{pmatrix} \varphi_a \\ \chi^{\dot{a}} \end{pmatrix}(x) = \begin{pmatrix} \varphi_a \\ (i\sigma_2\varphi^*)^{\dot{a}} \end{pmatrix}(x) \quad .$$

Explicitly we have

$$\psi_\alpha(x) = \sum_r \int d\mu(p) \left\{ u_\alpha(p,r)\, a(\vec{p},r)\, e^{-ipx} + v_\alpha(p,r)\, a^+(\vec{p},r)\, e^{ipx} \right\} \quad .$$

Note: the Majorana field is formally identical to a Dirac field in the helicity representation and setting $b = a$. It satisfies the Dirac equation is self–conjugate, however. It

necessarily describes a neutral spin 1/2 particle, usually called Majorana–neutrino.

Majorana spinors $\psi_M^c = \psi_M$ can describe only particles which do not carry any absolutely (locally) conserved charge: From the known particles only neutrinos can be Majorana particles. In the SM only the sterile right-handed singlets $\nu_R \equiv \nu_s$ (s=sterile) cam be Majorana particles. The left-handed ν_L being members of weak iso-doublets ($I = 1/2$) and carying also hypercharge ($Y = -1$) cannot be Majorana particles.

$$\psi_m = \begin{pmatrix} \varphi \\ -i\sigma_2\varphi^* \end{pmatrix}$$

is a spinor describing only two independent degrees of freedom. Important: the reality condition

$$\boxed{\psi_M^c = \mathcal{C}\phi_M^* = \phi_M}$$

impies that, like for a real scalar field, for example, there is no free choice of a phase for a Majorana field.

Action of Majorana field:

$$\boxed{\mathcal{L}_{\mathrm{Majorana}} = \frac{1}{4}\bar{\psi}_M\gamma^\mu \overleftrightarrow{\partial}_\mu \psi_M}$$

(2 degrees of freedom)

Majorana mass term:

$$\boxed{m\bar{\psi}_M \psi_M = m \varphi_L^T \left(-i\sigma_2\right) \varphi_L + \text{h.c.}}$$

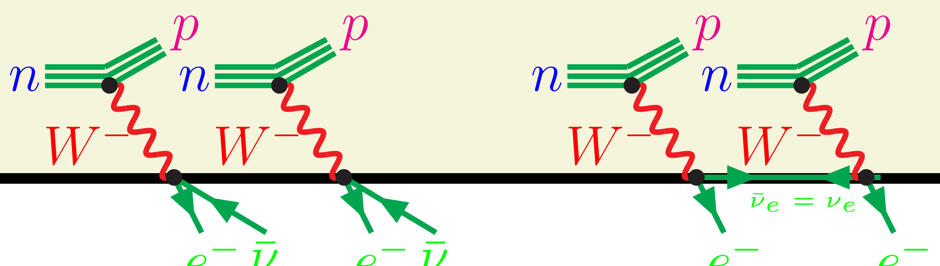
such terms are not invariant under (global) phase transformations

$$\boxed{\varphi_L \to e^{i\alpha} \varphi_L}$$

$$\Downarrow$$

Majorana mass terms violate Fermion number conservation. In particular ν_M's violate lepton number $\not{L}_\ell$

Dirac fermion lines are oriented (directed arrow lines) and conserve Fermion number, Majorana fermion lines are unoriented (double directed) and violate Fermion number conservation (by two units: $\Delta L\ell = 2$) $\Rightarrow$ neutrinoless double β–decay.



③ The Yukawa sector and quark flavor mixing

The fact that there exist three families of fermions which are made up of fields with identical $SU(2)_L \otimes U(1)_Y$ transformation properties allows us to form invariant Yukawa couplings for arbitrary combinations of fields from the different families. Such **flavor mixing** has been known to occur in the quark sector since a long time (Cabibbo 1962) while for leptons/neutrinos family mixing and thus lepton-flavor violation has been established only recently.

For the quarks we have four **horizontal vectors** in "family space" with identical quantum numbers with respect to the local gauge group. These are the left–handed and right–handed versions of the up and down family vectors $u_i = (u, c, t)$ and $d_i = (d, s, b)$ and we denote them by $u_{iL}, d_{iL}, u_{iR}, d_{iR}$. The general form of the Yukawa term reads

$$\mathcal{L}^q_{\text{Yukawa}} \quad = \quad -\sum_{i,j=1}^{3} \left[G^{ij}_{tq} \bar{L}_{qi} \Phi_t u_{jR} + G^{ij}_{bq} \bar{L}_{qi} \Phi_b d_{jR} + \text{h.c.} \right]$$

where Φ_b is the Higgs doublet field and $\Phi_t = i\tau_2 \Phi_b$ its Y–charge conjugate and with G^{ij}_{tq} and G^{ij}_{bq} arbitrary 3 x 3 matrices. If CP is not a good symmetry they are necessarily complex! The mass–matrix we obtain by inserting $\Phi_b = \frac{v}{\sqrt{2}} \binom{0}{1}$ and $\Phi_t = \frac{v}{\sqrt{2}} \binom{1}{0}$. Thus

$$\mathcal{L}^q_{\text{mass}} \quad = \quad -\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} \left[G^{ij}_{tq} \bar{u}_{iL} u_{jR} + G^{ij}_{bq} \bar{d}_{iL} d_{jR} + \text{h.c.} \right]$$

$$= \quad -\sum_{i,j=1}^{3} \left[m_{tq}^{ij} \bar{u}_{iL} u_{jR} + m_{bq}^{ij} \bar{d}_{iL} d_{jR} + \text{h.c.} \right]$$

with

$$m_{\cdot}^{ij} = \frac{v}{\sqrt{2}} G_{\cdot}^{ij} \quad .$$

The quark fields considered up to now are in the **weak interaction basis** in which the matter field Lagrangian is diagonal in the families. We now have to find the **physical fields** in the **mass eigenstate basis** in which the mass–matrix is diagonal such that each field has a fixed mass.

The mass eigenstates are the ones which propagate (propagators of physical fields to be utilized in Feynman rules) while in interactions the weak or flavor eigenstates are the ones destroyed or created.

To this end we have to perform global unitary transformation on the horizontal vectors. Unitary because they have to leave the kinetic terms of the quarks invariant and global because we have to diagonalize the constant coupling matrices $G_{\cdot\,q}$.

Thus, we have to perform the transformations

$$\begin{aligned} u_{jR} &\rightarrow \left(U_R^u\right)_{jk} u_{kR} \\ d_{jR} &\rightarrow \left(U_R^d\right)_{jk} d_{kR} \end{aligned}$$

$$\begin{aligned} u_{jL} &\rightarrow (U_L^u)_{jk}\, u_{kL} \\ d_{jL} &\rightarrow (U_L^d)_{jk}\, d_{kL} \quad . \end{aligned}$$

As a result of this basis change, the interaction of the gauge fields with the quarks, which used to be family diagonal, now no longer are necessarily so. Neutral current interactions, since they involve always $(U^u_{L,R})^+ U^u_{L,R} = 1$ or $(U^d_{L,R})^+ U^d_{L,R} = 1$, continue to be diagonal. However, for charged current interactions what enters after the basis change is the unitary matrix

$$V \equiv V_{CKM} = (U_L^u)^+ U_L^d \ ,$$

or its adjoint. This gives rise to family mixing in the charged current.

