Summary Lectures 1 to 7

1. Cosmology as a observational science

   Olbers Paradox, Cosmological Principle, Hubble Expansion, Age of the Universe, Nuclear Age Dating

2. Spectral fingerprints, Cosmic Distance Scales, Variable stars, Birth and Dead of stars, Hertzsprung Russell Sequence (Stellar Evolution). The solar Neutrino problem and how stars produce energy

3. Modern Cosmology, Cosmological principle, Gravitation and Geometry, Hubble's law and the universal expansion and the Big Bang, observational Horizons, Sketch of cosmological evolution

4. Special relativity in a Nutshell, Summary and concepts of general relativity, Status of experimental tests, Gravitational Lensing, Black Holes

5. Cosmological Models, Robertson-Walker Universes, Weyl's postulate and the cosmic time, the cosmic scale factor $S(t)$, Measuring distances in curved space, Cosmological observables: Hubble constant $H(t)$ and deceleration parameter $q(t)$

6. Relativistic Hydrodynamics, The Energy-Momentum Tensor of the universes, dynamics of the universe: Solving Einstein's Equation for the isotropic Universe: Friedmann's Equations, Equations of state and energy balance. Simplest examples: radiation and matter denominated Universes.

7. Detailed discussion of the solutions of Friedmann's equations. Big Bang Cosmology, Cosmic Black-body radiation, Einstein-de Sitter Universe as a limiting case, critical matter density

Radiation and Matter

Thermal History

A more realistic consideration of the evolution of the state of the universe requires to take into account both radiation and matter as well as other contributions to the true energy-momentum tensor of the universe. In this lecture we consider radiation and matter in more detail. In view of the structure of the Friedmann equation

$$\boxed{\frac{\dot{S}^2}{c^2} = \frac{c_1}{S^2} + \frac{c_2}{S} - k + \frac{\Lambda}{3} S^2}$$

the evolution is expected to be dominated

t small	radiation $\sim \frac{c_1}{S^2}$
t medium	matter $\sim \frac{c_2}{S}$
t large	curvature or cosmological constant $\sim -k + \frac{\Lambda}{3} S^2$

In the Big Bang case $S(t) \to 0 \;\; (t \to 0)$.

The simplest possible model is that of an ideal gas in thermal equilibrium together with the radiation:
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$$
\begin{aligned}
p &= n\,k_{\mathrm{B}}\,T + \frac{1}{3}\,a\,T^4 \\
\rho/c^2 &= n\,m + (\gamma-1)^{-1}\,n\,k_{\mathrm{B}}\,T + a\,T^4
\end{aligned}
$$

n	number of gas atoms	;	m mass of a gas atom
k_{B}	Boltzmann constant	;	a Stefan-Boltzmann constant
γ	specific heat per gas atom	;	$\gamma = 5/3$ single atom gas

Note: the crucial point why we can apply equilibrium thermodynamics to the expanding universe, which is **not** in equilibrium itself!, is the fact that the expansion is slow relative to characteristic particle interaction times. i.e. process rates are large enough to contiguously restore equilibrium in comoving volumes. With other words, at each stage in general there was plenty of time available for equilibrium to be reached. Interaction rates are energy dependent and the energy is determined by the energy density which is decreasing due to the expansion. Typically, below some threshold interaction rates for specific processes drop to zero, such that subsystems may follow their own equilibrium distributions once they were decoupled (Examples: photon CMB, neutrino CB, see below)

Particle number conservation: $n\,S^3 = n_0\,S_0^3$

Energy balance:
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$$\frac{\mathrm{d}}{\mathrm{d}S}\left\{n\,m\,S^3 + (\gamma-1)^{-1}\;n\,k_\mathrm{B}\,T\,S^3 + a\,T^4\,S^3\right\} = -3\,n\,k_\mathrm{B}\,T\,S^2 - a\,T^4\,S^2\,.$$

With $n\,S^3 = \mathrm{constant}$ we obtain

$$\frac{S}{T}\,\frac{\mathrm{d}T}{\mathrm{d}S} = -\frac{\sigma+1}{\sigma+\frac{1}{3}\,(\gamma-1)^{-1}}$$

where σk_B is the photon entropy per gas atom:

$$\sigma \doteq \frac{4\,a\,T^3}{3\,n\,k_\mathrm{B}} \quad \triangleright \quad n = \frac{4\,a\,T^3}{3\,k_\mathrm{B}\,\sigma}$$

Boarder cases: i) little radiation: $\sigma \ll 1$

$$\underline{T = T_0 \left(\frac{S}{S_0}\right)^{-3\,(\gamma-1)}}$$

is the usual form of adiabatic expansion of an ideal gas.

ii) much radiation: $\sigma \gg 1$

$$\underline{T = T_0 \left(\frac{S}{S_0}\right)^{-1}}$$

"hot universe" situation. This is the <u>stable case</u>, if σ ever is big it remains big

$$\sigma = \frac{4\,a\,T_0^3\,\left(\frac{S}{S_0}\right)^{-3}}{3\,n_0\,k_{\rm B}\left(\frac{S_0}{S}\right)^3} = \frac{4\,a\,T_0^3}{3\,n_0\,k_{\rm B}} \gg 1 \quad \checkmark$$

As long as radiation and matter are in equilibrium: $T_{\rm mat}(t) = T_{\rm rad}(t)$ i.e.

