Essentials of the Muon $g-2$

Fred Jegerlehner*
fjeger@physik.hu-berlin.de

CALC 2012, JINR Dubna, Russia, July 23 - August 2, 2012

* Humboldt University Berlin/DESY Zeuthen

Introductory Remarks

- The electron and muon anomalous magnetic moments ($a_\ell = (g_\ell - 2)/2$) belong to the most precisely measured quantities in particle physics.
Actual precision: e: .24ppb, μ: .54 ppm

- They are pure relativistic quantum correction effects (vanishing at tree level) and hence test the concept of relativistic quantum field theory in general and the Standard Model (SM) of elementary particle physics in particular with highest sensitivity (up to the leading 5-loop effects)

- The high precision is an extraordinary challenge both for theory and experiment

- The last muon $g-2$ experiment (BNL 2004) has reached a precision at which non-perturbative hadronic effects have to be known with high precision. Hadronic vacuum polarization (HVP) about 11 SD's, hadronic light-by-light scattering (HLBL) about 2 SD's

- Experiments in design/progress will improve the accuracy by a factor 5 which

represents a tremendous challenge to theory in the coming years. Most important are improvements in the calculation of the hadronic effects, a particular challenge for lattice QCD. Real progress recently.

Outline of lecture:

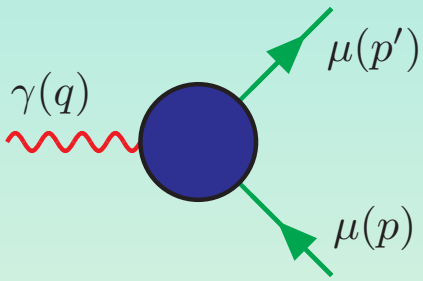
- $g-2$ introduction, history, muon properties, lepton moments
- $g-2$ experimental principles, the Muon $g-2$ experiments
- Standard Model Prediction for a_μ
- Evaluation of $a_\mu^{\rm had}$
- About the hadronic light-by-light scattering contribution
- Theory vs Experiment; do we see New Physics?
- Summary and Outlook

Muon $g-2$ introduction, history, muon properties, lepton moments

Particle with spin $\vec{s} \Rightarrow$ magnetic moment $\vec{\mu}$ (internal current circulating)

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\,\vec{s} \;\; ; \quad g_\mu = 2\,(1 + a_\mu)$$

Dirac: $g_\mu = 2 \quad , \quad a_\mu = \frac{\alpha}{2\pi} + \cdots$ muon anomaly



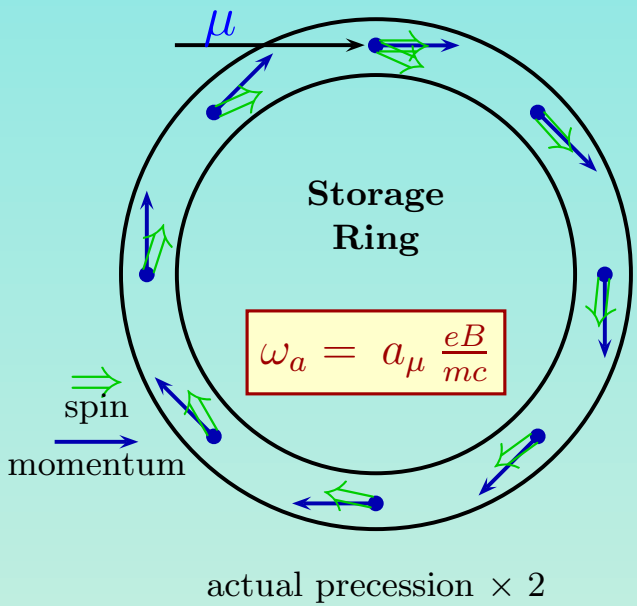
Electromagnetic Lepton Vertex

$$= (-\mathrm{i}e)\,\bar{u}(p')\left[\gamma^\mu F_1(q^2) + \mathrm{i}\,\frac{\sigma^{\mu\nu}q_\nu}{2m_\mu}F_2(q^2)\right]u(p)$$

$$F_1(0) = 1 \;\; ; \quad F_2(0) = a_\mu$$

a_μ responsible for the Larmor precession

Larmor precession $\vec{\omega}$ of beam of spin particles in a homogeneous magnetic field $\vec{B}$



Spin precession in the $g-2$ ring
($\sim 12'$/circle)

Magic Energy: $\vec{\omega}$ is directly proportional to $\vec{B}$ at magic energy $\sim$ 3.1 GeV

$$\vec{\omega}_a = \frac{e}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\vec{\beta}\times\vec{E}\right]^{E\sim 3.1\text{GeV}}_{\text{at "magic } \gamma\text{"}} \simeq \frac{e}{m}\left[a_\mu \vec{B}\right]$$

CERN, BNL g-2 experiments

Stern, Gerlach 22: $g_e = 2$; Kusch, Foley 48: $g_e = 2\,(1.00119 \pm 0.00005)$

Basic principle of experiment: measure Larmor precession of highly polarized muons circulating in a ring

$a_\mu = 0$ would mean no rotation of spin relative to muon momentum!

Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2} = E/mc^2$, $\gamma_{\rm mag} = \sqrt{1 + 1/a_\mu} \simeq 29.3 \Rightarrow$ muon lifetime $\tau_\mu = 2.19711\,\mu s$ at rest $\rightarrow \tau_\mu = 64.435\,\mu s$ in motion.

For the measurement of the anomalous magnetic moment we need to look at the

❒ equation of motion of a charged Dirac particle in an external field $A_\mu^{\rm ext}(x)$:

$$\left(i\hbar\gamma^\mu\partial_\mu + Q_\ell\tfrac{e}{c}\gamma^\mu(A_\mu(x) + A_\mu^{\rm ext}(x)) - m_\ell c\right)\psi_\ell(x) = 0$$
$$\left(\Box g^{\mu\nu} - \left(1 - \xi^{-1}\right)\partial^\mu\partial^\nu\right)A_\nu(x) = -Q_\ell e\,\bar\psi_\ell(x)\gamma^\mu\psi_\ell(x)\ .$$

Neglecting the radiation field (2nd eq.) in a first step: Dirac equation (1st eq.) as a relativistic one–particle problem

$$i\hbar\frac{\partial\psi}{\partial t} = \boldsymbol{H}\psi\ , \quad \boldsymbol{H} = c\,\vec\alpha\left(\vec p - \frac{e}{c}\vec A\right) + \beta\,mc^2 + e\,\Phi$$

with

$$\beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \vec{\alpha} = \gamma^0 \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} .$$

Interpretation:

1. Non–relativistic limit

Dipole moments (static): orbiting particle with electric charge e and mass m exhibits a magnetic dipole moment

$$\vec{\mu}_L = \frac{e}{2m} \vec{L}$$

where $\vec{L} = m\, \vec{r} \times \vec{v}$ is the orbital angular momentum ($\vec{r}$ position, $\vec{v}$ velocity). An electrical dipole moment can exist due to relative displacements of the centers of positive and negative electrical charge distributions. Magnetic and electric moments contribute to the electromagnetic interaction Hamiltonian with magnetic

$\vec{B}$ and electric $\vec{E}$ fields

$$\mathcal{H} = -\vec{\mu}_m \cdot \vec{B} - \vec{d}_e \cdot \vec{E}$$

where $\vec{\mu}_m$ and $\vec{d}_e$ the magnetic and electric dipole moment operators.

In the absence of an external field spin is a conserved quantity in the rest frame, i.e. the Dirac equation must be equivalent to the Pauli equation via a unitary transformation (Foldy-Wouthuysen):

$$\psi' = \boldsymbol{U}\psi\,, \quad \boldsymbol{H}' = \boldsymbol{U}\left(\boldsymbol{H} - \mathrm{i}\hbar\frac{\partial}{\partial t}\right)\boldsymbol{U}^{-1} = \boldsymbol{U}\boldsymbol{H}\boldsymbol{U}^{-1}$$

where the time–independence of $\boldsymbol{U}$ has been used, and we obtain

$$\mathrm{i}\hbar\frac{\partial\psi'}{\partial t} = \boldsymbol{H}'\psi'\,; \quad \psi' = \begin{pmatrix} \varphi' \\ 0 \end{pmatrix},$$

where φ' is the Pauli spinor. In fact $\boldsymbol{U}$ is a Lorentz boost matrix

$$\boldsymbol{U} = \mathbf{1}\cosh\theta + \vec{n}\,\vec{\gamma}\,\sinh\theta = \mathrm{e}^{\theta\vec{n}\vec{\gamma}}$$

with

$$\vec{n} = \frac{\vec{p}}{|\vec{p}|}\,, \quad \theta = \frac{1}{2}\mathrm{arccosh}\frac{p^0}{mc} = \mathrm{arcsinh}\frac{|\vec{p}|}{mc}$$

and we obtain, with $p^0 = \sqrt{\vec{p}^2 + m^2c^2}$,

$$\boldsymbol{H}' = cp^0\beta\,; \quad [\boldsymbol{H}', \vec{\Sigma}] = 0\,, \quad \vec{\Sigma} = \vec{\alpha}\,\gamma_5 = \begin{pmatrix} \vec{\sigma} & \mathbf{0} \\ \mathbf{0} & \vec{\sigma} \end{pmatrix}$$

where $\vec{\Sigma}$ is the spin operator. The v/c–expansion simply follows by expanding the matrix $\boldsymbol{U}$:

$$\boldsymbol{U}(\vec{p}\,) = \exp\theta\frac{\vec{p}}{|\vec{p}|}\vec{\gamma} = \exp\theta\frac{\vec{p}\vec{\gamma}}{2mc}\,; \quad \theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{\vec{p}^2}{m^2c^2}\right)^n\,.$$

2. Non–relativistic lepton with $A_\mu^{\rm ext} \neq 0$

To get non–relativistic representation for small velocities we have to split off the phase of the Dirac field due to the rest energy of the lepton $\psi = \hat{\psi}\, {\rm e}^{-{\rm i}\,\frac{mc^2}{\hbar}\,t}$. Consequently, the Dirac equation takes the form

$$ {\rm i}\hbar\frac{\partial\hat{\psi}}{\partial t} = \left(\boldsymbol{H} - mc^2\right)\hat{\psi}\ ;\ \ \hat{\psi} = \left(\begin{array}{c}\hat{\varphi}\\ \hat{\chi}\end{array}\right)\ , $$

and describes the coupled system of equations

$$ \begin{aligned} \left({\rm i}\hbar\frac{\partial}{\partial t} - e\,\Phi\right)\hat{\varphi} &= c\,\vec{\sigma}\,(\vec{p} - \frac{e}{c}\vec{A}\,)\,\hat{\chi} \\ \left({\rm i}\hbar\frac{\partial}{\partial t} - e\,\Phi + 2mc^2\right)\hat{\chi} &= c\,\vec{\sigma}\,(\vec{p} - \frac{e}{c}\vec{A}\,)\,\hat{\varphi}\ . \end{aligned} $$

For $c \to \infty$ we obtain

$$\hat{\chi} \simeq \frac{1}{2mc}\vec{\sigma}\,(\vec{p} - \frac{e}{c}\vec{A}\,)\,\hat{\varphi} + O(v^2/c^2)$$

and hence

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - e\,\Phi\right)\,\hat{\varphi} \quad \simeq \quad \frac{1}{2m}\left(\vec{\sigma}\,(\vec{p} - \frac{e}{c}\vec{A}\,)\right)^2\,\hat{\varphi}\,.$$

As $\vec{p}$ does not commute with $\vec{A}$, we may use the relation

$$(\vec{\sigma}\vec{a})(\vec{\sigma}\vec{b}) = \vec{a}\vec{b} + \mathrm{i}\vec{\sigma}\,(\vec{a} \times \vec{b})$$

to obtain

$$\left(\vec{\sigma}\,(\vec{p} - \frac{e}{c}\vec{A}\,)\right)^2 = (\vec{p} - \frac{e}{c}\vec{A}\,)^2 - \frac{e\hbar}{c}\,\vec{\sigma}\cdot\vec{B}\;; \quad \vec{B} = \mathrm{rot}\vec{A}\,.$$