V_{ij} not directly observable:

physics invariant under redefinition of the phase of Dirac fields (phase is not observable)

$u_i \rightarrow e^{i\phi_i} u_i,\, d_j \rightarrow e^{i\phi_j} d_j \Rightarrow V_{ij} \rightarrow e^{i(\phi_i - \phi_j)} V_{ij}$

Observable must be invariants under phase transformations:

$V_{ij} V_{ij}^* = |V_{ij}|^2$

$V_{ij} V_{ik}^* V_{lk} V_{lj}^*$

...

etc. see below

❸ The CKM matrix

In the Standard Model: the basic quark charged-current interactions is

$$\mathcal{L}_{\mathrm{quark-W}} = \frac{g}{2\sqrt{2}}\bar{u}_i\gamma_\mu(1-\gamma_5)V_{ij}d_jW^\mu \;+\;\mathrm{h.c.}\,,$$

where u_i are the charged 2/3 quarks (u, c, t), d_j the charged $-1/3$ quarks (d, s, b) and g is the $SU(2)_L$ weak coupling constant ($G_F/\sqrt{2} = g^2/8M_W^2$, where G_F is the Fermi constant). V is the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix. V may be parametrized in terms of 3 rotation angles and one phase (a unitary $n \times n$–matrix has n^2 parameters, between the n up type quarks and the n down type quarks there are $2n-1$ relative phases which affect the mixing matrix but are unobservable (transforming all fields with the same phase of course does not change the mixing matrix). Hence, there remain $n^2-(2n-1)$ observable mixing parameters: For $n=2$: 1 rotation angle (the Cabibbo angle), for $n=3$: 3 rotation angles and 1 phase). A useful convention reads:

$$\begin{aligned} V &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} . \end{aligned}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ with i and j being the family labels. Without loss of generality one may assume all c_{ij} and s_{ij} to be positive and the phase δ_{13} to lie in the range $0 \leq \delta_{13} < 2\pi$.

A non-degenerate phase

$$\delta \equiv \delta_{13} \neq 0\;,\quad \pi$$

leads to complex effective couplings in the charge changing current which violate CP-invariance. It is important to notice that this specific kind of "standard model CP-violation" is only possible for more than two families. In fact, the mixing matrix has to satisfy a number of conditions in order that CP-violation occurs. The basic observation is that a unitary 3 by 3 matrix which has a zero matrix element somewhere is necessarily real. Therefore mixing must be non-degenerate:

$$\theta_{ij} \neq 0\;,\quad \frac{\pi}{2} \quad ij = 12, 23, 13\;\;.$$

For the same reason, **CP-violation can only occur this way if all the states are distinguishable**. This means that **all states with the same charge must have different masses**:

$$m_u < m_c < m_t \quad \text{and} \quad m_d < m_s < m_b\;\;.$$

which happens to be so in Nature.

There is no mixing effect in the neutral current. In the weak interaction basis the neutral current is diagonal in helicities, flavors and families and the unitary CKM–transformation

$$\sum_{i=1}^{3} \bar{d}_{iL} \cdots d_{iL} \;\rightarrow\; \sum_{i=1}^{3} \bar{\tilde{d}}_{iL} \cdots \tilde{d}_{iL} \equiv \sum_{i=1}^{3} \bar{d}_{iL} \cdots d_{iL}$$

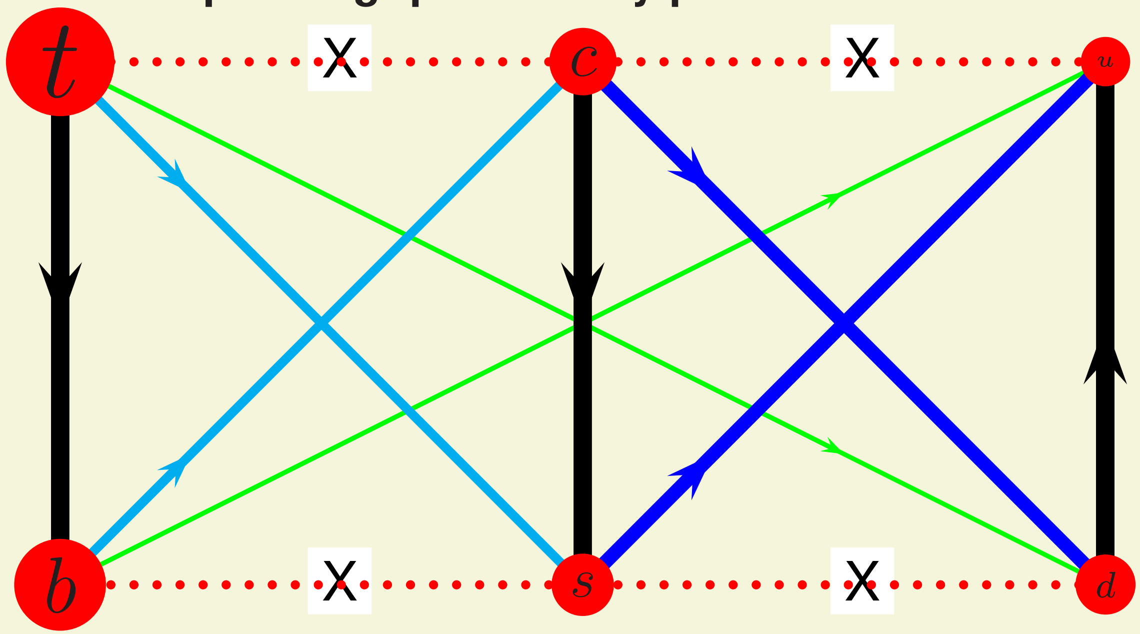
leaves its form invariant. This is called the Glashow-Iliopoulos-Maiani (GIM)–mechanism explaining the absence of flavor-changing neutral currents (FCNC).

The CKM mixing hierarchy

Empirically the CKM matrix elements may be expanded in $\lambda = \sin\theta_{\mathrm{Cabibbo}} \simeq 0.22$ with approximate size

$$|V| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

The corresponding quark decay pattern is illustrated in the following diagram:



Note: **The u quark is stable, the s and b quarks are metastable.** Flavor changing neutral currents (FCNC) at tree level · · X · · are forbidden by the GIM–mechanism. Flavor changing neutral current transitions are allowed, however, as second (or higher) order transitions: e.g. $b \longrightarrow s$ is in fact $b \longrightarrow (u^*, c^*, t^*) \longrightarrow s$, where the asterix indicates “virtual transition”.

The Wolfenstein parametrization

Set $s_{12} = \lambda$, $s_{23} = A\lambda^2$ and $s_{13} = \sigma A\lambda^3$ where $A, \sigma \sim O(1)$ is assumed. Usually one writes $\sigma e^{-i\delta} = \rho - i\eta$ i.e. $\sigma = \sqrt{\rho^2+\eta^2}$, $\rho = \sigma\cos\delta$ and $\eta = \sigma\sin\delta$. Expanding in λ up to and including terms of order λ^3 (λ^5) for the real (imaginary) part, V is given by

$$V = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3\,(\rho-i\eta) \\ -\lambda\,(1+A^2\lambda^4(\rho+i\eta)) & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3\left[1-(1-\lambda^2/2)\,(\rho+i\eta)\right] & -A\lambda^2\,(1+\lambda^2(\rho+i\eta)) & 1 \end{pmatrix} .$$

In this particular (quark) phase convention, the imaginary part of the matrix appears at order λ^3.