$$T_{\rm mat} = T_{\rm rad} = T_{0\gamma}\left(\frac{S}{S_0}\right)^{-1} .$$

As long as $\sigma \gg 1$ we have

$$\left.\begin{array}{lcl} p & = & \frac{a}{3}\left(\frac{4}{\sigma}+1\right)T^4 \simeq \frac{a}{3}T^4 \\ \rho/c^2 & = & a\left(\frac{4\,m}{3\,\sigma\,k_{\rm B}\,T}+\frac{4}{3\,(\gamma-1)\,\sigma}+1\right)T^4 \simeq a\,T^4 \end{array}\right\} p \simeq \frac{\rho}{3}$$

In the following we assume to have : $\sigma \gg 1$

"Hot Universe"

- early universe: $T_{\rm mat} = T_{\rm rad} = T_{0\gamma}\left(\frac{S}{S_0}\right)^{-1}$
  $\sigma = \frac{4\,a\,T^3}{3\,k_{\rm B}\,n} = \text{constant} \gg 1\ ;\quad p = \rho/3$

- Big-bang solution universal for small t!
  $S(t) \simeq \left(4\,\bar{C}\right)^{1/4}(ct)^{1/2}$ respectively $\underline{S = S_0\,(t/t_0)^{1/2}}$
  independent of $k, \Lambda,$ etc.

$$\triangleright \qquad T = T_0\left(\frac{S_0}{S}\right) \to \infty \quad (t \to 0)$$

$$\rho = \rho_{\rm rad} = \rho_{0,\rm rad}\left(\frac{S_0}{S}\right)^4 \to \infty \quad (t \to 0)$$

Fireball: T, ρ extremely big

At some point (Planck Temperature) classical picture must break down!

- isotropic background radiation
  $T_{0\gamma}$, $\rho_{0\gamma}$ black body spectrum at any time

$$\boxed{\rho_{0\gamma} = a\, T_{0\gamma}^4 \cdot c^2}$$

$$T_{0\gamma} \sim 2.725\ ^\circ\mathrm{K}\ ,\ \rho_{0\gamma} \simeq 4.64 \times 10^{-34}\ \mathrm{gr/cm^3}\ \cdot c^2$$

State of matter: Density inside nuclei is $\rho^* = \rho_{\mathrm{nuclei}} \simeq 2 \times 10^{15}\ \mathrm{gr/cm^3}\ \cdot c^2$

Corresponding radiation temperature is $\rho^* = \rho_{0\gamma} \left(\frac{T^*}{T^*_{0\gamma}}\right)^4 \ \triangleright\ \ T^* \sim 5.4 \times 10^{11}\ ^\circ\mathrm{K}$

Which means that for $T > T^* \simeq 10^{12}\ ^\circ\mathrm{K}$ transition to quark-gluon plasma takes place. This we will consider later.

For $T \leq T^* \simeq 10^{12}\ ^\circ\mathrm{K}$ one can understand the thermal history of the universe rather well, under the following conditions

- isotropic background radiation is truly the relict of the Big-Bang fireball
- universe is hot: $\sigma \gg 1$

which we take for granted (check for self-consistency).

By the expansion the universe cools down. As long as the photon energy $E_\gamma = \mathrm{h}\,\nu_\gamma = k_\mathrm{B}\,T_\gamma$ is bigger than the rest mass $m_i\,c^2$ of an elementary particle of species i a permanent creation and annihilation of particle-antiparticle pairs takes place:

$$\gamma + \gamma \leftrightarrow e^+e^-, \mu^+\mu^-, p\bar{p}, n\bar{n}, \cdots \ ,$$

such that a plasma in thermal equilibrium exists. We assume here that charge Q, baryon-number B and lepton-number L are conserved and that in average there is

"neutrality"

$$n_Q = n_B = n_L = 0 \; .$$

This means that in the thermal equilibrium the chemical potentials vanish: $\mu_i = 0$.

Chemical potential is conjugate to particle number (concentration),
like temperature is conjugate to entropy and pressure to the volume.

The equilibrium distributions for Fermions/Bosons are given by

$$n_i(q) = 4\,\pi\,\mathrm{h}^{-3}\,g_i\,q^2\,\frac{1}{\left(\exp\left(\frac{E_i(q)}{k_\mathrm{B}T}\right) \pm 1\right)} \; ; \quad \pm \begin{array}{l}\text{fermions}\\ \text{bosons}\end{array}$$

q	:	modulus of momentum
g_i	:	number of spin states
$E_i(q) = \left(m_i^2 + q^2\right)^{1/2}$	:	for relativistic particles

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Comments concerning the chemical potentials μ:
Thermodynamics: the change in internal energy U due to a change in particle number (species i)

$$\mathrm{d}U = \sum_i \mu_i \, \mathrm{d}N_i \; .$$

Thus

$$\mu_i = \left(\frac{\partial U}{\partial N_i}\right)_{\mathscr{S},V} = -T \left(\frac{\partial \mathscr{S}}{\partial N_i}\right)_{U,V}$$

at constant entropy $\mathscr{S}$ or constant internal energy U, respectively, and constant volume V. Note: internal energy $U = U(\mathscr{S}, V, N_1, \cdots, N_r)$

$$\mathrm{d}U = T \, \mathrm{d}\mathscr{S} - p \, \mathrm{d}V + \sum_i \mu_i \, \mathrm{d}N_i \; ,$$

for thermal, mechanical [work] and chemical reaction energy.

- Thermal equilibrium: $T = \left(\frac{\partial U}{\partial \mathscr{S}}\right)_{V,N} \text{constant}$,

- Mechanical equilibrium: $p = -\left(\frac{\partial U}{\partial V}\right)_{\mathscr{S},N} \text{constant}$,

- Chemical equilibrium for species i: $\mu_i = \left(\frac{\partial U}{\partial N_i}\right)_{\mathscr{S},V,N_{j\neq i}} \text{constant}$

In the early universe the different species of elementary particles made up the "chemical composition" of the state of matter and radiation.