This leads us to the *Pauli equation* (W. Pauli 1927)

$$\mathrm{i}\hbar\frac{\partial\hat{\varphi}}{\partial t} = \hat{H}\,\hat{\varphi} = \left(\frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 + e\,\Phi - \frac{e\hbar}{2mc}\,\vec{\sigma}\cdot\vec{B}\right)\hat{\varphi}$$

which up to the spin term is nothing but the non–relativistic Schrödinger equation. The last term is the one this lecture is about: it has the form of a potential energy of a magnetic dipole in an external field. In leading order in v/c the lepton behaves as a particle which has besides a charge also a magnetic moment

$$\vec{\mu} = \frac{e\hbar}{2mc}\,\vec{\sigma} = \frac{e}{mc}\,\vec{s}\,; \quad \vec{s} = \hbar\frac{\vec{\sigma}}{2}$$

with $\vec{s}$ the angular momentum. For comparison: the orbital angular momentum reads

$$\vec{\mu}_{\text{orbital}} = \frac{Q}{2m}\,\vec{L} = g_l\,\frac{Q}{2m}\,\vec{L}\,; \quad \vec{L} = \vec{r}\times\vec{p} = -\mathrm{i}\hbar\,\vec{r}\times\vec{\nabla} = \hbar\vec{l}$$

and thus the total magnetic moment is

$$\vec{\mu}_{\rm total} = \frac{Q}{2m}\left(g_l\,\vec{L} + g_s\,\vec{s}\right) = Q\,\frac{m_e}{m}\,\mu_{\rm B}\,\left(g_l\,\vec{l} + g_s\,\vec{s}\right)$$

where

$$\mu_{\rm B} = \frac{e\hbar}{2m_e c}$$

is Bohr's magneton. As a result for the electron $m = m_e$:

$$g_l = 1 \quad \text{and} \quad \boxed{g_s = 2}\ .$$

The last remarkable result is due to Dirac (1928) and tells us that the gyromagnetic ratio ($\frac{e}{mc}$) is twice as large as the one from the orbital motion.

The Foldy-Wouthuysen transformation for arbitrary A_μ cannot be performed in closed analytic form. However, the expansion in v/c can be done in a systematic

way (see e.g Landau-Lifschitz, Bjorken-Drell) and yields the effective Hamiltonian

$$
\begin{aligned}
H' &= \beta\left(mc^2 + \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} - \frac{\vec{p}^4}{8m^3c^2}\right) + e\,\Phi - \beta\,\frac{e\hbar}{2mc}\,\vec{\sigma}\cdot\vec{B} \\
&\quad -\frac{e\hbar^2}{8m^2c^2}\,\mathrm{div}\vec{E} - \frac{e\hbar}{4m^2c^2}\,\vec{\sigma}\cdot\left[(\vec{E}\times\vec{p} + \frac{\mathrm{i}}{2}\mathrm{rot}\vec{E})\right] + O(v^3/c^3)\ .
\end{aligned}
$$

Origin of additional terms:

- $\frac{\vec{p}^4}{8m^3c^2}$ leading relativistic correction,
- $\mathrm{div}\vec{E}$ Darwin term - fluctuations of the electrons position
- $\vec{\sigma}\cdot\left[(\vec{E}\times\vec{p} + \frac{\mathrm{i}}{2}\mathrm{rot}\vec{E})\right]$ spin–orbit interaction

- experimental setup $\mathrm{div}\vec{E} = 0\ ;\ \ \mathrm{rot}\vec{E} = 0\ .$

- besides a homogeneous magnetic field an electric quadrupole field is required for focusing the beam

For the magnetic term $\propto \vec{\sigma}$ we then have

$$H_{\rm mag} = -\vec{\mu}\cdot\left\{\vec{B} + \underbrace{\frac{1}{2}}_{g_l/g_s}\,\frac{\vec{E}\times\vec{v}}{c^2}\right\}\;;\;\;\vec{\mu} = \frac{e\hbar}{2mc}\,\vec{\sigma} = \frac{e}{m}\,\vec{s} = \frac{e}{2m}\,g_s\,\vec{s}$$

- in fact full relativistic kinematics is required (tuning to magic energy)

The correct relativistic formula [$g_2 = 2 \to 2\,(1+a_\mu)$ and appropriate γ factors] for the spin precession in transversal fields is

$$\frac{{\rm d}\vec{P}}{{\rm d}t} = \vec{\omega}_s \times \vec{P}\;;\;\;\vec{\omega}_s = -\frac{e}{\gamma m}\left\{(1+\gamma a)\;\vec{B} + \gamma\left(a + \frac{1}{\gamma+1}\right)\frac{\vec{E}\times\vec{v}}{c^2}\right\},$$

where $a = g/2 - 1$ is the anomaly term. While the cyclotron motion

$$\frac{{\rm d}\vec{v}}{{\rm d}t} = \vec{\omega}_c \times \vec{v}\,,\;\;\vec{\omega}_c = -\frac{e}{\gamma m}\left(\vec{B} + \frac{\gamma^2}{\gamma^2-1}\,\frac{\vec{E}\times\vec{v}}{c^2}\right).$$

The velocity $\vec{v}$ thus rotates, without change of magnitude, with the relativistic cyclotron frequency $\vec{\omega}_c$. The precession of the polarization $\vec{P}$=muon spin $\vec{s}_\mu$, transversal fields is

$$\vec{\omega}_a = \vec{\omega}_s - \vec{\omega}_c = -\frac{e}{m}\left\{a\,\vec{B} + \left(a - \frac{1}{\gamma^2-1}\right)\frac{\vec{E}\times\vec{v}}{c^2}\right\} .$$

This establishes the key formula for measuring a_μ. The motion is simple only for the magic energy $a - \frac{1}{\gamma^2_{\rm mag}-1} = 0$.

Future:

❒ Fermilab E969 follow up experiment of BNL E821, traditional, working at magic energy

❒ New measurements of muon g-2 and EDM with ultra-cold muon beam at J-PARC (works with $\vec{E} = 0$) new concept, vastly different kinematics region (slow muons) providing important cross check

The role of a_μ in precision physics

Precision measurement of a_μ provides most sensitive test of magnetic helicity flip transition

$$\bar{\psi}_L \, \sigma_{\mu\nu} \, F^{\mu\nu} \, \psi_R \qquad (\text{ dim 5 operator })$$

such a term must be absent for any fermion in any renormalizable theory at tree level (no adjustable parameter)

$$\Downarrow$$

a_μ is a pure "quantum correction" effect:
a finite model-specific prediction in any renormalizable quantum field theory (QFT)

– test of quantum structure

– monitor for new physics

Most fascinating aspect highly complex mathematics meets reality !

Note that in higher orders the form factors in general acquire an imaginary part. One may write therefore an effective dipole moment Lagrangian with complex "coupling"

$$\mathcal{L}_{\rm eff}^{\rm DM} = -\frac{1}{2}\left\{\bar\psi\,\sigma^{\mu\nu}\left[D_\mu\,\frac{1+\gamma_5}{2} + D_\mu^*\,\frac{1-\gamma_5}{2}\right]\psi\right\}\,F_{\mu\nu}$$

with ψ the muon field and

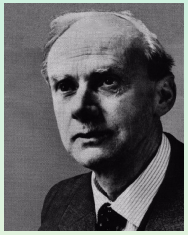
$$\text{Re}\,D_\mu = a_\mu\,\frac{e}{2m_\mu}\quad,\quad \text{Im}\,D_\mu = d_\mu = \frac{\eta_\mu}{2}\frac{e}{2m_\mu}\;.$$

Thus the imaginary part of $F_{\rm M}(0)$ corresponds to an electric dipole moment. The latter is non–vanishing only if we have T violation. Highly suppressed in the SM.

Some $g-2$ history

It started with atomic spectra in magnetic fields!
The electron:

- 1924 Stern-Gerlach see level splitting due to electron spin,
- 1925 Gouldsmit-Uhlenbeck postulate electron spin $\frac{1}{2}\hbar$ and spin angular momentum implying a magnetic moment $e\hbar/2m_e =$ Bohr magneton,
- 1927 Pauli QM of spin,
- 1928 Dirac relativistic QM Dirac electron, surprisingly $g_e = 2$, twice the value known from orbital angular momentum



- 1934 Kinster & Houston supports strongly $g_e \simeq 2$

- 1936 Anderson & Neddermeyer discovery of the muon in cosmic rays. Rabi: "Who ordered that?"

- 1948 Tomonaga, Schwinger, Feynman renormalization of QED [Nobel Prize 1965] (curing the notorious infinities)

$\Rightarrow$ Feynman rules, Feynman diagrams and all that

- 1947 Nafe et al., Nagle et al. HFS of H and D differ by 2×10^{-3} from Fermi Theory; Breit maybe $g \neq 2$.

- 1947 Kusch, Foley atomic precession in a constant magnetic field $\Rightarrow$ first precision determination of the magnetic moment of the electron $g_e = 2 \times [1.00119(5)]$.

Anomaly $a_e = \frac{g_e - 2}{2}$, $a_e \neq 0 \rightarrow$ structure of object!

- 1948 Schwinger unambiguous prediction of a higher order effects, leading (one–loop diagram) contribution to the anomalous magnetic moment $a_\ell^{\mathrm{QED}(1)} = \frac{\alpha}{2\pi} \simeq 0.00116$ (which accounts for 99 % of the anomaly), contribution is due to quantum fluctuations via virtual electron photon interactions [universal ($\ell = e, \mu, \tau$)]

  Together with Schwinger's result the first tests of the virtual quantum corrections, predicted by a relativistic quantum field theory [together with (Lamb–shift)].

> A triumph which established
> QFT is the basic structure of elementary particle theory

- 1987 Dehmelt et al. [U. of Washin.] $a_e^{\mathrm{exp}} = 1.159\,652\,1883(42) \times 10^{-3}$ [3.62 ppb] Penning Trap

- 2007 Gabrielse et al. [Harvard Univ.] $a_e^{\mathrm{exp}} = 1.159\,652\,180\,85(76) \times 10^{-3}$ [.66 ppb] Quantum Cyclotron

- 2008 Gabrielse et al. [Harvard Univ.] $a_e^{\rm exp} = 1.159\,652\,180\,73(76)\times 10^{-3}$ [.24 ppb]

The muon:

- 1956 Berestetskii et al.

$$\delta a_\ell \propto \frac{\alpha}{\pi}\frac{m_\ell^2}{M^2} \qquad (M \gg m_\ell)\ ,$$

where M may be

- the mass of a heavier SM particle, or
- the mass of a hypothetical heavy state beyond the SM, or
- an energy scale or an ultraviolet cut–off where the SM ceases to be valid.

$\Rightarrow$ muon much better monitor for heavy physics! enhanced by factor $(m_\mu/m_e)^2 \sim 43000$

But how to measure a_μ?

- 1957 Lee & Yang parity violation in weak transitions $\Rightarrow$ polarized muons!
- 1957 Garwin, Lederman & Weinrich determined $g_\mu = 2.00$ within 10%

  Friedman & Telegdi point out CP conserved with high accuracy, while P and C are maximally violated
- 1960 Columbia precession experiment $a_\mu = 0.00122(8)$ at a precision of about 5%
- 1961 first CERN cyclotron muon $g-2$ experiment $\rightarrow$ nothing special was observed within the 0.4% level of accuracy of the experiment $\Rightarrow$ first real evidence the muon is just a heavy electron!
- 1962 1st CERN muon storage ring, μ^+ and μ^- at the same machine, CPT test!