The unitarity of the CKM matrix implies

$$\sum_q V_{iq} V^*_{jq} = \delta_{ij} \, .$$

In particular, considering the condition

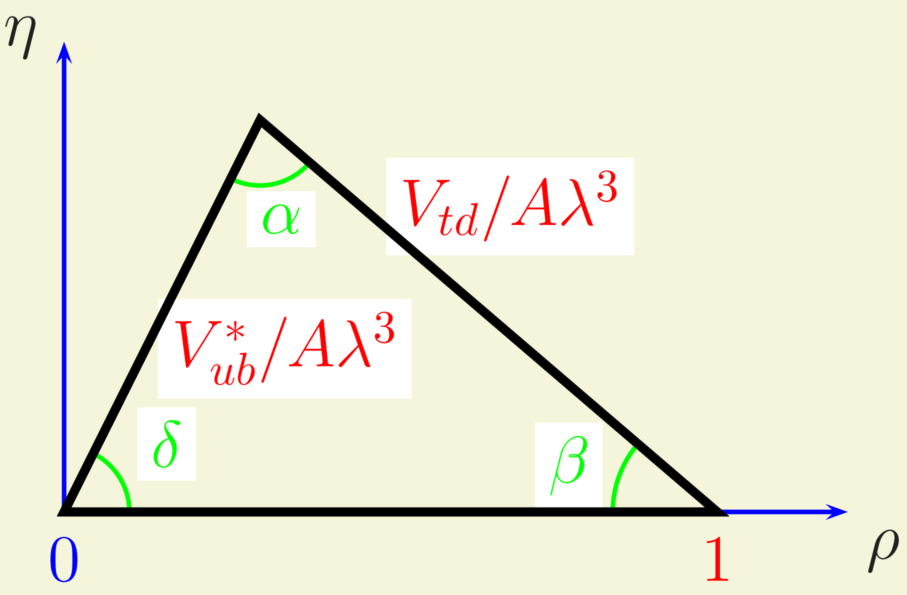
$$V^*_{ub} V_{ud} + V^*_{cb} V_{cd} + V^*_{tb} V_{td} = 0$$

in the approximation $V_{ud} \simeq V_{tb} \simeq 1$, one obtains

$$\frac{V^*_{ub}}{A\lambda^3} + \frac{V_{td}}{A\lambda^3} - 1 = 0 \, .$$

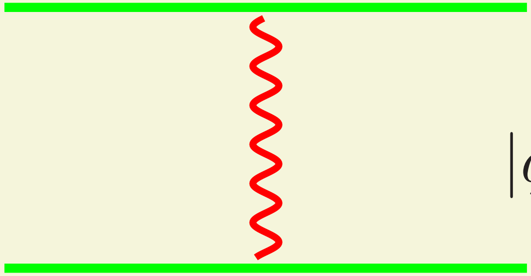
This relation identifies a triangle in the ρ–η plane. The angles of this

triangle, α, β and δ, are measures of CP-violation.

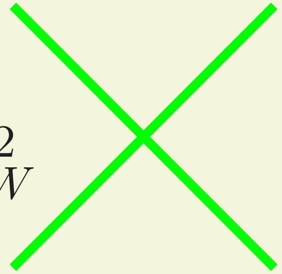


The unitarity triangle in the ρ–η plane.

Low energy effective charged current (CC) interaction:



$$|q^2| \overset{\equiv}{\ll} M_W^2$$



$$\left(\frac{g}{2\sqrt{2}}\right)^2 \frac{1}{M_W^2} \equiv \frac{G_F}{\sqrt{2}}$$

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{q_i=u,c,t} \lambda_{q_i} \left(\bar{d}\gamma_\mu(1-\gamma_5)\, q_i\right) \left(\bar{q}_i\gamma^\mu(1-\gamma_5)\, s\right)$$

$$\lambda_{q_i} \equiv V^*_{q_i s} V_{q_i d}$$

$$H_{\text{eff}} = H(s \to d) = \lambda_u H_u + \lambda_c H_c + \lambda_t H_t$$

Unitarity: $\lambda_u + \lambda_c + \lambda_t = 0$ i.e. λ_i's form a closed triangle in the complex plane.

Unitarity triangles: K versus B

K-system: **Effective Hamiltonian:**

$$H(s \to d) \simeq \lambda_u H_u + \lambda_c H_c + \lambda_t H_t$$

where

$$\begin{aligned} \lambda_u &= V_{us}^* V_{ud} = \lambda\ \left(1-\lambda^2/2\right) \\ \lambda_c &= V_{cs}^* V_{cd} = -\lambda\ \left(1-\lambda^2/2\right) \\ \lambda_t &= V_{ts}^* V_{td} = -A^2\lambda^5\ (1-\rho-i\eta) \end{aligned}$$

$$\begin{aligned} H_u &= \left(\bar{d}\gamma_\mu(1-\gamma_5)u\right)\ \left(\bar{u}\gamma^\mu(1-\gamma_5)s\right) \\ H_c &= \left(\bar{d}\gamma_\mu(1-\gamma_5)c\right)\ \left(\bar{c}\gamma^\mu(1-\gamma_5)s\right) \\ H_t &= \left(\bar{d}\gamma_\mu(1-\gamma_5)t\right)\ \left(\bar{t}\gamma^\mu(1-\gamma_5)s\right) \end{aligned}$$

$$\frac{H(s \to d)_{\mathrm{CPV}}}{H(s \to d)_{\mathrm{CPC}}} \simeq \frac{A^2\lambda^5\eta}{\lambda} \sim O(\lambda^4) \simeq 2.3 \times 10^{-3} \quad \text{miliweak}$$

B-System: Effective Hamiltonian:

$$H(b \to d) \simeq \lambda_u H_u + \lambda_c H_c + \lambda_t H_t$$

where

$$\begin{aligned} \lambda_u &= V_{ub}^* V_{ud} = \ A\lambda^3\ (\rho + i\eta) \\ \lambda_c &= V_{cb}^* V_{cd} = -A\lambda^3 \\ \lambda_t &= V_{tb}^* V_{td} = A\lambda^3\ (1 - \rho - i\eta) \end{aligned}$$

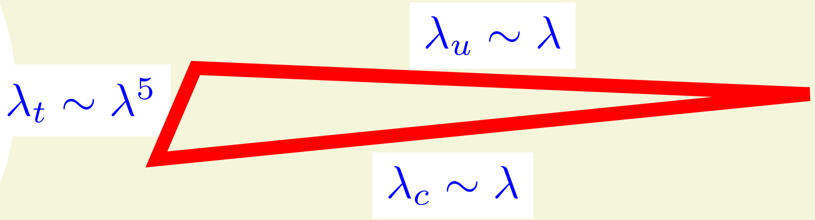
$$\begin{aligned} H_u &= \left(\bar{d}\gamma_\mu(1-\gamma_5)u\right)\ \left(\bar{u}\gamma^\mu(1-\gamma_5)b\right) \\ H_c &= \left(\bar{d}\gamma_\mu(1-\gamma_5)c\right)\ \left(\bar{c}\gamma^\mu(1-\gamma_5)b\right) \\ H_t &= \left(\bar{d}\gamma_\mu(1-\gamma_5)t\right)\ \left(\bar{t}\gamma^\mu(1-\gamma_5)b\right) \end{aligned}$$

$$\frac{H(b \to d)_{\mathrm{CPV}}}{H(b \to d)_{\mathrm{CPC}}} \simeq \frac{A\lambda^3\eta}{A\lambda^3} \sim O(1) \quad !$$

$\eta = O(1)$ **assumed !** **confirmed by BABAR and BELLE 01 !**

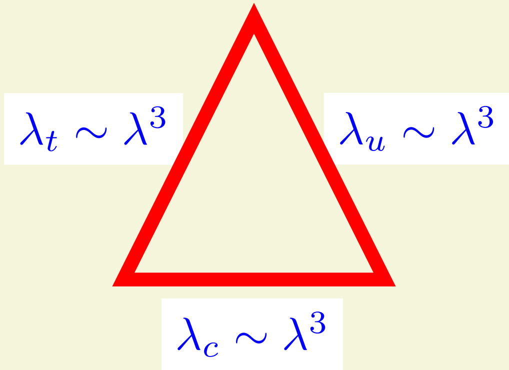
Unitarity Δ

for K–decays



Unitarity Δ

for B–decays



Properties of unitarity triangles

1) Phase transformations on quark fields induce a rotation of the triangles only.