1) Photons: can be emitted and absorbed without constraint, i.e., n_γ is arbitrary, such that $\mu_\gamma \equiv 0$; $n_\gamma = \rho_\gamma / \mathrm{h}\nu_\gamma$; $q = E_\gamma = \mathrm{h}\nu_\gamma$

   ▷ Planck distribution

2) Particle-antiparticle creation and annihilation:

$$\gamma + \gamma \leftrightarrow X + \bar{X} \; ;$$

in thermal equilibrium

$$\triangleright \quad \mu_\gamma = 0 = n_X + n_{\bar{X}}$$

hence we always have

$$\boxed{\mu_{\text{antiparticle}} = -\mu_{\text{particle}}}$$

3) In weak processes like

$$e^- + \mu^+ \leftrightarrow \nu_e + \bar{\nu}_\mu \; , \quad e^- + p \leftrightarrow \nu_e + n \;\; \text{etc.}$$

we have

$$\triangleright \quad \mu_{e^-} - \mu_{\nu_e} = \mu_{\mu^-} - \mu_{\nu_\mu} = \mu_n - \mu_p \ \text{ etc.}$$

4) charge, baryon number and lepton number densities

$$\begin{aligned} n_Q &= n_p - n_{\bar{p}} - n_{e^-} + n_{e^+} + \cdots \\ n_B &= n_p + n_n - n_{\bar{p}} - n_{\bar{n}} \quad \text{essentially nucleons only} \\ n_M &= n_{\mu^-} + n_{\nu_\mu} - n_{\mu^+} - n_{\bar{\nu}_\mu} \\ n_E &= n_{e^-} + n_{\nu_e} - n_{e^+} - n_{\bar{\nu}_e} \ . \end{aligned}$$

In the early universe $n_\gamma \gg n_B, n_M, n_E$, equivalent to $\sigma \gg 1$. First approximation

$$\boxed{n_B = n_M = n_E = \cdots = 0}$$

In the approximation of separate lepton-number L_ℓ $(\ell = e, \mu, \tau)$ conservation.

Distributions:
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$$n_i(q) \propto \frac{1}{\exp\left(\frac{E_i-\mu_i}{k_B T}\right)+1} \; ; \;\; \bar{\mu}_i = -\mu_i \;\; \text{particle} \leftrightarrow \text{antiparticle}$$

Hence: n_Q, n_B, n_M, n_E odd function of four independent chemical potentials $\mu_p, \mu_{e^-}, \mu_{\nu_e}, \mu_{\nu_\mu}$, such that

$$n_Q = n_B = n_M = n_E = 0 \rhd \mu_i = 0 \, \forall i \; .$$

In more details: in the very early universe (energies above GUT scales) particle-antiparticle symmetry (pair creation via energetic photons) $n_Q = 0$ for individual particle species like

$$n_{Q_e} = n_{e^+} - n_{e^-} = 0$$

in terms of distributions:

$$n_{e^+}(E) = \frac{1}{\pi^2\hbar^3}\frac{E^2}{e^{\frac{E+\mu_{e^-}}{k_B T}}+1}\ ;\ \ n_{e^-}(E) = \frac{1}{\pi^2\hbar^3}\frac{E^2}{e^{\frac{E-\mu_{e^-}}{k_B T}}+1}$$

$$n_{Q_e} = \int \mathrm{d}E\ (n_{e^+}(E) - n_{e^-}(E)) = \frac{1}{\pi^2}\left(\frac{k_B T}{\hbar}\right)^3 h\left(\frac{\mu_{e^-}}{k_B T}\right)$$

with

$$\boxed{h(x) = \int_0^\infty \mathrm{d}y\, y^2\left[(e^{y+x}+1)^{-1} - (e^{y-x}+1)^{-1}\right]\ .}$$

Thus, $n_{Q_e} = 0$ requires $x = 0$ i.e.

$$\mu_{e^-} = -\mu_{e^+} = 0\ .$$

Furthermore, $n_{Q_e} = 0$ implies $n_E = n_{\nu_e} - n_{\bar{\nu}_e}$ and $n_E = 0$ requires

$$\mu_{\nu_e} = -\mu_{\bar{\nu}_e} = 0 \ .$$

Analogous conclusions follow for n_{Q_μ} and n_M. Note that the conservation of n_X implies

$$\boxed{n_X = n_{X0} \left(\frac{S_0}{S}\right)^3}$$

Back to the early universe: basic dynamical equations

$$\boxed{\frac{\dot{S}^2}{c^2} = \frac{\kappa}{3}\rho S^2}$$

universal for early universe.

Equilibrium thermodynamics:

$$\rho(T) = \sum_i \int E_i(q)\, n_i(q,T)\, \mathrm{d}q$$

$$p(T) = \sum_i \int \left(\frac{q^2}{3\, E_i(q)}\right) n_i(q,T)\, \mathrm{d}q$$

Applying 2nd law of thermodynamics: $\mathscr{S}(V,T)$ entropy

$$
\begin{aligned}
T\,\mathrm{d}\mathscr{S}(V,T) &= \mathrm{d}U + p\mathrm{d}V\ ; \quad U = \rho\cdot V \\
\frac{\partial\mathscr{S}(V,T)}{\partial V} &= \frac{1}{T}\,(\rho(T)+p(T)) \\
\frac{\partial\mathscr{S}(V,T)}{\partial T} &= \frac{V}{T}\,\frac{\mathrm{d}\rho(T)}{\mathrm{d}T}
\end{aligned}
$$

with integrability condition (Maxwell relations)

$$
\begin{aligned}
\frac{\partial\mathscr{S}(V,T)}{\partial V\,\partial T} &= \frac{\partial\mathscr{S}(V,T)}{\partial T\,\partial V} \quad \triangleright \\
\frac{\mathrm{d}p(T)}{\mathrm{d}T} &= \frac{1}{T}\,(\rho(T)+p(T))\ .
\end{aligned}
$$