- 1969 2nd CERN muon storage ring, precision of 7 ppm reached
- 2001 BNL E821 experiment 20 years later
- 2004 BNL $g-2$ experiment closed, precision of 0.54 ppm reached (14–fold improvement)

The begin of E821 in 1984:
G. Danby, J. Field, F. Farley,
E. Picasso, F. Krienen, J. Bailey,
V. Hughes, F. Combley

Lepton properties:

- most puzzling replica of identical particles
- 3 families required to get CP violation via CKM flavor mixing

- Leptons $\ell = e, \mu, \tau$ in SM interact via gauge bosons γ electromagnetically and Z, W weakly

- Masses: $m_e = 0.511$ MeV, $m_\mu = 105.658$ MeV and $m_\tau = 1776.99$ MeV mass patterns are a big puzzle!

As masses differ by orders of magnitude the leptons show very different behavior, the most striking being the very different lifetimes.

- Lifetimes: $\tau_e = \infty$, $\tau_\mu = 2.197 \times 10^{-6}$ sec, $\tau_\tau = 2.906 \times 10^{-13}$ sec

❖ Decays:

- μ decays very close to 100% in $e\bar{\nu}_e\nu_\mu$
- τ decays to about 65% into hadronic states $\pi^-\nu_\tau$, $\pi^-\pi^0\nu_\tau$, , $\cdots$
  17.36% $\mu^-\bar{\nu}_\mu\nu_\tau$ and 17.85% $e^-\bar{\nu}_e\nu_\tau$

The **intrinsic** magnetic moment of a particle is proportional to the *spin operator*

$$\vec{L} \to \vec{s} = \frac{\hbar\vec{\sigma}}{2}$$

$\Rightarrow$ defines gyromagnetic ratio g (g-factor $\Rightarrow$ Zeeman effect) and its electric pendant η

$$\vec{\mu}_m = g\, Q\, \mu_0 \frac{\vec{\sigma}}{2} \quad , \quad \vec{d}_e = \eta\, Q\, \mu_0 \frac{\vec{\sigma}}{2}$$

$\mu_0 = e\hbar/2m$, σ_i ($i = 1, 2, 3$) are the Pauli spin matrices, Q is the electrical charge in units of e, $Q = -1$ for the leptons $Q = +1$ for the antileptons and m the mass.

Anomalous magnetic moment $a_\ell \equiv \frac{g_\ell - 2}{2}$

Production and Decay of Muons

- Muon $g-2$ experiment requires polarized muons
- Maximum P violating weak decays (no right–handed neutrinos can be produced) allows to do this easily from pion decay
- Pions are produced by shooting protons on a target [at Brookhaven the 24 GeV proton beam extracted from the AGS with 60×10^{12} protons per AGS cycle of 2.5 s impinges on a Nickel target of one interaction length]
- Pions are momentum selected in forward direction

Relevant decay chain in muon $g-2$ experiment: $\pi \quad \rightarrow \quad \mu + \nu_\mu$

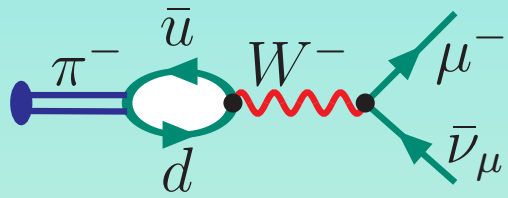
$\hookrightarrow \; e + \nu_e + \nu_\mu$

producing polarized muons which decay into electrons which carry along their direction of motion the knowledge of the muon's polarization

Illustration B. Touchek

1) Pion decay:

The π^- is a pseudoscalar bound state $\pi^- = (\bar{u}\gamma_5 d)$ of a d quark and a u antiquark $\bar{u}$. The main decay channel is via the diagram:



π–decay .

Two–body decay of the charged spin zero pseudoscalar meson $\rightarrow$ lepton energy is fixed (monochromatic) $E_\ell = \sqrt{m_\ell^2 + p_\ell^2} = \frac{m_\pi^2+m_\ell^2}{2m_\pi}$, $p_\ell = \frac{m_\pi^2-m_\ell^2}{2m_\pi}$.

Fermi type effective Lagrangian:

$$\mathcal{L}_{\text{eff,int}} = -\frac{G_\mu}{\sqrt{2}} V_{ud} \left(\bar{\mu}\gamma^\alpha (1-\gamma_5)\, \nu_\mu\right) (\bar{u}\gamma_\alpha (1-\gamma_5)\, d) + \text{h.c.}$$

G_μ Fermi constant, $V_{ud} \sim 1$ CKM matrix element

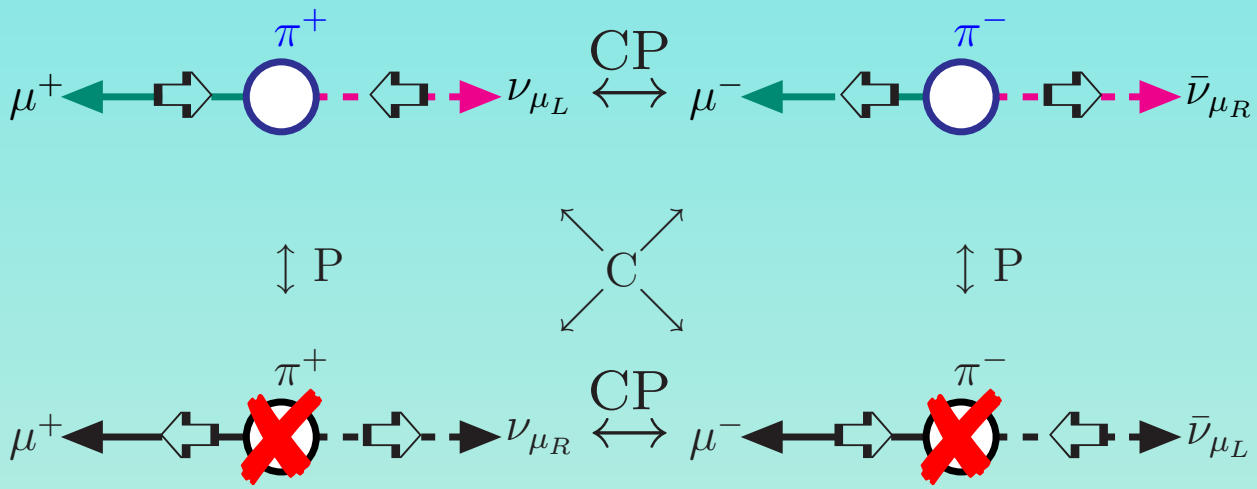
Transition matrix–element:

$$T \quad = \quad {}_{\rm out}\!< \mu^-, \bar{\nu}_\mu | \pi^- >_{\rm in} = -{\rm i}\frac{G_\mu}{\sqrt{2}}\, V_{ud}\, F_\pi \left(\bar{u}_\mu \gamma^\alpha\, (1-\gamma_5)\, v_{\nu_\mu}\right)\, p_\alpha$$

hadronic matrix–element $\left\langle 0 |\, \bar{d}\, \gamma_\mu \gamma_5\, u\, | \pi(p) \right\rangle \doteq {\rm i} F_\pi p_\mu$, F_π pion decay constant. As π pseudoscalar $\rightarrow$ only A of weak charged $V-A$ current couples to the pion.

Pion decay rate [$\delta_{\rm QED}$ = electromagnetic correction]

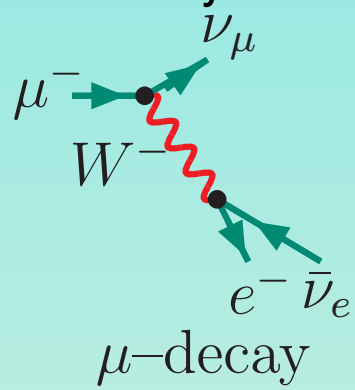
$$\Gamma_{\pi^- \to \mu^- \bar{\nu}_\mu} = \frac{G_\mu^2}{8\pi}\, |V_{ud}|^2 F_\pi^2\, m_\pi\, m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \times (1 + \delta_{\rm QED}) \ ,$$



Pion decay is a parity violating weak decay where leptons of definite handedness are produced depending on the given charge. CP is conserved while P and C are violated maximally (unique handedness). μ^- [μ^+] is produced with positive [negative] helicity $h = \vec{s} \cdot \vec{p}/|\vec{p}|$. The existing μ^- and μ^+ decays are related by a CP transformation. The decays obtained by C or P alone are inexistent in nature.

2) Muon decay:

Muon decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ is a three body decay



.

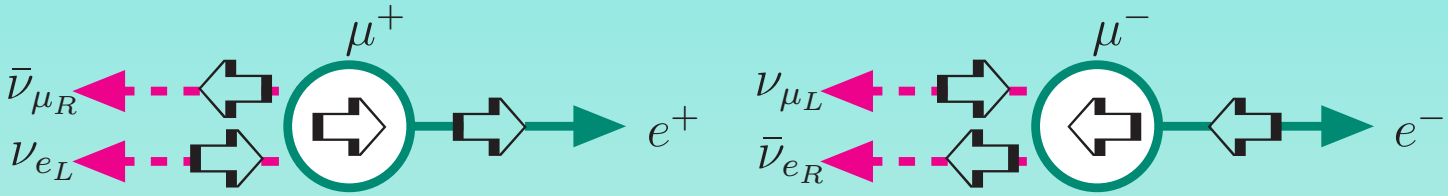
Effective Lagrangian:

$$\mathcal{L}_{\text{eff,int}} = -\frac{G_\mu}{\sqrt{2}} \left(\bar{e}\gamma^\alpha \left(1-\gamma_5\right) \nu_e\right) \left(\bar{\nu}_\mu \gamma_\alpha \left(1-\gamma_5\right) \mu\right) + \text{h.c.}$$

and

$$T \quad = \quad {}_{\text{out}}\!< e^-, \bar{\nu}_e \nu_\mu | \mu^- >_{\text{in}} = \frac{G_\mu}{\sqrt{2}} \left(\bar{u}_e \gamma^\alpha \left(1-\gamma_5\right) v_{\nu_e}\right) \left(\bar{u}_{\nu_\mu} \gamma_\alpha \left(1-\gamma_5\right) u_\mu\right)$$

$\Rightarrow$ μ^- and the e^- have both the same left–handed helicity [the corresponding anti–particles are right–handed] in the massless approximation:



In μ^- [μ^+] decay the produced e^- [e^+] has negative [positive] helicity, respectively

The electrons are thus emitted in the direction of the muon spin, i.e. measuring the direction of the electron momentum provides the direction of the muon spin.

After integrating out the two unobservable neutrinos, the differential decay probability to find an $e^\pm$ with reduced energy between x and $x + dx$ emitted at an angle between θ and $\theta + d\theta$ reads

$$\frac{d^2\Gamma^\pm}{dx\, d\cos\theta} = \frac{G_\mu^2 m_\mu^5}{192\pi^3}\, x^2 \left(3 - 2x \pm P_\mu \cos\theta\,(2x-1)\right)$$

and typically is strongly peaked at small angles. The reduced $e^\pm$ energy is

$x = E_e/W_{\mu e}$ with $W_{\mu e} = \max E_e = (m_\mu^2 + m_e^2)/2m_\mu$, the $e^\pm$ emission angle θ is the angle between the e momentum $\vec{p}_e$ and the muon polarization vector $\vec{P}_\mu$. The result above holds in the approximation $x_0 = m_e/W_{e\mu} \sim 9.67 \times 10^{-3} \simeq 0$.