2) Invariant measures of the triangles are angles, sides and area.

3) Knowledge of sides (i.e. moduli of the quark mixing matrix elements) suffices to fix the angles. Thus the magnitude and structure of CP–violating effects is predictable on the basis of CP–conserving observables !

4) Different types of transitions are characterized by different effective Hamiltonians:

$$H(s \to d), H(b \to d), H(b \to s),$$

$$H(c \to u), H(t \to u), H(t \to c).$$

Each of these has its own unitarity triangle.

5) However: by virtue of the unitarity of the CKM–matrix, all unitarity triangles have the same area.

④ The Yukawa sector and neutrino flavor mixing

The SM Yukawa sector for the leptons and neutrinos formally has a very similar structure as the one for the quarks. The difference in the corresponding quantum numbers is of course important (see Slide [4]).

$$\mathcal{L}^{\ell}_{\text{Yukawa}} = -G_{b\ell}\left(\bar{L}_{\ell}\Phi_b \ell_R + \text{h.c.}\right) - G_{t\ell}\left(\bar{L}_{\ell}\Phi_t \nu_{\ell R} + \text{h.c.}\right)$$

where Φ_b is the Higgs doublet field and $\Phi_t = i\tau_2\Phi_b$ its Y–charge conjugate and with $G^{ij}_{t\ell}$ and $G^{ij}_{b\ell}$ arbitrary 3 x 3 matrices. If CP is not a good symmetry they are necessarily complex! The mass–matrix we obtain by inserting $\Phi_b = \frac{v}{\sqrt{2}}\binom{0}{1}$ and $\Phi_t = \frac{v}{\sqrt{2}}\binom{1}{0}$. Thus

$$\begin{aligned} \mathcal{L}^{\ell}_{\text{mass}} &= -\frac{v}{\sqrt{2}}\sum_{i,j=1}^{3}\left[G^{ij}_{t\ell}\bar{\nu}_{iL}\nu_{jR} + G^{ij}_{b\ell}\bar{\ell}_{iL}\ell_{jR} + \text{h.c.}\right] \\ &= -\sum_{i,j=1}^{3}\left[m^{ij}_{t\ell}\bar{\nu}_{iL}\ell_{jR} + m^{ij}_{b\ell}\bar{\ell}_{iL}\ell_{jR} + \text{h.c.}\right] \end{aligned}$$

with

$$m^{ij}_{\cdot} = \frac{v}{\sqrt{2}} G^{ij}_{\cdot} \quad .$$

The fields considered up to now are in the **weak interaction basis** in which the matter field Lagrangian is diagonal in the families. We now have to find the **physical fields** in the **mass eigenstate basis** in which the mass–matrix is diagonal such that each field has a fixed mass.

The mass eigenstates are the ones which propagate (propagators of physical fields to be utilized in Feynman rules) while in interactions the weak or flavor eigenstates are the ones destroyed or created.

To this end we have to perform global unitary transformation on the horizontal vectors. Unitary because they have to leave the kinetic terms of the quarks invariant and global because we have to diagonalize the constant coupling matrices $G_{\cdot\, q}$.

Thus, we have to perform the transformations

$$\begin{array}{lcl} \nu_{jR} & \rightarrow & (U_R^\nu)_{jk}\, \nu_{kR} \\ \ell_{jR} & \rightarrow & \left(U_R^\ell\right)_{jk} \ell_{kR} \\ \nu_{jL} & \rightarrow & (U_L^\nu)_{jk}\, \nu_{kL} \\ \ell_{jL} & \rightarrow & \left(U_L^\ell\right)_{jk} \ell_{kL} \quad . \end{array}$$

As a result of this basis change, the interaction of the gauge fields with the quarks, which used to be family diagonal, now no longer are necessarily so. Neutral current interactions, since they involve always $(U^\nu_{L,R})^+ U^\nu_{L,R} = 1$ or $(U^\ell_{L,R})^+ U^\ell_{L,R} = 1$, continue to be diagonal. However, for charged current interactions what enters after the basis change is the unitary matrix

$$V \equiv V_{CKM} = (U_L^\nu)^+ U_L^\ell \;\;,$$

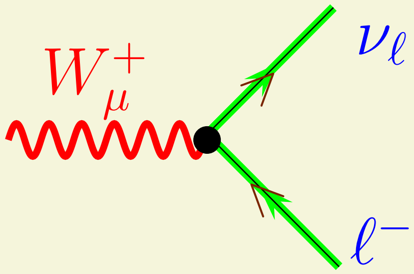
or its adjoint. This gives rise to family mixing in the charged current. By convention we assume the leptons (e, μ, τ) to coincide in the weak and the mass eigen basis, like (u, c, t) in case of the quarks, then the neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ are mixing in an analogous way as the quarks (d, s, b): The weak eigenstates in terms of the mass eigenstates read:

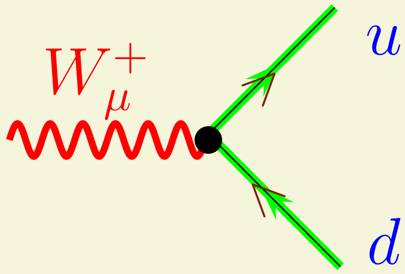
$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

- If neutrinos have a mass the charged current interactions of leptons are not flavor diagonal and mix similar as quarks:

Affects charged current (CC) structure (no FCNC)

$$\frac{g}{\sqrt{2}} W_\mu^+ \sum_{ij} \left(U_{\mathrm{MNS}}^{ij} \bar{l}_L^i \gamma_\mu \nu_L^j \;+\; U_{\mathrm{CKM}}^{ij} \bar{U}_L^i \gamma_\mu D_L^j \right) \;+\; h.c.$$





$$\sqrt{2}\frac{M_W}{v}\gamma^\mu \frac{1-\gamma_5}{2}\; V_{\nu_\ell \ell} \qquad \sqrt{2}\frac{M_W}{v}\gamma^\mu \frac{1-\gamma_5}{2}\; V_{ud}$$

(Maki, Nakagawa, Sakata 1962) also: Pontecorvo 1957, Gribov&Pontecorvo 1969, Bilenky&Pontecorvo 1978

Quark flavor mixing:

$$(\bar{\psi}_u, \bar{\psi}_c, \bar{\psi}_t) \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} \psi_d \\ \psi_s \\ \psi_b \end{pmatrix}$$

6 charged massive Fermion fields $\Rightarrow$ must be Dirac fields $\Rightarrow$ global phase not observable $\Rightarrow$ V_{ij} not physical observable, depends on phase conventions.

unitary $N \times N$ matrix has $2N^2 - N^2 = N^2$ free parameters

if all fields multiplied by the same phase no effect on V

number of relative phases between the $2N$ fields: $2N - 1$

number of physical parameters: $N^2 - 2N + 1$ = $\begin{cases} 1 & \text{for} \quad N = 2 \\ 4 & \text{for} \quad N = 3 \\ 9 & \text{for} \quad N = 4 \end{cases}$

number of pairs of different fields = number of posible mixing angles = number of upper half non-diagonal elements: $(N^2 - N)/2 = N\,(N-1)/2 = \begin{cases} 1 & \text{for} \quad N = 2 \\ 3 & \text{for} \quad N = 3 \\ 6 & \text{for} \quad N = 4 \end{cases}$

number of phases: number of physical parameters - number of rotaion angles:

$$N^2-2N+1-N(N-1)/2=\frac{1}{2}(N^2-3N+2)=\frac{1}{2}(N-1)(N-2)=\begin{cases} 0 & \text{for} \quad N=2 \\ 1 & \text{for} \quad N=3 \\ 3 & \text{for} \quad N=4 \end{cases}$$

Neutrino flavor mixing:

3 charged massive Fermion fields $\Rightarrow$ must be Dirac fields 3 neutral massive Fermion fields,

$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ singletts, may be Dirac or Majorana type.