Taking into account the energy balance equation

$$S^3(t)\,\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[S^3(t)\;(\rho(t)+p(t))\right]$$

altogether we obtain
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$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{S^3(t)}{T(t)}\;(\rho(t)+p(t))\right] = 0}$$

and up to a constant

$$\mathscr{S}(V,T) = \frac{V}{T}\;(\rho(T)+p(T))$$

is the total entropy of the system.

Result: the entropy in a volume $S^3(t)$,
$s \equiv \mathscr{S}\left(S^3(t), T\right) = \frac{S^3(t)}{T(t)}\,(\rho(t) + p(t)) = \text{constant}$
is a constant

This provides an important test for the applicability of the thermal equilibrium properties.

For highly relativistic particles, we have $E_i = q$, such that for the case hat all particles are highly relativistic indeed

$$p(T) = \tfrac{1}{3}\rho(T)$$

and from

$$T\,\frac{\mathrm{d}p}{\mathrm{d}T} = \rho + p \quad \triangleright \quad \rho(T) \propto T^4 \;\text{ and }\; T \propto S^{-1}(t)\,. \;\checkmark$$

In the highly relativistic case we get for the densities of the different particle

species (Fermions) because of $E_i = q$:

$$\rho_i/c^2 = \int_0^\infty \mathrm{d}q\, q\, n_i(q,T)/c^2 = \frac{4\pi}{\mathrm{h}^3\, c^2}\, g_i \int_0^\infty \mathrm{d}q\, q^3 \frac{1}{\exp\left(\frac{q}{k_\mathrm{B}\, T}\right)+1}$$

$$= \frac{7\,\pi^5}{30\,\mathrm{h}^3 c^2}\, g_i\, (k_\mathrm{B}\, T)^4 = \frac{7}{16}\, a\, T^4 \cdot g_i$$

where $a = \frac{\pi^2\, k_\mathrm{B}^4}{15\,\hbar^3\, c^2}$ is the Stefan-Boltzmann constant.

The contributions thus distinguish from each other by the spin multiplicity:

$$\rho_\nu = \tfrac{7}{16}\, a\, T^4 c^2\ ;\quad \nu = \nu_e,\ \bar\nu_e,\ \nu_\mu,\ \bar\nu_\mu,\ \nu_\tau,\ \bar\nu_\tau,\ \text{ more ?}$$

$$\rho_e = 2\,\rho_\nu \ \text{ etc.}$$

whereas

$$\rho_\gamma = a\, T^4 c^2\ .$$

Note: $\rho_\mathrm{fermion} = \frac{7}{8}\,\rho_\mathrm{boson}$ per spin degree of freedom. This is the effect of the Pauli

exclusion principle, which blocks part of the state space for fermions.

Formally, it is the ratio of the integrals for massless fermions and bosons $\int_0^\infty \mathrm{d}y\, y^3/(\exp y \pm 1) = \frac{\pi^4}{15} \cdot \left\{ \begin{array}{c} \frac{7}{8} \\ 1 \end{array} \right.$, respectively.

The entropy is then given by: ($p = \rho/3$)

$$s = \frac{4}{3}\frac{S^3(t)}{T(t)}\rho\ ;\quad \rho = \rho_\gamma + 6\rho_\nu + \rho_{e^-} + \rho_{e^+} + \cdots = \frac{g^*}{2}\rho_\gamma\ .$$

Furthermore, because of $\rho \propto S^{-4}$ it follows

$$\frac{\dot\rho}{\rho} = -4\frac{\dot S}{S} = -4\left(\frac{\kappa\rho}{3}\right)^{1/2}$$

which is Friedmann's equation for $k = 0\,(\Lambda = 0)$,

and therefore

$$\boxed{t = \frac{1}{\left(4\,\frac{\kappa}{3}\,\rho\right)^{1/2}} + \text{constant}\ .}$$

In the following we discuss in more detail the evolution of the universe at temperatures $T \lesssim T^* \simeq 10^{12}\ ^\circ\mathrm{K}$, after the annihilation of the nucleons and after μ-pair annihilation had taken place:

$N + \bar{N} \to \gamma + \gamma\ (N = p, n)$	$: m_p c^2 \simeq 938$ MeV	$\hat{=}$	$1.1 \times 10^{13}\ ^\circ\mathrm{K}$
$\pi^\pm \to \mu^\pm + \nu_\mu, \pi^0 \to 2\gamma$	$: m_\pi c^2 \simeq 139$ MeV	$\hat{=}$	$1.6 \times 10^{12}\ ^\circ\mathrm{K}$
$\mu^+ + \mu^- \to \gamma + \gamma$	$: m_\mu c^2 \simeq 106$ MeV	$\hat{=}$	$1.2 \times 10^{12}\ ^\circ\mathrm{K}$
$e^+ + e^- \to \gamma + \gamma$	$: m_e c^2 \simeq 511$ keV	$\hat{=}$	$5.93 \times 10^{9}\ ^\circ\mathrm{K}$