Result: since parity is violated maximally in this weak decay there is a strong correlation between the muon spin direction and the direction of emission of the positrons. The differential decay rate for the muon in the rest frame is given by and

$$d\Gamma/\Gamma = N(E_e)\left(1 + \frac{1-2\,x_e}{3-2\,x_e}\,\cos\theta\right)\,d\Omega\ ,$$

in which E_e is the positron energy, x_e is E_e in units of the maximum energy $m_\mu/2$, $N(E_e)$ is a normalization factor

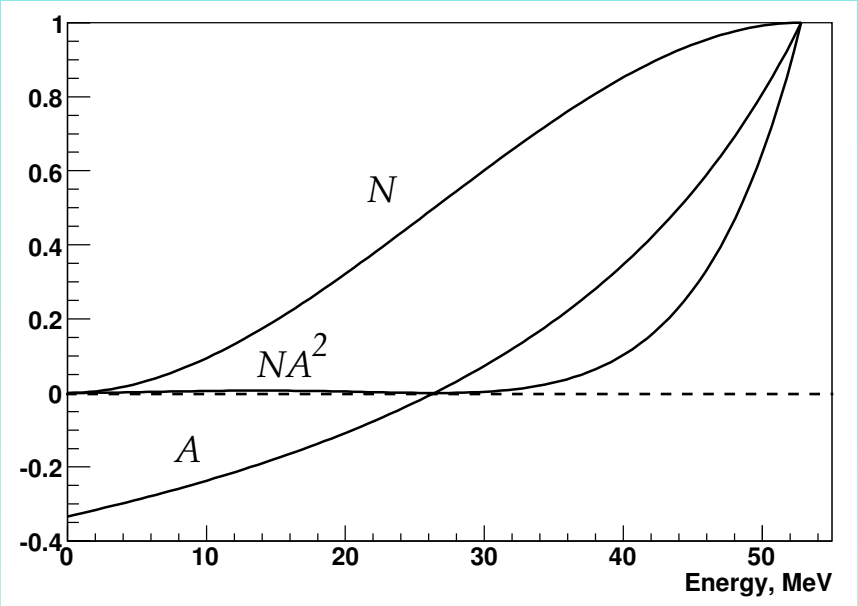
$$N(E_e) = 2\,x_e^2\,(3-2\,x_e)$$

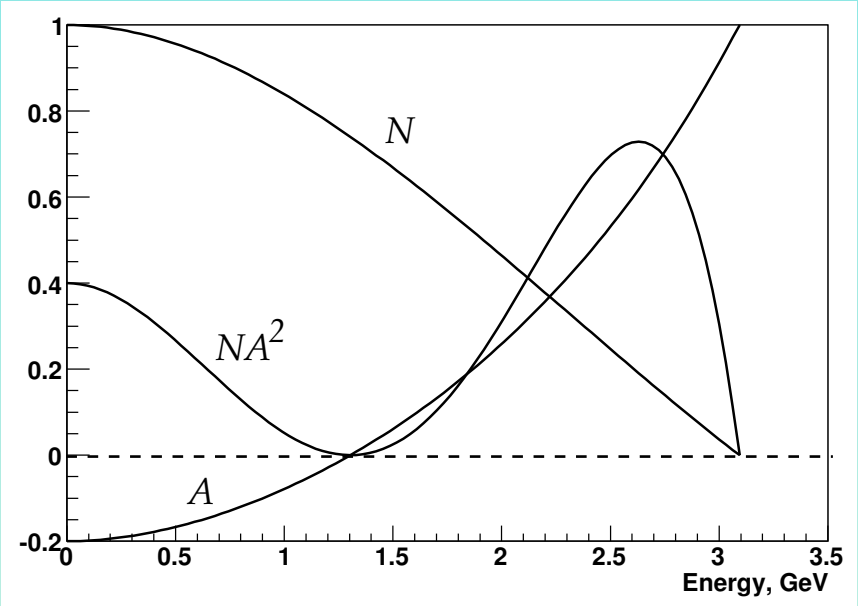
and θ the angle between the positron momentum in the muon rest frame and the

muon spin direction. The μ^+ decay spectrum is peaked strongly for small θ due to the non–vanishing coefficient of $\cos\theta$

$$A(E_e) \doteq \frac{1-2\,x_e}{3-2\,x_e}\,,$$

which is called asymmetry factor and reflects the *parity violation*





[Muon rest frame (left), laboratory frame (right)]

Number of decay electrons per unit energy, N (arbitrary units), value of the asymmetry A, and relative figure of merit NA^2 (arbitrary units) as a function of electron energy. The polarization is unity. For the third CERN experiment and E821, $E_{max} \approx 3.1$ GeV ($p_\mu = 3.094$ GeV/c) in the laboratory frame

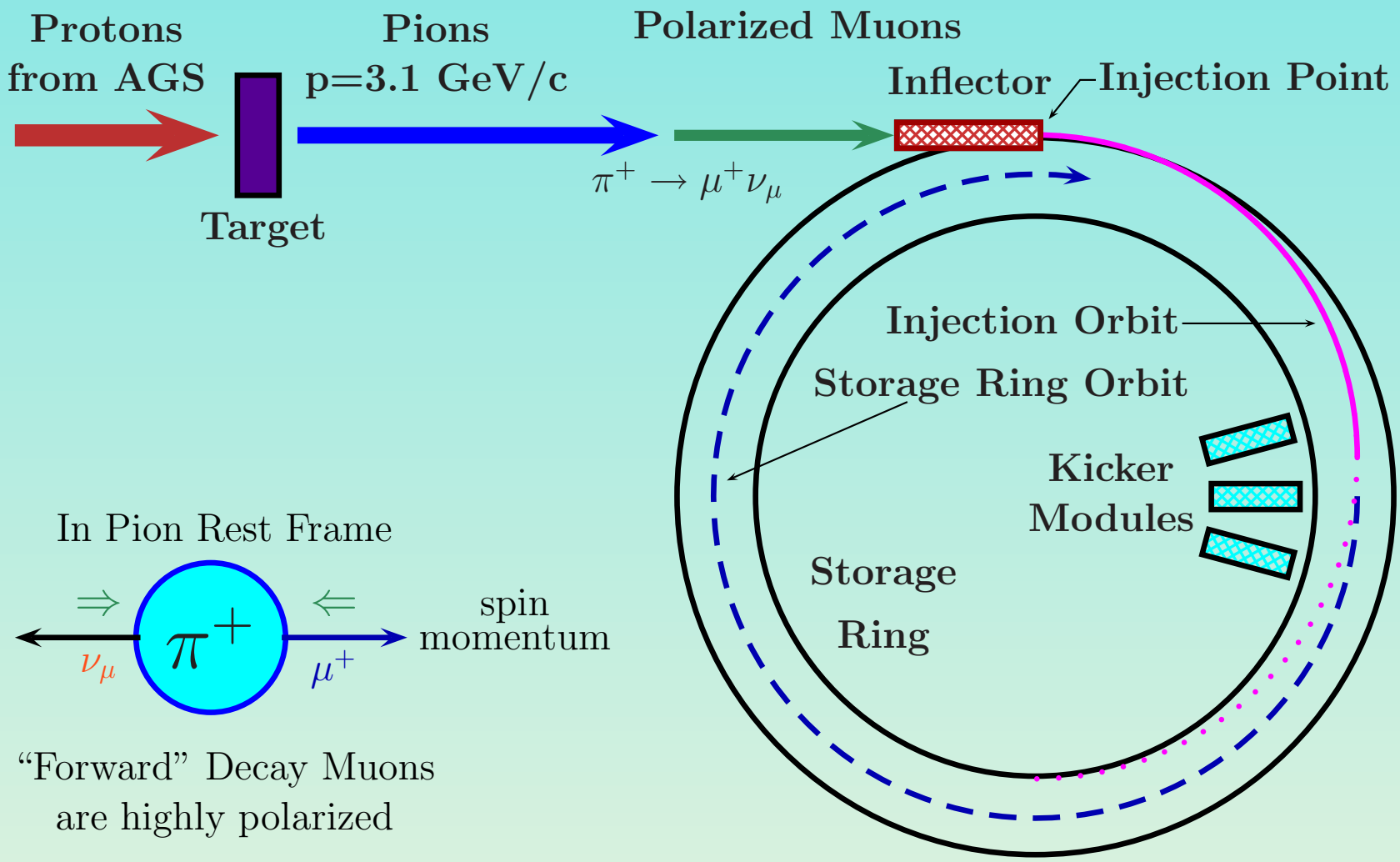
$g-2$ experimental principles, the Muon $g-2$ experiments

Principle of CERN and BNL muon g – 2 experiment:

Polarized muons circulating at magic energy in a storage ring

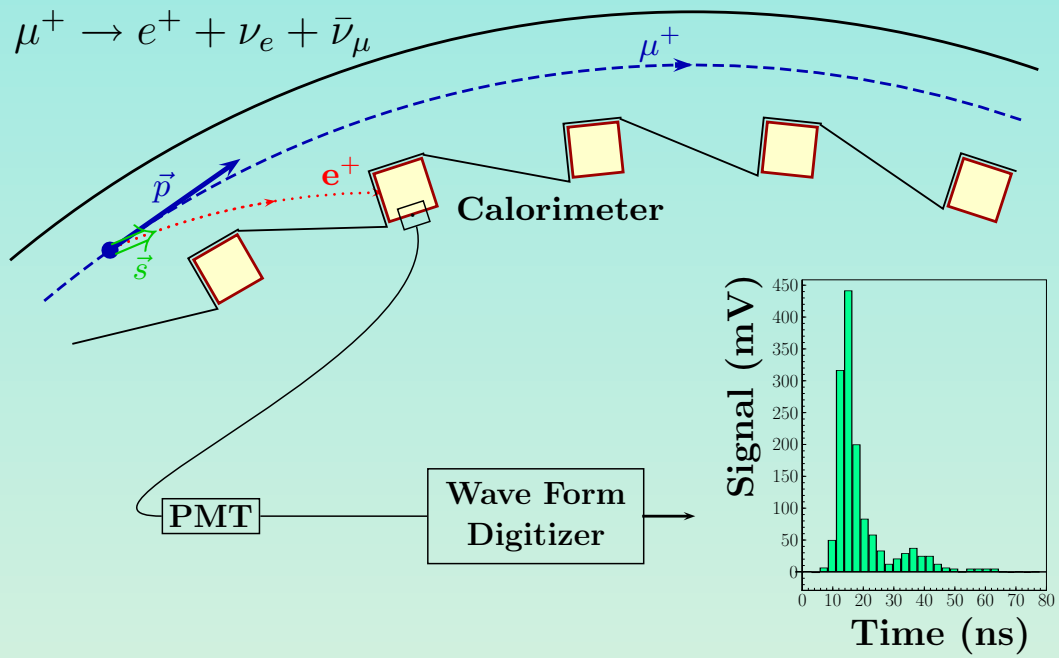
- improvements with E821
  - very high intensity of the primary proton beam from the proton storage ring AGS (Alternating Gradient Synchrotron) → much higher statistics
  - the injection of muons instead of pions into the storage ring → much less background
  - a super–ferric storage ring magnet → improved homogeneous magnetic field

BNL muon storage ring: r= 7.112 meters, aperture of the beam pipe 90 mm, field 1.45 Tesla, momentum of the muon $p_\mu = 3.094$ GeV/c (see `http://www.g-2.bnl.gov/`)



The schematics of muon injection and storage in the $g-2$ ring

Muons are circling in the ring many times before they decay into a positron plus two neutrinos: $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$. Maximal parity violation impies that the positron is emitted along the spin axis of the muon.



Decay of μ^+ and detection of the emitted e^+ (PMT=Photomultiplier)

The decay positrons detected by 24 lead/scintillating fiber calorimeters inside the

muon storage ring and the measured positron energy provides the direction of the muon spin.