Majorana $\equiv$ self-conjugate Fermion: $\nu^c = \eta\nu$; η a phase

phase not arbitrary, in fact observable: N free phases instead of $2N-1$ which means: $N-1$ extra phases.

$$U^{\mathrm{Dir}} = R_{23}\, R_{13}\, R_{12}$$

$$N = \mathrm{diag}\left\{e^{i\alpha_1} e^{i\alpha_2} e^{i\alpha_3}\right\}$$

$$U^{\mathrm{Maj}} = U^{\mathrm{Dir}}\, N$$

which mans

$$U^{\mathrm{Maj}}_{\alpha j} = U^{\mathrm{Dir}}_{\alpha j} \times e^{i\alpha_j}$$

- To fully determine the lepton flavour sector one has to find:
  - * How many, N, neutral states and their masses m_i
  - * Their mixings: # angles = $\frac{N(N-1)}{2} = 3$ for $N = 3$
  - * Their CP properties:

    Dirac: $\nu^C \neq \nu$ # phases = $\frac{(N-1)(N-2)}{2} = 1$ for $N = 3$

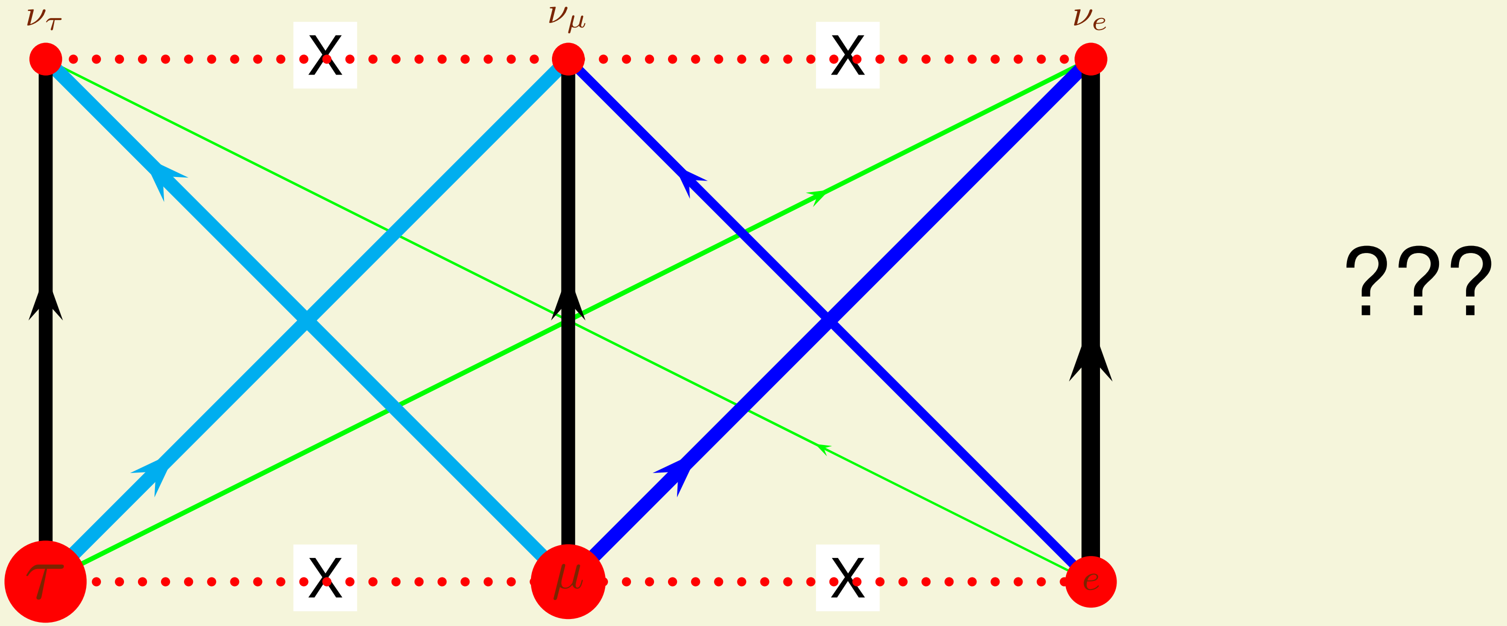
    Majorana: $\nu^C = \eta\nu$ (η=phase) # extra phases = $(N-1) = 2$ for $N = 3$

Neutrions travel as mass eigenstaes, in interaction they “regenerate” to flavor (weak) eigenstates

diagonal mass matrix $\Leftrightarrow$ only masses as parameters

non-diagonal mass matrix $\Leftrightarrow$ masses, mixing angles and CP violating phases

The corresponding lepton/neutrino transition pattern is illustrated in the following diagram: (neutrino mixing)



Note: The e is stable (lightest charged particle), the μ is metastable τ .... Flavor changing neutral currents (FCNC) at tree level $\cdots$X$\cdots$ are forbidden by the GIM–mechanism. Flavor changing neutral current transitions are allowed, however, as second (or higher) order transitions: e.g. $\mu \longrightarrow e$ is in fact $\mu \longrightarrow (\nu_e^*, \nu_\mu *, \nu_\tau^*) \longrightarrow e$, where the asterix indicates "virtual transition".

⑤ Neutrino mass terms

Three type of L-invariant mass terms posible:

Dirac mass	$m_D \bar{\nu}_L \chi_R + h.c.$
LH Majorana mass	$m_L \bar{\nu}_L \nu_R^c + h.c. = m_L^* \nu_L^T C \nu_L + h.c.$
RH Majorana mass	$m_R \bar{\chi}_L^c \chi_R + h.c. = m_R \chi_R^T C \chi_R + h.c.$

C **is the charge conjugation matrix; RH(LH) = right(left)-handed**

Dirac and RH Majorana mass terms require the existence of EW singlet neutrinos

It is a convention to label RH-neutrinos χ_R "right-handed neutrino singlets", since they are singlets we also could write them as $\chi_L' = (\chi_R)^c$

Dirac mass terms conserve total lepton number $L_{\rm tot}$, while Majorana mass terms violate $L_{\rm tot}$.