At this stage the relativistic particles in thermal equilibrium are

$$\gamma, e^\pm, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau\ .$$

The density of non-relativistic particles, in particular relict nucleons now can be regarded as negligible, such that

$$\rho = \frac{g^*}{2}\rho_\gamma \; ; \;\; \rho_\gamma = \rho_{0\gamma}\left(\frac{T}{T_0}\right)^4$$

and

$$t = \frac{1}{\left(2\,\frac{\kappa}{3}\rho_{0\gamma}\right)^{1/2}}\frac{1}{\sqrt{g^*}}\left(\frac{T_0}{T}\right)^2 + \text{constant}$$

$$s = \frac{2}{3}\frac{S_0^3}{T_0}\rho_{0\gamma}\left(\frac{S(t)}{S_0}\right)^3\left(\frac{T}{T_0}\right)^3 \cdot g^*(T)$$

Because the number of relativistic degrees of freedom g^* is a function of T and at the same time the entropy must be conserved the relation

$$S(t)\cdot T(t) = \text{constant}$$

cannot generally be true if not $g^* = \text{constant}$.

In general

$$g^*(T(t))\, S^3(t)\, T^3(t) = \text{constant}$$

holds and expresses the conservation of entropy.

In general, thermal equilibrium is obtained as a result of a large number of collisions between the particles in interaction. In other words, the average time τ_i between two collisions satisfies $t/\tau_i \gg 1$. Highly relativistic particles, in particular the massless photon and the quasi massless neutrinos, however, don't change their equilibrium distribution also if no interaction is left: i.e., highly relativistic particles behave as if they would be in thermal equilibrium with the rest of the system.

The reason may be easily explained by checking the form of the distributions:

$$n_i = \frac{g_i}{2\,\pi^2\hbar^3}\int_0^\infty \frac{\left(E^2 - m_i^2\right)^{1/2}\,E\,\mathrm{d}E}{\mathrm{e}^{\frac{E}{k_\mathrm{B}T}} \pm 1}$$

particle density

$$\rho_i = \frac{g_i}{2\,\pi^2\hbar^3}\int_0^\infty \frac{\left(E^2-m_i^2\right)^{1/2}\,E^2\,\mathrm{d}E}{\mathrm{e}^{\frac{E}{k_\mathrm{B}T}}\pm 1}$$

energy density

$$p_i = \frac{g_i}{2\,\pi^2\hbar^3}\int_0^\infty \frac{\left(E^2-m_i^2\right)^{3/2}\,\mathrm{d}E}{\mathrm{e}^{\frac{E}{k_\mathrm{B}T}}\pm 1}$$

pressure

$$s_i = \frac{1}{T}\,(\rho_i+p_i)$$

entropy density .

For $m_i \simeq 0$ in fact $p_i = \rho_i/3$; $\rho_i = \frac{g_i}{2}\,a\,T^4$, follows, such that

under redshift of the energy

$$E \propto \frac{1}{S(t)} \quad \bowtie \quad k_{\rm B}T \propto \frac{1}{S(t)}$$

and

$$\rho_i \propto \frac{1}{S(t)^4} \quad ; \quad n_i \propto \frac{1}{S(t)^3}$$

i.e., the distributions exactly follow the expansion law in equilibrium!

At $T \simeq T^* \simeq 10^{12}\,^\circ\mathrm{K}$ the neutrinos still couple and the thermal equilibrium with γ, $e^\pm$ and $\nu's$ is kept by a sufficiently high number of collision processes
$e^+ + e^- \leftrightarrow \gamma + \gamma$, $e^\pm + \nu_\mu \leftrightarrow \mu^\pm + \nu_e$, $e^\pm + \nu_e \leftrightarrow e^\pm + \nu_e$,
such that

$$T_\gamma = T_{e^\pm} = T_\nu \propto \frac{1}{S(t)} \qquad (\maltese)$$

Concerning the neutrinos: below the τ and μ thresholds charged current (CC) processes like $e^\pm + \nu_\tau \leftrightarrow \tau^\pm + \nu_e$ and $e^\pm + \nu_\mu \leftrightarrow \mu^\pm + \nu_e$ are not any longer able to keep neutrinos in thermal equilibrium. However, the neutral current (NC) processes

$$\boxed{e^+e^- \leftrightharpoons \bar{\nu}_i \nu_i}$$

still have sufficiently high rate to keep neutrinos in equilibrium. As neutrino masses are far below m_e the NC processes are not phase space suppressed as the CC ones are.

Exercise: estimate the collision rate for the above NC process at $T \sim 10^{11}\ ^\circ\mathrm{K}$ and compare it with the CC muon capture (inverse muon decay) process rate.

In the next stage, in the range $10^{11}\ ^\circ\mathrm{K} \gtrsim T \gtrsim 10^{10}\ ^\circ\mathrm{K}$ all neutrinos decouple, as neutrino processes get very rare. This happens still before e^+e^--annihilation begins.