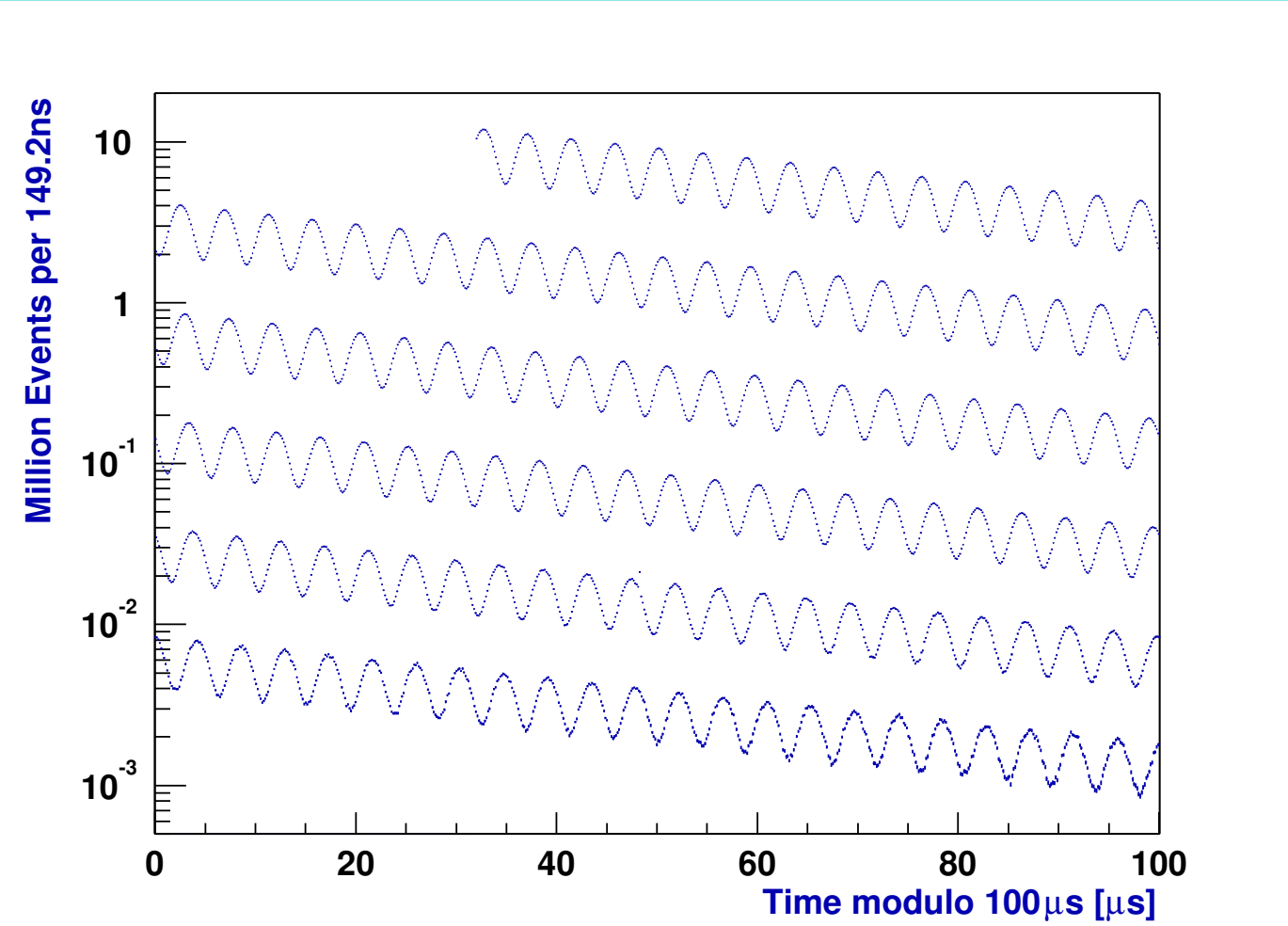
The number of decay positrons with energy greater than E emitted at time t after muons are injected into the storage ring is

$$N(t) = N_0(E)\ \exp\left(-t/\gamma\tau_\mu\right)\ [1 + A(E)\ \sin(\omega_a\, t + \phi(E))]\ ,$$

– $N_0(E)$ is a normalization factor, – τ_μ the muon life time, – $A(E)$ is the asymmetry factor for positrons of energy greater than E.

- exponential decay modulated by the $g-2$ angular frequency

- angular frequency ω_a neatly determined from the time distribution of the decay positrons observed with the electromagnetic calorimeters



Distribution of counts versus time for the 3.6 billion decays in the 2001 negative muon data–taking period

The magnetic field is measured by *Nuclear Magnetic Resonance* (NMR) using a standard probe of H_2O. This standard can be related to the magnetic moment of a free proton by

$$B = \frac{\hbar\omega_p}{2\mu_p} \,,$$

where ω_p is the Larmor spin precession angular velocity of a proton in water. Using this, the frequency ω_a and $\mu_\mu = (1 + a_\mu)\, e\hbar/(2m_\mu c)$, one obtains

$$a_\mu = \frac{R}{\lambda - R}$$

where

$$R = \omega_a/\omega_p \quad \text{and} \quad \lambda = \mu_\mu/\mu_p \,.$$

The quantity λ appears because the value of the muon mass m_μ is needed, and also because the B field measurement involves the proton mass m_p.

Measurements of the microwave spectrum of ground state muonium (μ^+e^-) at LAMPF at Los Alamos, in combination with the theoretical prediction of the Muonium hyperfine splitting $\Delta\nu$ (and references therein), have provided the precise value CODATA 2011: [raXiv:1203.5425v1]

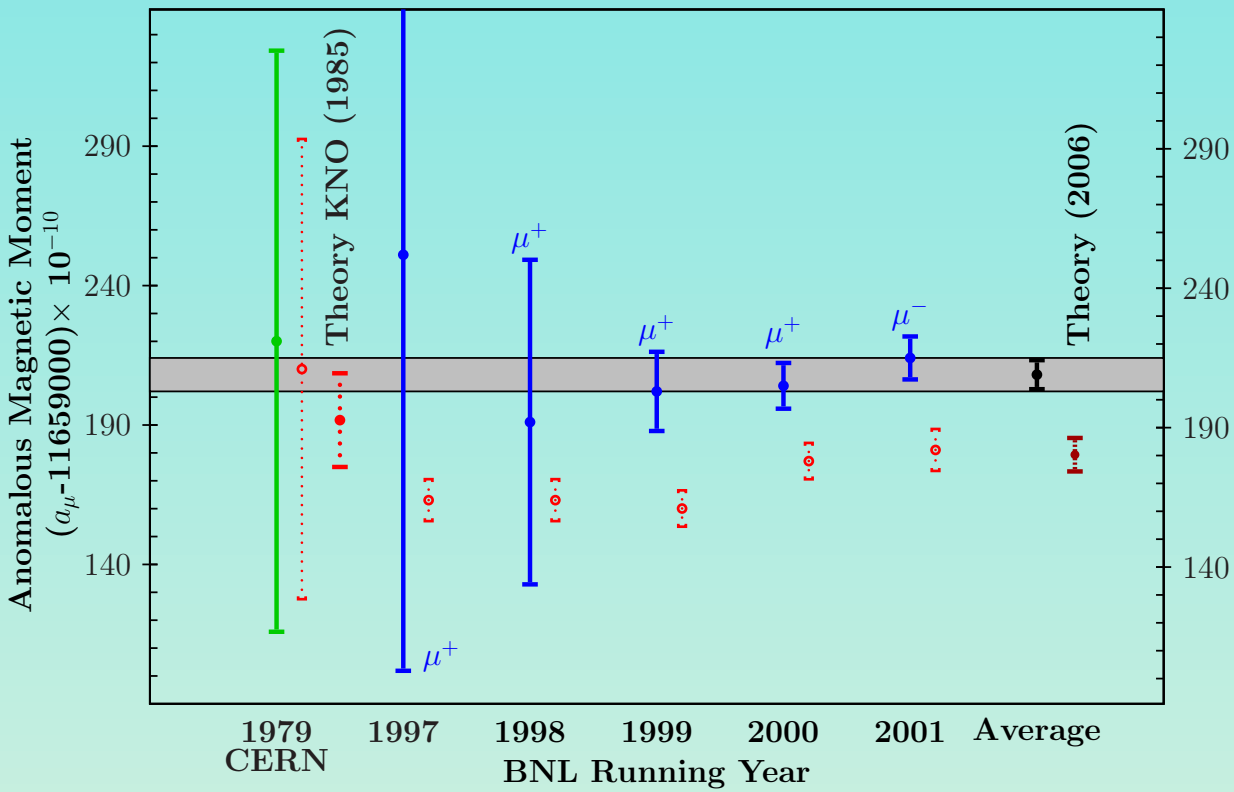
$$\frac{\mu_\mu}{\mu_p} = \lambda = 3.183\,345\,107(84)\ (25\ \text{ppb})\ ,$$

Since the spin precession frequency can be measured very well, the precision at which $g-2$ can be measured is essentially determined by the possibility to manufacture a constant homogeneous magnetic field $\vec{B}$ and to determine its value very precisely.

Final BNL determined $R = 0.0037072063(20)$, which yields new world average value

$$\mathbf{a}_\mu \quad = \quad \mathbf{11659209.1(5.4)(3.3)[6.3]} \times \mathbf{10^{-10}}\ ,$$

with a relative uncertainty of 0.54 ppm.



Results of individual E821 measurements, together with last CERN result and theory values quoted by the experiments

Standard Model Prediction for a_μ

What is new?

- new CODATA values for lepton mass ratios m_μ/m_e, m_μ/m_τ
- spectacular progress by Aoyama, Hayakawa, Kinoshita and Nio on 5–loop QED calculation (as well as improved 4–loop results) a number of leading terms checked analytically by Kataev!
  - ❒ $O(\alpha^5)$ electron $g-2$, substantially more precise $\alpha(a_e)$
  - ❒ Complete $O(\alpha^5)$ muon $g-2$, settles better the QED part

❒ QED Contribution

The QED contribution to a_μ has been computed through 5 loops

Growing coefficients in the α/π expansion reflect the presence of large $\ln\frac{m_\mu}{m_e} \simeq 5.3$ terms coming from electron loops. Input:

$a_e^{\rm exp} = 0.001\,159\,652\,180\,73(28)$ Gabrielse et al. 2008

$\alpha^{-1}(a_e) = 137.0359990842(331)(120)(370)(20)[0.37{\rm ppb}]$ Gabrielse et al 2007

$\alpha^{-1}(a_e) = 137.0359991657(331)(068)(046)(24)[0.25{\rm ppb}]$ Aoyama et al 2012

New: includes the universal 5-loop QED result for the first time!
Errors: from a_e input, α^4, α^5, hadronic
Used is SM prediction:

$$a_e^{\rm SM} \quad = \quad a_e^{\rm QED} + 1.691(13) \times 10^{-12} \text{ (hadronic \& weak)}\,.$$

dominated by LO hadronic: $a_e^{\rm had} = 1.652(13) \times 10^{-12}$, $a_e^{\rm weak} = 0.039 \times 10^{-12}$

$$a_\mu^{\rm QED} = 116\,584\,718.851\ \underbrace{(0.029)}_{\alpha_{\rm inp}}\ \underbrace{(0.009)}_{m_e/m_\mu}\ \underbrace{(0.018)}_{\alpha^4}\ \underbrace{(0.007)}_{\alpha^5}[0.36] \times 10^{-11}$$

The current uncertainty is well below the $\pm 60 \times 10^{-11}$ experimental error from E821

# n of loops	$C_i\ [(\alpha/\pi)^n]$	$a_\mu^{\rm QED} \times 10^{11}$
1	+0.5	116140973.289 (43)
2	+0.765 857 426(16)	413217.628 (9)
3	+24.050 509 88(32)	30141.9023 (4)
4	+130.8796(63)	381.008 (18)
5	+753.290(1.04)	5.094 (7)
tot		116584718.851 (0.036)

❶ 1 diagram Schwinger 1948

❷ 7 diagrams Peterman 1957, Sommerfield 1957

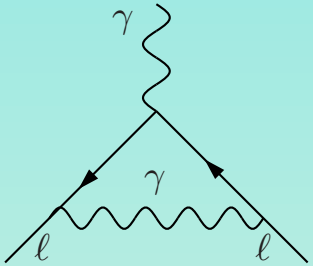
❸ 72 diagrams Lautrup, Peterman, de Rafael 1974, Laporta, Remiddi 1996