In the EWSM extended to include massive neutrino s:

i) **a Dirac mass term transforms as a weak $I = 1/2$ operator, and is coupled to the $I = 1/2$ Higgs field to make a gauge-invariant siglet**

ii) **a Majorana mass term involving the $I = 1/2$ left–handed neutrinos transform as $I = 1$ and must couple to an $I = 1$ and $Y = 2$ operator to make a gauge-invariant singlet**

iii) **a Majorana mass term involving the $SU(2) \otimes U(1)$ singlet neutral leptons, conventionally considered to be right-handed, is a singlet; it could be present as a bare mass term or couple to some other singlet operator**

The Seesaw mechanism

Simplest extension of the restricted SM Yukawa term $\mathcal{L}_{\text{Yukawa}}$ with zero neutrino masses (forbidden by imposing lepton number conservation via the 3 global flavor symmetries: $U(1)_e \otimes U(1)_\mu \otimes U(1)_\tau$): add right–handed singlets ν_R (= SM gauge group singlets): assuming 3 ν_R's as a minimum

$$\boxed{\mathcal{L}^{\text{ext}} = -\frac{1}{2} M_{Rij} \overline{\nu^c_{Ri}} \nu_{Rj} - G^\nu_{ij} \overline{L_{Li}}\, \Phi_t\, \nu_{Rj} + \text{h.c.}}$$

ν_R are electroweak siglets $\Rightarrow M_{Rij} \gg$ EW scale v

Mass term: $\Phi_t \Rightarrow \Phi_t = \frac{v}{\sqrt{2}}\binom{1}{0}$: general neutrino mass term

$$\mathcal{L}^\nu_m = -\frac{1}{2} \left(\begin{array}{cc} \bar{\nu}_\ell & \bar{\chi}^c_L \end{array} \right) = \left(\begin{array}{cc} m_L & m_D \\ m_D^T & M_R \end{array} \right) \left(\begin{array}{c} \nu^c_R \\ \chi_R \end{array} \right)$$

$m_L = m_L^T$ symmetric 3×3 left–handed Majorana mass matrix

$M_R = M_R^T$ symmetric $n_s \times n_s$ right–handed Majorana mass matrix

m_D $3 \times n_s$ Dirac mass matrix

Masses $m_i \propto v$ "light" $\ll M_R$ not protected by gauge symmetry, i.e., naturally much havier than those which get massless when $v \to 0$.

Diagonalization for $n_s = 3$:

$$\boxed{m_\nu = m_L - m_D^T \, M_R^{-1} \, m_D}$$

Hence, for $m_i \ll M_R$ the neutrino mass m_ν relative to m_ℓ appears suppressed by factor a m_{leptons}/M_R which could explain the smallnes of neutrino masses relative to the lepton masses, typically $M_R \sim 10^{-16}$ is required.

For $n_s = 3$:

3 masses + 3 angles + 1CP phase + 2 Majorana phases = 9 new observable parameters

Neutrino Mixing

Mixing matrix: 3 angles, 1 CP phase + (2 Majorana phases)

$$U_{\rm PMNS} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{atmospheric}} \begin{pmatrix} c_{13} & 0 & s_{13}e^{+i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{solar}}$$

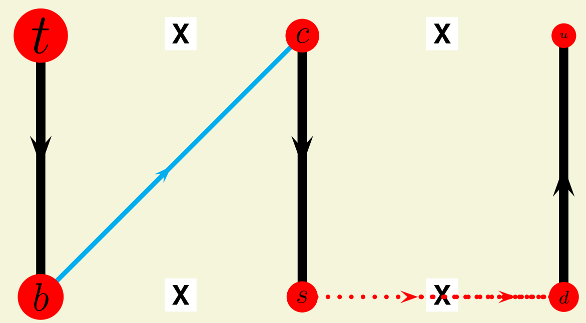
2ν oscillation analysis $\Rightarrow \Delta m^2 = \Delta m^2_\odot \ll \Delta M^2 \simeq \pm \Delta m^2_{\rm atm} \simeq \pm \Delta m^2_{23} \simeq \pm \Delta m^2_{13}$

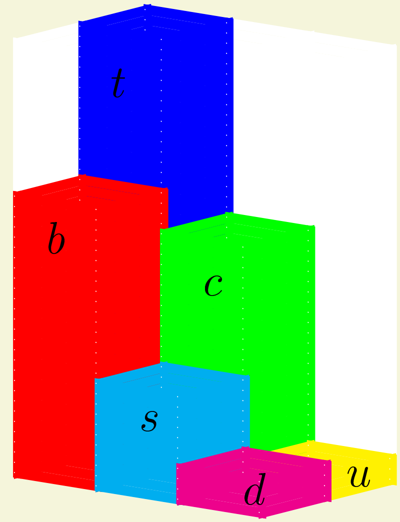
Mass matrix in flavor basis: $\boxed{m = U^* m^{\rm diag} U^+}$

$$\boxed{m^{\rm diag} = {\rm diag}(m_1 e^{-2i\rho}, m_2, m_3 e^{-2i\sigma})}$$

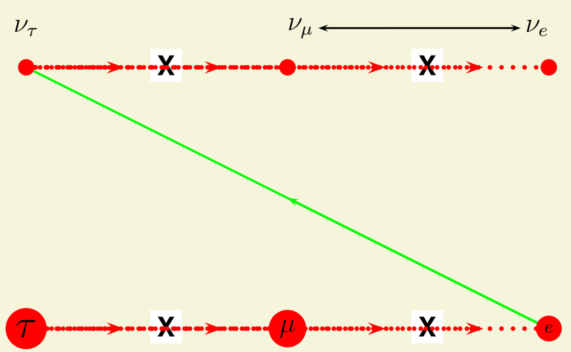
$$U = U(\theta_{ij}, \delta), m_2 = \sqrt{m_1^2 + \Delta m^2_{12}}, m_3 = \sqrt{m_1^2 + \Delta m^2_{13}}$$

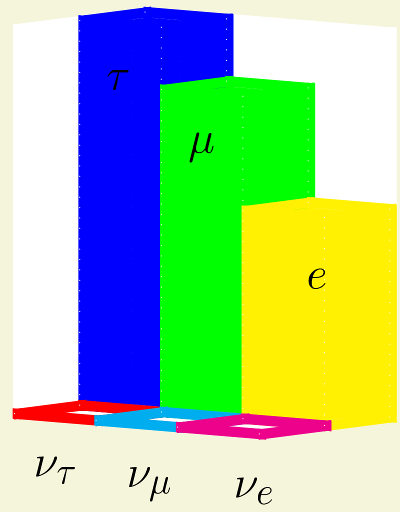
Quark mass hierarchy:





Lepton-neutrino mass hierarchy:





Hierarchy of quark (right top) and lepton masses (right bottom) [$\sim \log m_f$ in arbitrary units]

Neutrino Oscillations ($\cdots$, KamLAND 03)

- **If neutrinos have mass, a weak eigenstate $|\nu_\alpha>$ produced in $l_\alpha + N \to \nu_\alpha + N'$ is a linear combination of the mass eigenstates (ν_i) : $|\nu_\alpha> = \sum_{i=1}^n U_{\alpha i} |\nu_i>$**
- **After a distance L (or time t) it evolves as $|\nu_\alpha(t)> = \sum_{i=1}^n U_{\alpha i} e^{-iE_i t} |\nu_i>$ it can be detected with flavor β with probability $P_{\alpha\beta} = |<\nu_\beta|\nu_\alpha(t)>|^2$**

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{j\neq i}^n \mathrm{Re}\,\left[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right] \sin^2\left(\frac{\Delta_{ij}}{2}\right) + 2 \sum_{j\neq i}^n \mathrm{Im}\,\left[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right] \sin\left(\Delta_{ij}\right)$$

$$\frac{\Delta_{ij}}{2} = \frac{(E_i - E_j)\,L}{2} 1.27 \frac{(m_i^2 - m_j^2)}{\mathrm{eV}^2} \frac{L/E}{\mathrm{Km/GeV}}$$

- $P_{\alpha\beta}$

depends on Theoretical Parameters and on two Experimental parameters:

- $\Delta m^2_{ij} = m_i^2 - m_j^2$ **the mass differences**
- $U_{\alpha i}$ **the mixing angles (and Dirac phases)**
- E **the neutrino energy**
- L **the distance source - detector**

- **No information on absolute mass scale nor Majorana phases**

Solar Neutrinos

Δm^2_{12}	=	7×10^{-5} eV^2
$\tan^2\Theta_{12}$	=	0.4
$\sin^2 2\Theta_{13}$	$<$	0.067