As argued above (✠) hold until e^+e^- pair-annihilation sets in, thus effectively

$$g^* = \underset{\substack{\gamma \\ 2 \text{ spins}}}{2} + \frac{7}{8}\Big(\underset{\substack{e^\pm \\ 2 \text{ spins} \\ \text{each}}}{4} + \underset{\substack{\nu\text{'s} \\ 1 \text{ spin} \\ \text{each}}}{6}\Big) = \frac{43}{4} = \text{constant}$$

[9mm]

In order to describe in detail what happens when the electrons become non-relativistic, especially in the range where pair-annihilation takes place, we may use the following facts

1. charged particles (here $e^\pm$) are in thermal equilibrium all the time with the photons (Compton scattering)

2. for relativistic particles (massless particles) what we said above holds

3. entropy remains conserved

$$s = \frac{S^3(t)}{T(t)} \left\{ \frac{4}{3} a T_\gamma^4 + 6 \frac{4}{3} \frac{7}{16} a T_\nu + \rho_{e^-} + \rho_{e^+} + p_{e^-} + p_{e^+} \right\}$$

If one assumes that the chemical potentials of $e^\pm$ are negligible, we have $\rho_{e^+} = \rho_{e^-}$, $p_{e^+} = p_{e^-}$ and

$$\rho_e = \frac{1}{\pi^2 \hbar^3} \int_{m_e}^{\infty} \frac{\left(E^2 - m_e^2\right)^{1/2} E^2 \, \mathrm{d}E}{\mathrm{e}^{\frac{E}{k_\mathrm{B} T}} \pm 1}$$

$$p_e = \frac{1}{3 \pi^2 \hbar^3} \int_{m_e}^{\infty} \frac{\left(E^2 - m_e^2\right)^{3/2} \mathrm{d}E}{\mathrm{e}^{\frac{E}{k_\mathrm{B} T}} \pm 1}$$

A transformation of variables yields

$$\rho_{e^-} + \rho_{e^+} + p_{e^-} + p_{e^+} = 2\,\frac{(k_\mathrm{B}T)^4}{\pi^2\hbar^3}\cdot\frac{7\pi^4}{90}\,f\left(\frac{m_e}{k_\mathrm{B}T}\right) = 4\cdot\frac{4}{3}\,\frac{7}{16}\,a\,T^4\,f\left(\frac{m_e}{k_\mathrm{B}T}\right)$$

$$f(z) = \frac{90}{7\pi^4}\int\limits_0^\infty \mathrm{d}y\,y^2\left(\sqrt{y^2+z^2} + \frac{y^2}{3\,\sqrt{y^2+z^2}}\right)\frac{1}{\exp\,\sqrt{y^2+z^2}+1}\;;\quad f(0) = 1\;.$$

Herewith, provided all particles are relativistic and in thermal equilibrium,

$$s = S^3(t)\cdot T^3(t)\cdot\frac{4}{3}\,a\left\{\underbrace{1}_{\gamma} + \underbrace{\frac{21}{8}}_{6\nu's} + \underbrace{\frac{7}{4}\,f\left(\frac{m_e}{k_\mathrm{B}T}\right)}_{e^\pm}\right\} = \text{constant}$$

When electrons get non-relativistic, such that $f \neq 1$, only $T_{e^\pm} = T_\gamma$ remains equal, while T_ν must go its own way.

Since the neutrinos by themselves satisfy an equilibrium distribution with temperature T_ν and $p_\nu = \rho_\nu/3$ their contribution to the entropy is always

$$s_\nu = S^3(t) \cdot T^3(t) \cdot \frac{4}{3} a \cdot \frac{21}{8} \, .$$

As mentioned before, the cosmic redshift lets the energy scale like $E \propto \frac{1}{S(t)}$ which implies $s_\nu = \text{constant}$ as the distribution follows $T_\nu \propto \frac{1}{S(t)}$.

We thus can work out the neutrino temperature as follows:

$$
\begin{aligned}
s &= \underbrace{s_\nu}_{\text{const.}} + \frac{4}{3}\, a \left(S\, T_\gamma\right)^3 \left\{1 + \frac{7}{4} f\left(\frac{m_e}{k_{\mathrm{B}} T}\right)\right\} = \text{constant} \\
s - s_\nu &= S^3(t) \cdot T_\nu^3(t) \cdot \frac{4}{3}\, a \cdot \left(1 + \frac{7}{4}\right) \\
&\quad \text{valid at first for } t \text{ small and is independent of } t \\
&= S^3(t) \cdot T_\gamma^3(t) \cdot \frac{4}{3}\, a \cdot \left(1 + \frac{7}{4} f\left(\frac{m_e}{k_{\mathrm{B}} T}\right)\right)
\end{aligned}
$$

which yields for T_ν

$$
\boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \cdot \left(1 + \frac{7}{4} f\left(\frac{m_e}{k_{\mathrm{B}} T}\right)\right)^{1/3}}
$$

The time dependence is obtained with the Friedmann equation

$$\frac{\mathrm{d}S}{S} = \left(\frac{\kappa\rho}{3}\right)^{1/2} \mathrm{d}t$$

with

$$\begin{aligned}\rho &= \rho_\gamma + \rho_{e^+} + \rho_{e^-} + 6\,\rho_\nu \\ &= \underbrace{a\,T_\gamma^4}_{\text{photon}} + \underbrace{\frac{21}{8}\,a\,T_\nu^4}_{\text{neutrinos}} + \underbrace{\frac{7}{4}\,a\,T_\gamma^4\,g\left(\frac{m_e}{k_\mathrm{B}T}\right)}_{\text{electron-positron}}\end{aligned}$$

where

$$g(z) = \frac{4}{7}\frac{30}{\pi^4}\int_0^\infty \mathrm{d}y\, y^2\,\sqrt{y^2+z^2}\,\frac{1}{\exp\sqrt{y^2+z^2}+1}\;;\quad g(0)=1\;.$$

We are now able to calculate the relationship between the elapsed time and the density, given earlier to be $t = \frac{1}{(4\,\frac{\kappa}{3}\rho)^{1/2}}$ up to a constant, with the exact pair-annihilation taken into account. The pair-annihilation heats up the plasma, such that during this era $S(t) \cdot T(t) \neq \text{constant}$.