❹ about 1000 diagrams Kinoshita 1999, Kinoshita, Nio 2004, Ayoama et al. 2009/2012

❺ estimates of leading terms Karshenboim 93, Czarnecki, Marciano 00, Kinoshita, Nio 05

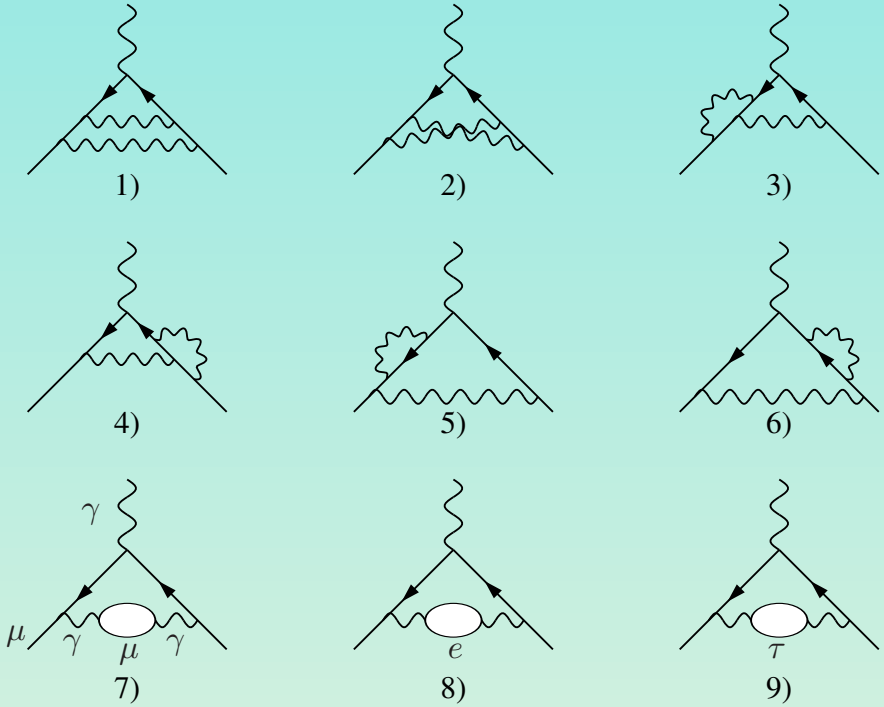
- all 12672 diagrams (fully automated numerical) Ayoama et al. 2012

Universal contributions: a_μ internal muons loops only



$$a^{(2)}_{\ell\,\text{universal}} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right)$$

Schwinger 1948



$$a^{(4)}_{\ell\,\text{universal}}[1-7] = \left[\frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3)\right]\left(\frac{\alpha}{\pi}\right)^2$$

Peterman 57, Sommerfield 57

compact dispersive calculation by Terentev 1962

Universal 3–loop contribution:
(Remiddi et al., Remiddi, Laporta 1996 [27 years for 72 diagrams])

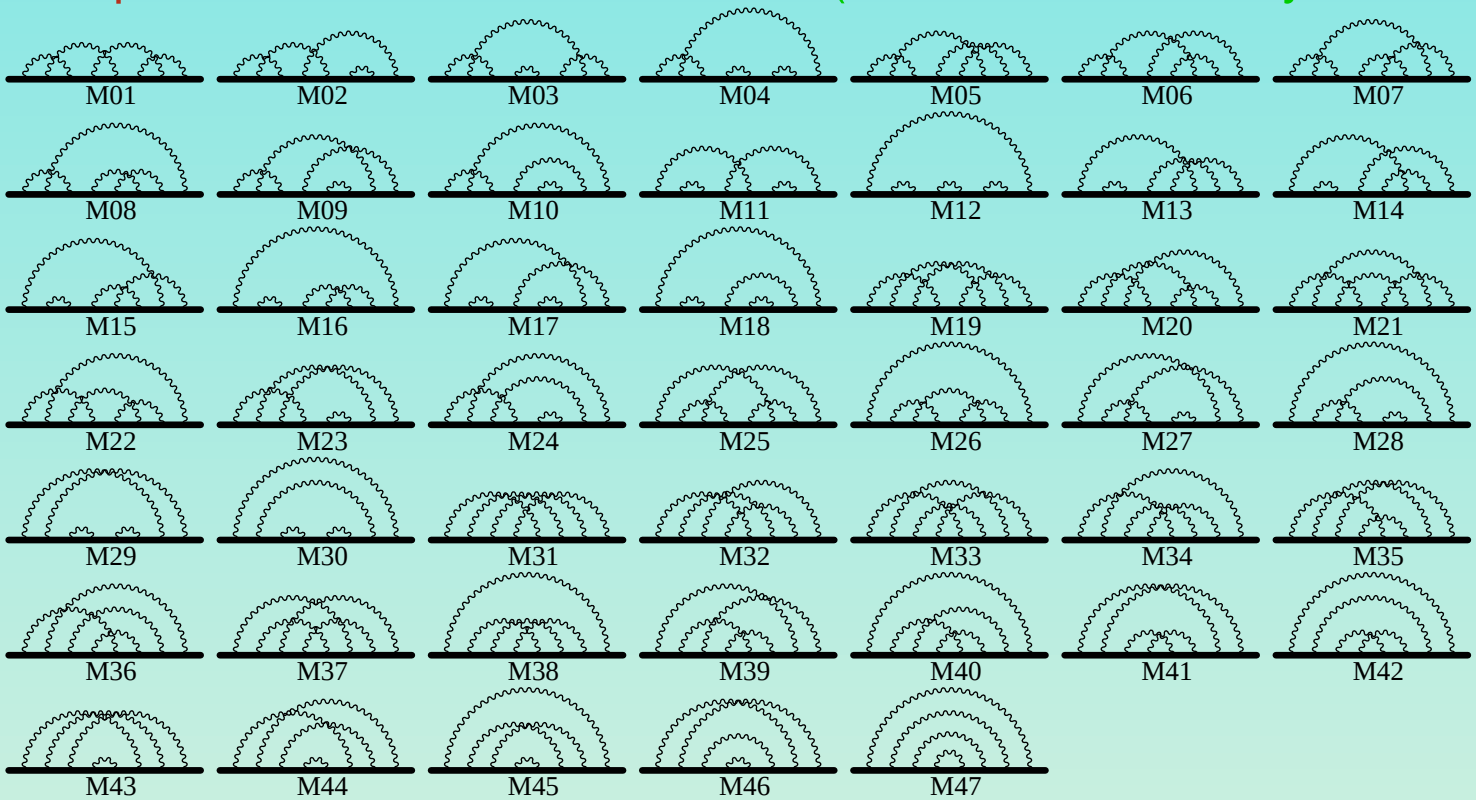
Result turned out to be surprisingly compact

$$a^{(6)}_{\ell\,\text{universal}} = \Big[\frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2 \ln 2 + \frac{139}{18}\zeta(3) + \frac{100}{3}\left\{\mathrm{Li}_4(\frac{1}{2}) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2 \ln^2 2\right\} - \frac{239}{2160}\pi^4 + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5)\Big]\left(\frac{\alpha}{\pi}\right)^3$$

Laporta & Remiddi 96

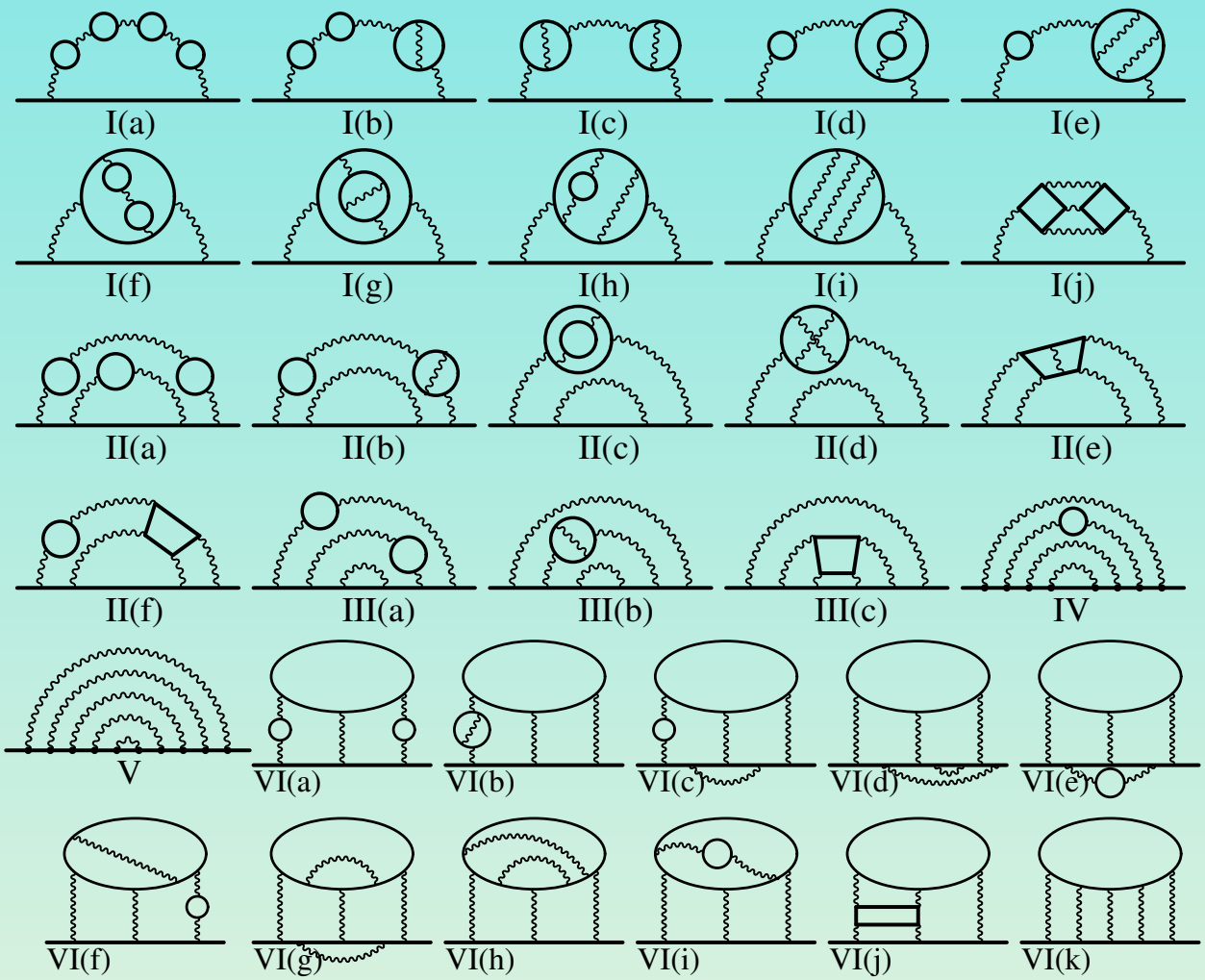
a monument!

Note on 4–loop contribution: (Kinoshita et al., Aoyama et al. 2007)



4-loop Group V diagrams. 47 self-energy-like diagrams of $M_{01} - M_{47}$ represent 518 vertex diagrams [by inserting the external photon vertex on the virtual muon lines in all possible ways].

30 years of heroic effort and succesful improvements!.