⋮

SNO

Atmospheric Neutrinos

Δm^2_{23}	=	$(1.3 - 3.0) \times 10^{-3}$ eV^2
$\sin^2 2\Theta_{12}$	$>$	0.9

Super Kamiokande

MACRO, SOUDAN

K2K

Main features

- **Smallness of ν masses:** $\boxed{m_\nu < (1-2)\ \text{eV}}$ $\quad m_\nu \ll m_\ell, m_q$

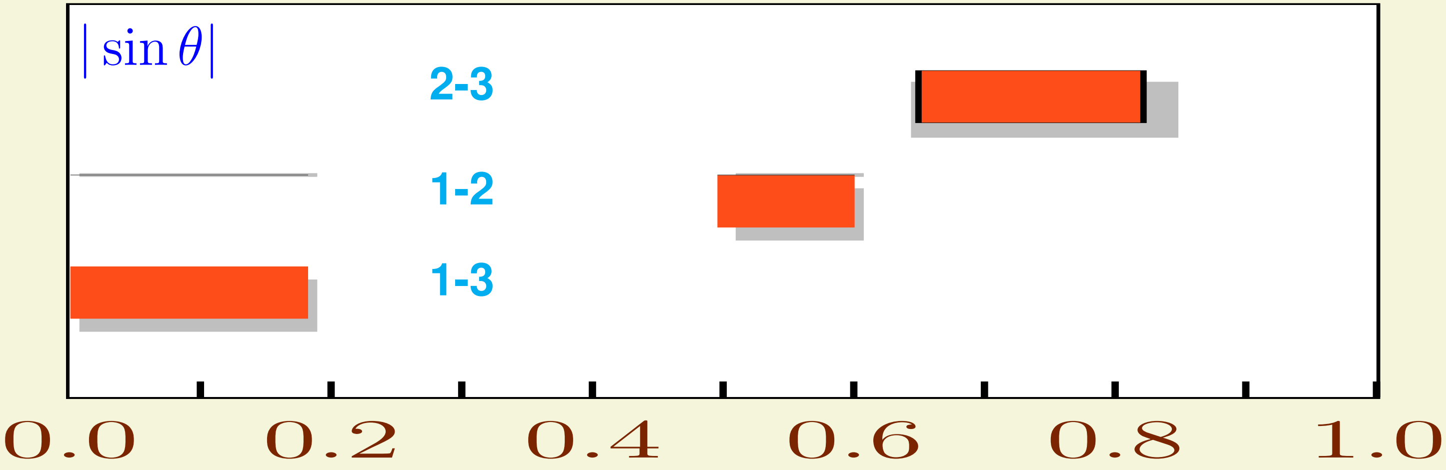
  $m_\nu > \sqrt{\Delta m^2_{23}} > 0.04$ **eV** **(at least for one mass)**

- **Hierarchy of Δm^2's :** $\boxed{|\Delta m^2_{12}/\Delta m^2_{23}| = 0.01 - 0.15}$

- **No strong hierarchy of masses:** $|m_2/m_3| > |\Delta m_{12}/\Delta m_{23}| = 0.18^{+0.22}_{-0.08}$

- **Bi–large or maximal mixing between neighboring families (1-2)(2-3):**

- **Small mixing between remote families (1-3):**

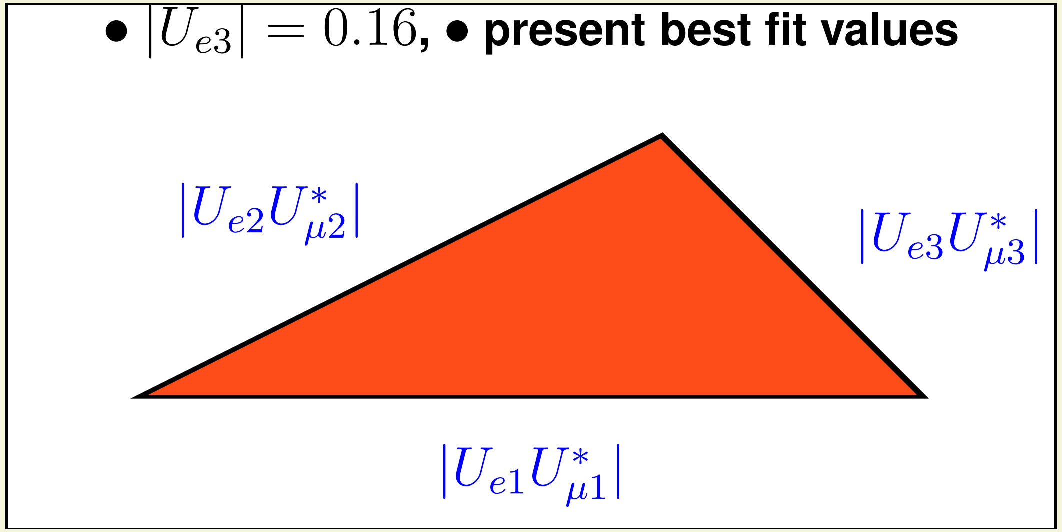


Leptonic Unitarity Triangle

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} 0.79 - 0.86 & 0.50 - 0.61 & 0.00 - 0.16 \\ 0.24 - 0.52 & 0.44 - 0.69 & 0.63 - 0.79 \\ 0.26 - 0.52 & 0.47 - 0.71 & 0.60 - 0.77 \end{pmatrix}$$

Global fit of the oscillation data at 1σ

(Gonzales-Garcia, Pena-Garay 03)



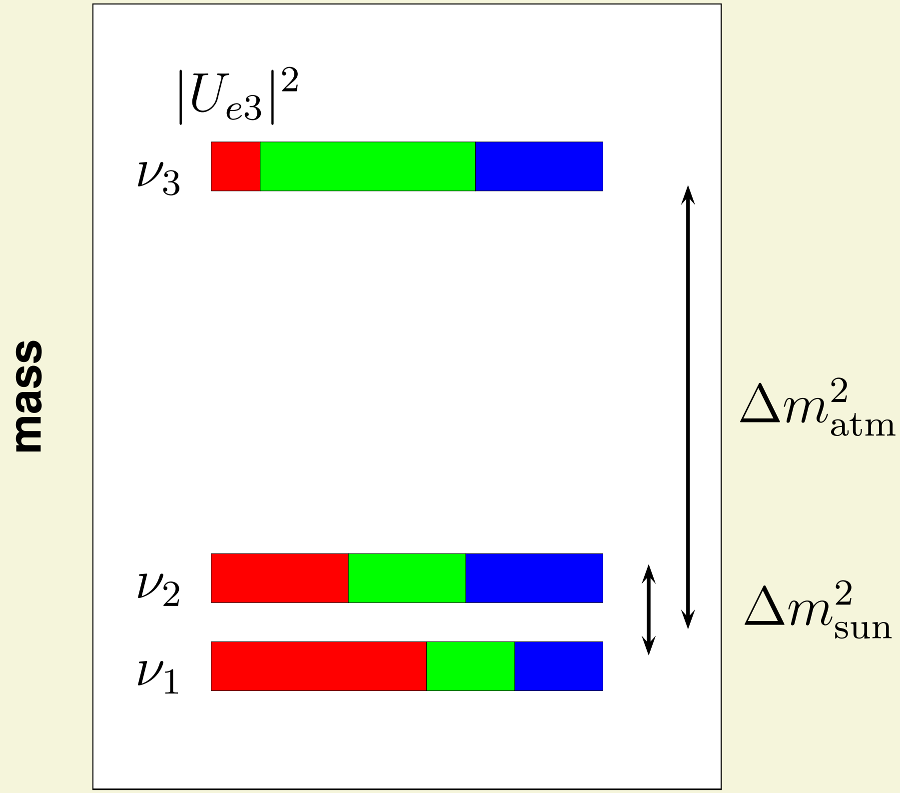
???