The precise relationship is obtained as follows: first introduce the auxiliary functions F and G

$$F(z) \doteq 1 + \frac{7}{4} f(z) \; ; \quad G(z) \doteq 1 + N_\nu \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} F^{4/3}(z) + \frac{7}{4} g(z)$$

where $f(z)$ and $g(z)$ were given above and $N_\nu = 3$ the number of neutrino species. Then from

$$T_\nu = T_{0\nu} \frac{S_0}{S} \; ; \quad \mathrm{d}T_\nu = -T_\nu \frac{\mathrm{d}S}{S}$$

together with

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} F\left(\frac{m_e}{k_B T}\right)^{1/3} T_\gamma \; ; \quad \mathrm{d}T_\nu = T_\nu \frac{\mathrm{d}T_\gamma}{T_\gamma} + \frac{1}{3} T_\nu \frac{\mathrm{d}F}{F} \; ,$$

we obtain

$$\frac{\mathrm{d}S}{S} = -\frac{\mathrm{d}T_\nu}{T_\nu} = -\left(\frac{\mathrm{d}T_\gamma}{T_\gamma} + \frac{1}{3} T_\nu \frac{\mathrm{d}F}{F}\right) = \left(\frac{\kappa\rho}{3}\right)^{1/2} \mathrm{d}t \; ; \quad \rho = a\, T_\gamma^4\, G\left(\frac{m_e}{k_B T}\right) \; .$$

The integration yields the following important relation between the elapsed time t and T_γ:
[3mm]

$$t = -\int \left(\frac{\kappa}{3} a\, T_\gamma^4\, G\left(\frac{m_e}{k_B T}\right)\right)^{-1/2} \left(\frac{\mathrm{d}T_\gamma}{T_\gamma} + \frac{1}{3} T_\nu \frac{\mathrm{d}F\left(\frac{m_e}{k_B T}\right)}{F\left(\frac{m_e}{k_B T}\right)}\right)$$

Denoting $z \doteq \frac{m_e}{k_B T}$ we have

$$\frac{dF(z)}{dT} = \frac{dF(z)}{dz}\frac{dz}{dT} = -\frac{z}{T}\dot{F}(z) = -\frac{7}{4}\frac{z}{T}\dot{f}(z) \ ,$$

and with $\frac{dT}{T} = -\frac{dz}{z}$ we may write

$$\boxed{t = t_e \int \frac{dz\, z}{\sqrt{G(z)}} \left(3 - \frac{z\dot{F}}{F}(z)\right)}$$

where

$$\boxed{t_e = (3\,\kappa\, a\, c^2\, (m_e/k_B)^4)^{-1/2} = 4.3694 \text{ sec.}}$$

The integral can be evaluated numerically; see the Table

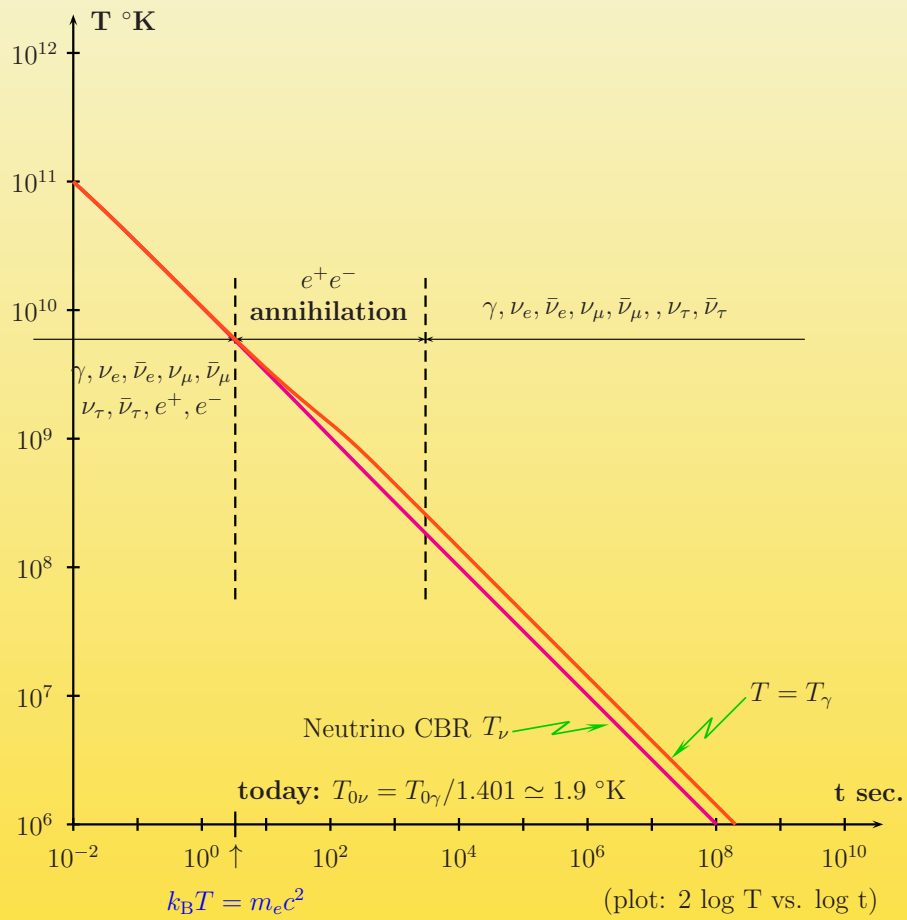
$T(^\circ\mathrm{K})$	T/T_ν	$t(\mathrm{sec})$
3×10^{11}	1.0000	0
2×10^{11}	1.0000	1.3807×10^{-3}
1×10^{11}	1.0001	8.8375×10^{-3}
6×10^{10}	1.0002	2.6519×10^{-2}
3×10^{10}	1.0009	1.0952×10^{-1}
2×10^{10}	1.0020	2.4829×10^{-1}
1×10^{10}	1.0080	1.0070
6×10^{9}	1.0219	2.8675
3×10^{9}	1.0804	12.671
2×10^{9}	1.1587	32.301
1×10^{9}	1.3454	168.11
3×10^{8}	1.4010	1979.6
1×10^{8}	1.4010	1.7779×10^{4}
1×10^{7}	1.4010	1.7774×10^{6}
1×10^{6}	1.4010	1.7774×10^{8}