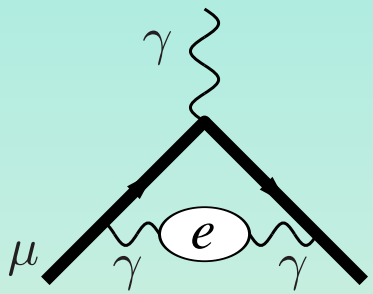


First complete 5-loop calculation! (Aoyama et al. 2012)

Mass dependent contributions:

electron and tau loops bringing in mass ratios me/m_μ and m_μ/m_τ

– LIGHT internal masses $\Rightarrow$ large logarithms [of mass ratios] singular in the limit $m_{\rm light} \to 0$



$$a_\mu^{(4)}(\mathrm{vap}, e) = \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + O\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2 .$$

note large log $\ln\frac{m_\mu}{m_e} \simeq 5.3$

exact two–loop result [errors due to uncertainty in mass ratio (m_e/m_μ)]

$$a_\mu^{(4)}(\mathrm{vap}, e) \simeq 1.094\,258\,3111(84)\left(\frac{\alpha}{\pi}\right)^2 = 5.90406007(5)\times 10^{-6} .$$

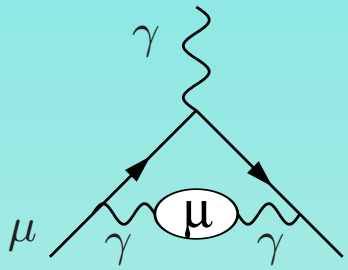
LL UV log; m_μ serves as UV cut–off, electron mass as IR cut–off, relevant integral

$$\int_{m_e}^{m_\mu} \frac{\mathrm{d}E}{E} = \ln \frac{m_\mu}{m_e}$$

may be obtained by renormalization group replace in one–loop result $\alpha \to \alpha(m_\mu)$

$$a_\mu = \frac{1}{2}\frac{\alpha}{\pi}\,(1 + \frac{2}{3}\frac{\alpha}{\pi}\,\ln\frac{m_\mu}{m_e})$$

– EQUAL internal masses yields pure number

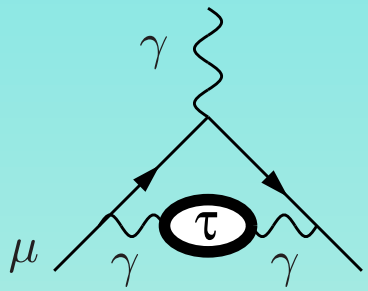


$$a_\mu^{(4)}(\mathrm{vap},\mu) = \left[\frac{119}{36} - \frac{\pi^2}{3}\right] \left(\frac{\alpha}{\pi}\right)^2 ,$$

large cancellation between rational [3.3055...] and transcendental π^2 term [3.2899...], result 0.5% of individual terms:

$$a_\mu^{(4)}(\mathrm{vap},\mu) \simeq 0.015\,687\,4219 \left(\frac{\alpha}{\pi}\right)^2 = 8.464\,1332 \times 10^{-8} .$$

– HEAVY internal masses decouple in the limit $m_{\mathrm{heavy}} \to \infty$, small power correction



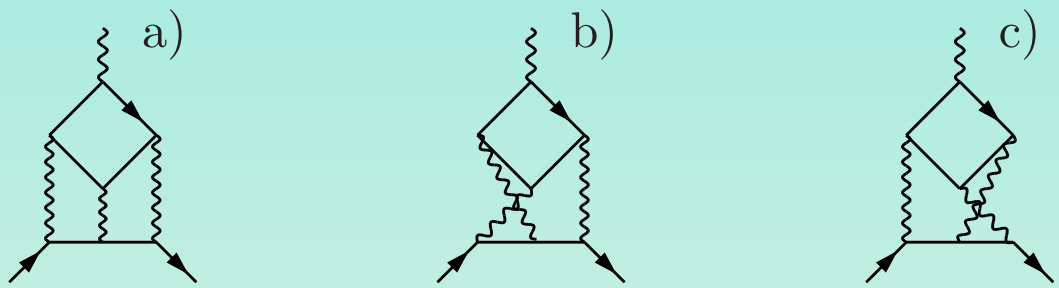
$$a_\mu^{(4)}(\mathrm{vap},\tau) = \left[\frac{1}{45}\left(\frac{m_\mu}{m_\tau}\right)^2 + O\left(\frac{m_\mu^4}{m_\tau^4}\ln\frac{m_\tau}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2 .$$

Note “heavy physics” contributions, from mass scales $M \gg m_\mu$, typically are proportional to m_μ^2/M^2. This means that besides the order in α there is an extra suppression factor, e.g. $O(\alpha^2) \to Q(\alpha^2\frac{m_\mu^2}{M^2})$ in our case. To unveil new heavy states thus requires a corresponding high precision in theory and experiment. τ contribution tiny

$$a_\mu^{(4)}(\mathrm{vap},\tau) \simeq 0.000\,078\,064(25)\left(\frac{\alpha}{\pi}\right)^2 = 4.211\,935\,34(87)\times 10^{-10},$$

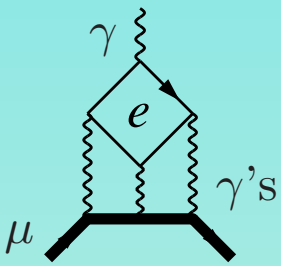
Light-by-Light scattering contribution to $g-2$

6 diagrams related by permutation of photon lines attached to muon:



Again, different regimes:

– LIGHT internal masses also in this case give rise to potentially large logarithms of mass ratios which get singular in the limit $m_{\rm light} \to 0$

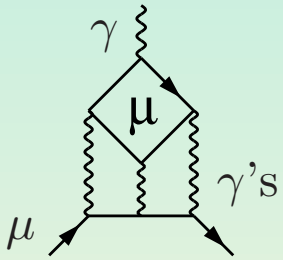


$$a_\mu^{(6)}(\mathrm{lbl}, e) = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \frac{59}{270}\pi^4 - 3\,\zeta(3) - \frac{10}{3}\pi^2 + \frac{2}{3} + O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^3 .$$

Again a light loop which yields a unexpectedly large contribution

$$a_\mu^{(6)}(\mathrm{lbl}, e) \simeq 20.947\,924\,89(16)\left(\frac{\alpha}{\pi}\right)^3 = 2.625\,351\,02(2)\times 10^{-7} .$$

– EQUAL internal masses case which yields a pure number which is usually included in the $a_\ell^{(6)}$ universal part:



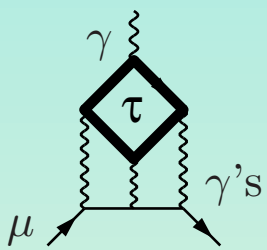
$$a_\mu^{(6)}(\mathrm{lbl}, \mu) = \left[\frac{5}{6}\,\zeta(5) - \frac{5}{18}\,\pi^2\,\zeta(3) - \frac{41}{540}\pi^4 - \frac{2}{3}\pi^2\ln^2 2 + \frac{2}{3}\ln^4 2 + 16a_4 - \frac{4}{3}\,\zeta(3) - 24\pi^2\ln 2 + \frac{931}{54}\pi^2 + \frac{5}{9}\right]\left(\frac{\alpha}{\pi}\right)^3 ,$$

where a_4 is a known constant. The single scale QED contribution is much smaller

$$a_\mu^{(6)}(\mathrm{lbl},\mu) \simeq 0.371005293 \left(\frac{\alpha}{\pi}\right)^3 = 4.64971651 \times 10^{-9}$$

but is still a substantial contributions at the required level of accuracy.

– HEAVY internal masses again decouple in the limit $m_{\mathrm{heavy}} \to \infty$ and thus only yield small power correction

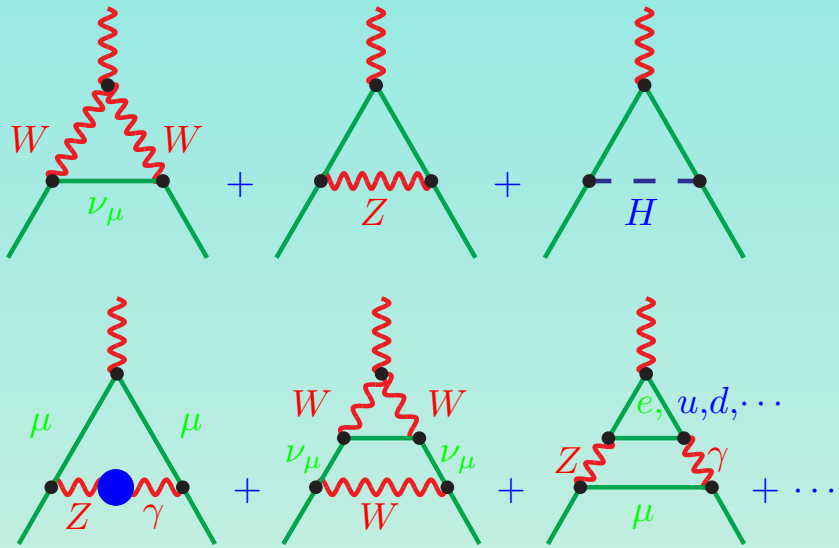


$$a_\mu^{(6)}(\mathrm{lbl},\tau) = \left[\left[\frac{3}{2}\,\zeta(3) - \frac{19}{16}\right]\left(\frac{m_\mu}{m_\tau}\right)^2 + O\left(\frac{m_\mu^4}{m_\tau^4}\ln^2\frac{m_\tau}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^3 .$$

As expected this heavy contribution is power suppressed yielding

$$a_\mu^{(6)}(\mathrm{lbl},\tau) \simeq 0.002\,142\,90(69) \left(\frac{\alpha}{\pi}\right)^3 = 2.685\,65(86) \times 10^{-11} .$$

❒ Weak contributions



Brodsky, Sullivan 67, ...,
Bardeen, Gastmans, Lautrup 72
Higgs contribution tiny!
$a_\mu^{\text{weak(1)}} = (194.82 \pm 0.02) \times 10^{-11}$

Kukhto et al 92
potentially large terms $\sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$
Peris, Perrottet, de Rafael 95
quark-lepton (triangle anomaly) cancellation
Czarnecki, Krause, Marciano 96

Heinemeyer, Stöckinger, Weiglein 04, Gribouk, Czarnecki 05 full 2–loop result

Most recent evaluations: improved hadronic part (beyond QPM)

$$a_\mu^{\text{weak}} = (154.0 \pm 1.0[\text{had}] \pm 0.3[\text{m}_\text{H}, \text{m}_\text{t}, 3-\text{loop}]) \times 10^{-11}$$

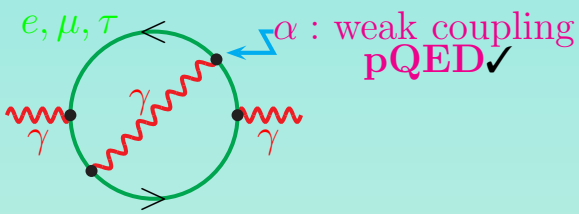
(Knecht, Peris, Perrottet, de Rafael 02, Czarnecki, Marciano, Vainshtein 02)

❒ Hadronic contributions

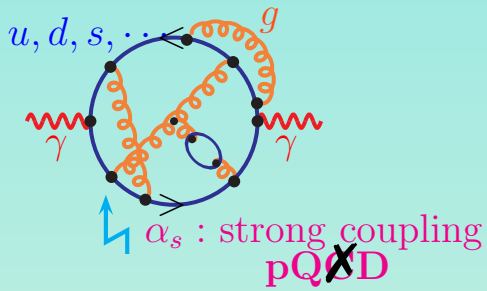
General problem in electroweak precision physics:

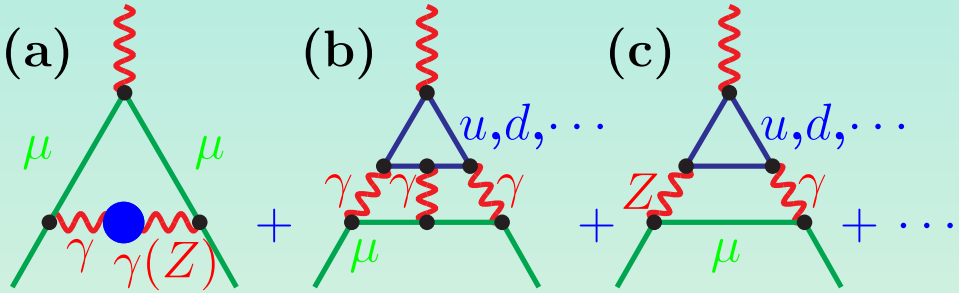
contributions from hadrons (quark loops) at low energy scales