very preliminary

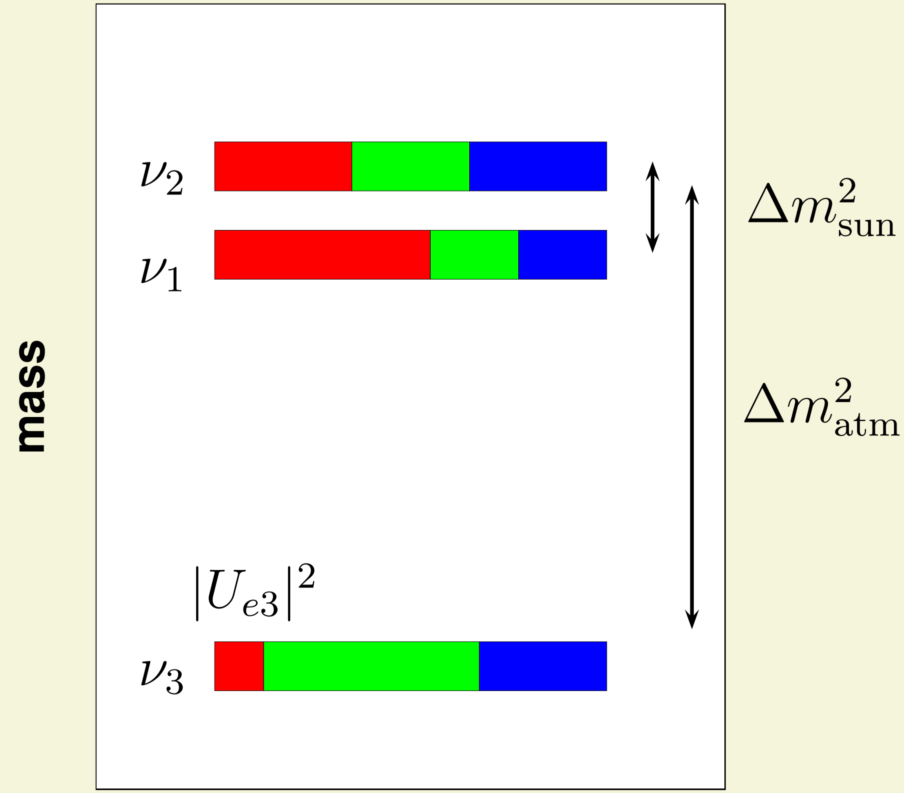
? θ_{13}, δ_{13} **?**

to be measured yet

Mass spectrum and mixing



normal mass ordering



inverted mass ordering

- **type of spectrum, ordering, absolute mass scale ?**
- $|U_{e3}|^2 =$ **?, Dirac vs. Majorana ($0\nu\beta$–decay)**

Constituents of matter:

Spin 1/2 fermions

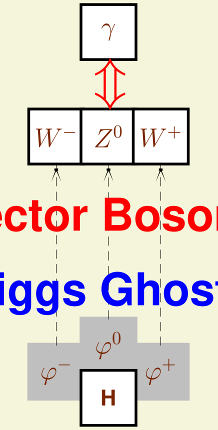
$SU(3)_c$

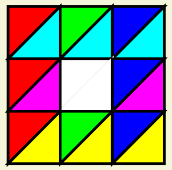
		siglets	color triplets	anti-triplets	
1st family		ν_{eL} / e^-_L	$\mathrm{u_L}$ $\mathrm{u_L}$ $\mathrm{u_L}$ / $\mathrm{d_L}$ $\mathrm{d_L}$ $\mathrm{d_L}$	$\bar{\mathrm{u}}_\mathrm{L}$ $\bar{\mathrm{u}}_\mathrm{L}$ $\bar{\mathrm{u}}_\mathrm{L}$ / $\bar{\mathrm{d}}_\mathrm{L}$ $\bar{\mathrm{d}}_\mathrm{L}$ $\bar{\mathrm{d}}_\mathrm{L}$	← $SU(2)_\mathrm{L}$ weak iso-spin doublets
	sterile ν_s ⇨	ν_{eR}	$\mathrm{u_R}$ $\mathrm{u_R}$ $\mathrm{u_R}$	$\bar{\mathrm{u}}_\mathrm{R}$ $\bar{\mathrm{u}}_\mathrm{R}$ $\bar{\mathrm{u}}_\mathrm{R}$	weak iso-spin singlets
		e^-_R	$\mathrm{d_R}$ $\mathrm{d_R}$ $\mathrm{d_R}$	$\bar{\mathrm{d}}_\mathrm{R}$ $\bar{\mathrm{d}}_\mathrm{R}$ $\bar{\mathrm{d}}_\mathrm{R}$	
2nd family		$\nu_{\mu L}$ / μ^-_L	$\mathrm{c_L}$ $\mathrm{c_L}$ $\mathrm{c_L}$ / $\mathrm{s_L}$ $\mathrm{s_L}$ $\mathrm{s_L}$	$\bar{\mathrm{c}}_\mathrm{L}$ $\bar{\mathrm{c}}_\mathrm{L}$ $\bar{\mathrm{c}}_\mathrm{L}$ / $\bar{\mathrm{s}}_\mathrm{L}$ $\bar{\mathrm{s}}_\mathrm{L}$ $\bar{\mathrm{s}}_\mathrm{L}$	
		$\nu_{\mu R}$	$\mathrm{c_R}$ $\mathrm{c_R}$ $\mathrm{c_R}$	$\bar{\mathrm{c}}_\mathrm{R}$ $\bar{\mathrm{c}}_\mathrm{R}$ $\bar{\mathrm{c}}_\mathrm{R}$	
		μ^-_R	$\mathrm{s_R}$ $\mathrm{s_R}$ $\mathrm{s_R}$	$\bar{\mathrm{s}}_\mathrm{R}$ $\bar{\mathrm{s}}_\mathrm{R}$ $\bar{\mathrm{s}}_\mathrm{R}$	
3rd family		$\nu_{\tau L}$ / τ^-_L	$\mathrm{t_L}$ $\mathrm{t_L}$ $\mathrm{t_L}$ / $\mathrm{b_L}$ $\mathrm{b_L}$ $\mathrm{b_L}$	$\bar{\mathrm{t}}_\mathrm{L}$ $\bar{\mathrm{t}}_\mathrm{L}$ $\bar{\mathrm{t}}_\mathrm{L}$ / $\bar{\mathrm{b}}_\mathrm{L}$ $\bar{\mathrm{b}}_\mathrm{L}$ $\bar{\mathrm{b}}_\mathrm{L}$	
		$\nu_{\tau R}$	$\mathrm{t_R}$ $\mathrm{t_R}$ $\mathrm{t_R}$	$\bar{\mathrm{t}}_\mathrm{R}$ $\bar{\mathrm{t}}_\mathrm{R}$ $\bar{\mathrm{t}}_\mathrm{R}$	
		τ^-_R	$\mathrm{b_R}$ $\mathrm{b_R}$ $\mathrm{b_R}$	$\bar{\mathrm{b}}_\mathrm{R}$ $\bar{\mathrm{b}}_\mathrm{R}$ $\bar{\mathrm{b}}_\mathrm{R}$	
		Leptons	Quarks		

Carrier of Forces:

Spin 1 Gauge Bosons

Photon





Octet of Gluons

Vector Bosons

“Cooper Pairs”:

Spin 0 Higgs Boson

Higgs Ghosts

Higgs Particle