Leptons



Quarks





(a) Hadronic vacuum polarization $O(\alpha^2), O(\alpha^3)$

(b) Hadronic light-by-light scattering $O(\alpha^3)$

(c) Hadronic effects in 2-loop EWRC $O(\alpha G_F m_\mu^2)$

Light quark loops

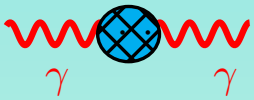
↓

Hadronic “blobs”

Hadronic vacuum polarization effects in $g-2$ [quark loops]

Role of hadronic two point correlator (non-perturbative):

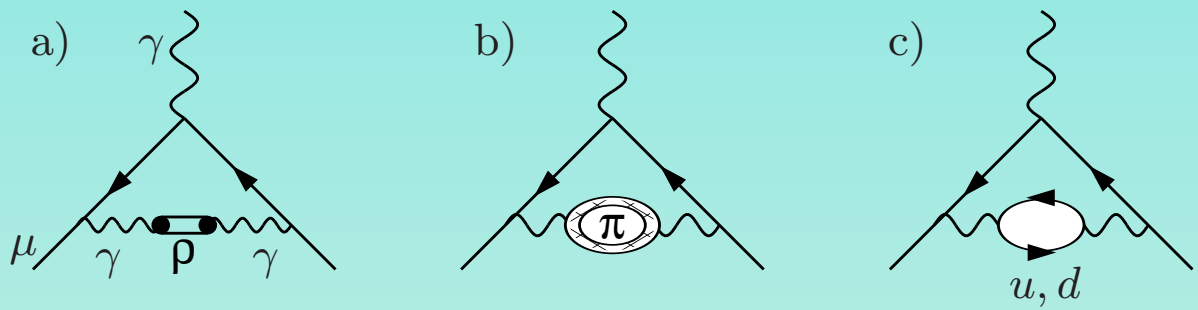
- key object $\langle 0|T\, j_{\rm em}^{\mu\,{\rm had}}(x)\; j_{\rm em}^{\nu\,{\rm had}}(0)|0\rangle$
- hadronic electromagnetic current



$$j_{\rm em}^{\mu\,{\rm had}} = \sum_c \left(\frac{2}{3}\bar{u}_c\gamma^\mu u_c - \frac{1}{3}\bar{d}_c\gamma^\mu d_c - \frac{1}{3}\bar{s}_c\gamma^\mu s_c + \frac{2}{3}\bar{c}_c\gamma^\mu c_c - \frac{1}{3}\bar{b}_c\gamma^\mu b_c + \frac{2}{3}\bar{t}_c\gamma^\mu t_c \right) ,$$

- hadronic part on photon self-energy $\Pi'^{\,{\rm had}}_\gamma(s) \Leftrightarrow \langle 0|j_{\rm em}^{\mu\,{\rm had}}(x)\; j_{\rm em}^{\nu\,{\rm had}}(0)|0\rangle$
- hadronic vacuum polarization due to the 5 "light" quarks $q = u, d, s, c, b$
- top quark [mass $m_t \simeq 173$ GeV] pQCD applies [$\alpha_s(m_t)$ small]
- in fact t is irrelevant by decoupling theorem [heavy particles decouple in QED/QCD], t like τ VP loop extra factor $N_c Q_t^2 = 4/3$:

Low energy effective theory: e.g. CHPT here equivalently scalar QED of pions



Low energy effective graphs a) [ρ-exchange] and b) [π-loop] and high energy graph c) [quark-loops]

Low energy effective estimates of the leading VP effects $a_\mu^{(4)}(\text{vap}) \times 10^8$
For comparison: 5.8420 for μ–loop, 590.41 for e–loop

data [280,810] MeV	ρ^0-exchange	$\pi^\pm$-loop	(u,d)-loops
4.2666	4.2099	1.4154	$2.2511[449.25]^*$

* current quarks: $m_u \sim 3\text{MeV}, m_d \sim 8\text{MeV}$

Often resorting to QPM using effective “constituent quark masses” [concept not

well-defined] e.g. $m_u \sim m_d \sim 300$ MeV (about 1/3 of the proton mass) one gets 2.2511×10^{-8} (ambiguous)

Quark and pion loops fail: missing is the pronounced ρ^0 spin 1 resonance $e^+e^- \to \rho^0 \to \pi^+\pi^-$ almost saturates the result based on dispersion relation and e^+e^--data.

Lesson:

- pQCD fails; QPM result arbitrary (quark masses)
- ChPT (only knows pions) fails; reason only converge for $p \lesssim 400$ MeV
- dominating is spin 1 resonance ρ^0 at $\simeq 775$ MeV (VDM); cries for large N_c QCD
- lattice QCD now on the way to solve the problem once one can simulate at physical quark masses
- resort on sum rule type semi-phenomenological approach Dispersion Relations (DR) and experimental data.

Dispersion relations and VP insertions in $g-2$

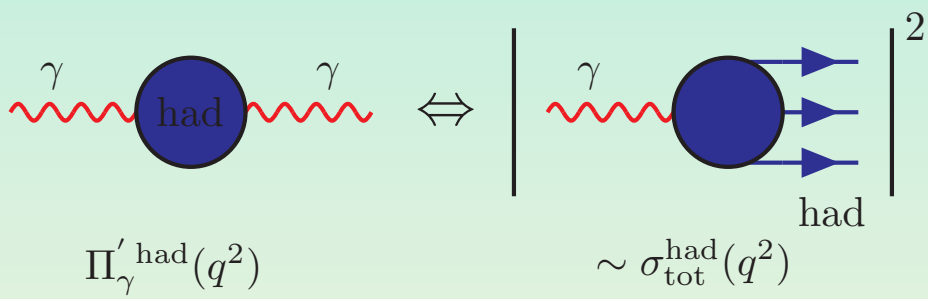
Starting point:

❒ *Optical Theorem* (unitarity) for the photon propagator

$$\mathrm{Im}\Pi'_\gamma(s) = \frac{s}{4\pi\alpha}\,\sigma_{\mathrm{tot}}(e^+e^- \to \text{anything})$$

❒ *Analyticity* (causality), may be expressed in form of a so–called (subtracted) dispersion relation

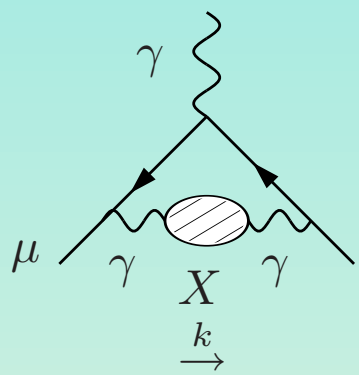
$$\Pi'_\gamma(k^2) - \Pi'_\gamma(0) = \frac{k^2}{\pi}\int_0^\infty \mathrm{d}s \frac{\mathrm{Im}\Pi'_\gamma(s)}{s\,(s-k^2-\mathrm{i}\varepsilon)} \quad .$$



– based on general principles

– holds beyond perturbation theory

Use of DRs in g-2 calculations, prototype example: diagram of the type



"blob" = full photon propagator $g^{\mu\nu}$ term of the full photon propagator, carrying loop momentum k, reads

$$\frac{-\mathrm{i}g^{\mu\nu}}{k^2\,(1+\Pi'_\gamma(k^2))} \simeq \frac{-\mathrm{i}g^{\mu\nu}}{k^2}\left(1-\Pi'_\gamma(k^2)+\left(\Pi'_\gamma(k^2)\right)^2-\cdots\right)$$

and the renormalized photon self–energy may be written as

$$-\frac{\Pi'_{\gamma\,\mathrm{ren}}(k^2)}{k^2} = \int_0^\infty \frac{\mathrm{d}s}{s}\frac{1}{\pi}\mathrm{Im}\,\Pi'_\gamma(s)\,\frac{1}{k^2-s}\,.$$

- k dependence under the convolution integral shows up in free propagator only
- free photon propagator in next higher order is replace by

$$-\mathrm{i}g_{\mu\nu}/k^2 \to -\mathrm{i}g_{\mu\nu}/(k^2-s)$$

  = exchange of a "massive photon" of mass square s.

- afterwards convoluted with imaginary part of the photon vacuum polarization

– calculate the contributions from the massive photon analytically

– this is possible to 3 loops in QED

The leading order result is

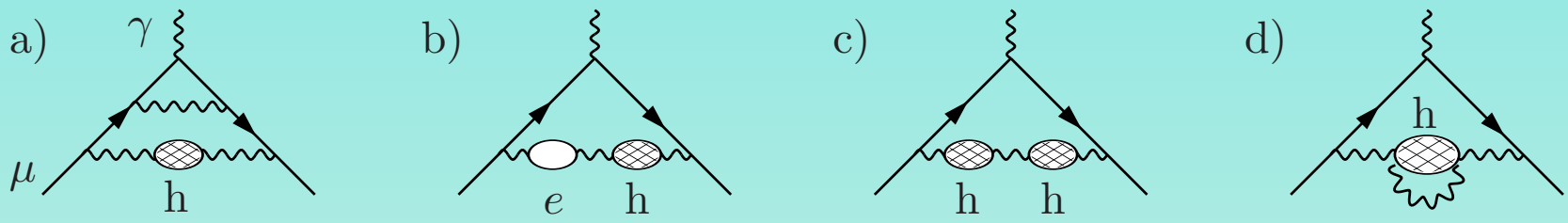
$$K_\mu^{(2)}(s) \equiv a_\mu^{(2)\,\text{heavy}\,\gamma} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\,\frac{x^2\,(1-x)}{x^2+(s/m_\mu^2)(1-x)}$$

second order contribution to a_μ from an exchange of a photon with square mass s ($s = 0$ Schwinger result).

The contribution from the "blob" to $g-2$ then reads

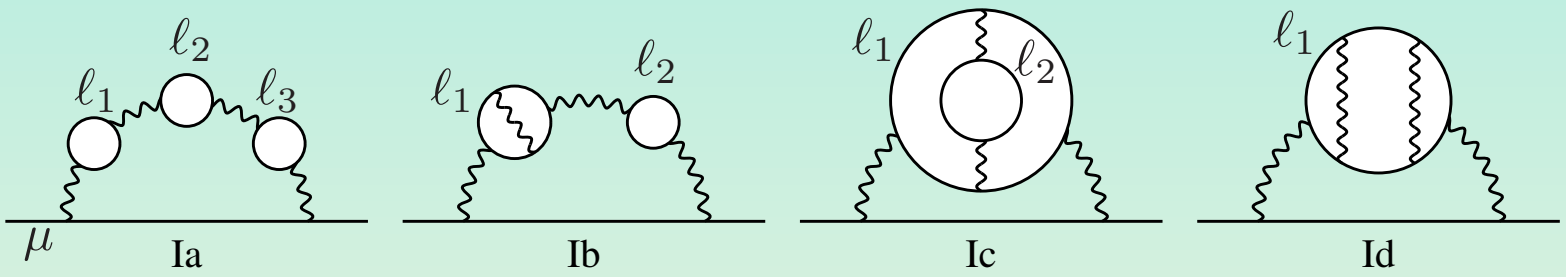
$$a_\mu^{(X)} = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}s}{s}\,\mathrm{Im}\,\Pi_\gamma^{\prime(X)}(s)\,K_\mu^{(2)}(s)\,.$$

“Trick” applies to higher order hadronic VP contributions



Kinoshita, Nizic, Okamoto 1985, Krause 1996, ...

as well as to analytic calculations of higher order diagrams like



3–loop: Hoang et al 95, 4–loop: Broadhurst, Kataev, Tarasov 93, Kinoshita et al