Until pair-annihilation sets in one has $t \tilde{\propto} 1/T^2$. After pair-annihilation we again have $t \tilde{\propto} 1/T^2$, however, with $T_\nu \neq T_\gamma$ since

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \cdot \left(1 + \frac{7}{4} f\left(\frac{m_e}{k_B T}\right)\right) \to T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

as $f\left(\frac{m_e}{k_B T}\right) \simeq 1$; $k_B T \gg m_e$ while $f\left(\frac{m_e}{k_B T}\right) \simeq 0$; $k_B T \ll m_e$. As a result [3mm]

$$\boxed{\| \; T_\gamma / T_\nu = 1.401 \quad \Rightarrow \quad T_\nu = 1.945 \; ^\circ\mathrm{K}}$$



Era of decoupling (freeze out) of neutrinos, e^+e^--annihilation reheating the photon sea

Comment on the evaluation of the integrals f and g:
by a change of integration variables $y \to x = \sqrt{y^2 + z^2}$ and $x \to u : x = zu$ we obtain

$$
\begin{aligned}
f &= \frac{90}{7\pi^4} z^4 \left(I_1(z) + I_2(z)\right) \\
g &= \frac{120}{7\pi^4} z^4 \left(I_1(z)\right) \\
I_1(z) &= = \int_1^\infty \mathrm{d}u\, u^2 \sqrt{u^2 - 1}\, \frac{1}{\mathrm{e}^{zu} + 1} \\
I_2(z) &= = \int_1^\infty \mathrm{d}u\, \frac{\left(\sqrt{u^2 - 1}\right)^3}{3} \frac{1}{\mathrm{e}^{zu} + 1} \,.
\end{aligned}
$$

Furthermore, we need the derivative of f:

$$\dot{f} = \frac{90}{7\pi^4}\left[4\,z^3\,\left(I_1(z) + I_2(z)\right) + z^4\,\left(\dot{I}_1(z) + \dot{I}_2(z)\right)\right] \;,$$

where the integrals $\dot{I}_i$ are obtained from I_i by

$$\frac{1}{\mathrm{e}^{zu}+1} \to \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\mathrm{e}^{zu}+1}\right] = -u\,\frac{\mathrm{e}^{zu}}{(\mathrm{e}^{zu}+1)^2} \;.$$

For the numerical evaluation one has to write

$$\frac{1}{\mathrm{e}^{zu}+1} = \frac{\mathrm{e}^{-zu}}{1+\mathrm{e}^{-zu}} \;.$$

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More on the evaluation of the integrals:

Electron density:

$$\rho_e = \frac{7}{4}\, a\, T^4\, g\left(\frac{m_e}{k_{\rm B} T}\right) \;;\quad g(z) = \frac{4}{7}\,\frac{30}{\pi^4}\, I(z)$$

We want to study the basic integral $I(z)$ defined by

$$I(z) = \int\limits_0^\infty {\rm d}y\, y^2\, \sqrt{y^2+z^2}\, \frac{1}{{\rm e}^{\sqrt{y^2+z^2}}+1}\;.$$

We perform two successive transformations
$y \to x: \; x = \sqrt{y^2 + z^2}, \, \mathrm{d}x = \frac{y}{x}\,\mathrm{d}y, \; y = \sqrt{x^2 - z^2}$ and $x \to u: \; uz = x, \mathrm{d}x = z\,\mathrm{d}u$, thus

$$
\begin{aligned}
I(z) &= \int_z^\infty \mathrm{d}x x^2 \sqrt{x^2 - z^2}\, \frac{1}{\mathrm{e}^x + 1} \\
&= z^4 \int_1^\infty \mathrm{d}u u^2 \sqrt{u^2 - 1}\, \frac{1}{\mathrm{e}^{zu} + 1} \\
&= z^4 \int_1^\infty \mathrm{d}u u^2 \sqrt{u^2 - 1}\, \mathrm{e}^{-zu}\, \frac{1}{1 + \mathrm{e}^{-zu}} \, .
\end{aligned}
$$

We note that the following bound apply:

$$
\frac{1}{2} \le \frac{1}{1 + \mathrm{e}^{-zu}} \le 1 \; ; \quad 0 < \mathrm{e}^{-zu} \le 1
$$

and we expand

$$\frac{1}{1+\mathrm{e}^{-zu}} \simeq 1 - \mathrm{e}^{-zu} + \cdots$$

Then the leading term reads

$$\begin{aligned}
& \int_1^\infty \mathrm{d}u\, u^2\, \sqrt{u^1 - 1}\, \mathrm{e}^{-zu} \\
= & \int_1^\infty \mathrm{d}u\, \left(u^2 - 1\right)^{3/2}\, \mathrm{e}^{-zu} + \int_1^\infty \mathrm{d}u\, \left(u^2 - 1\right)^{1/2}\, \mathrm{e}^{-zu} \\
= & \frac{\Gamma(2+\frac{1}{2})}{\sqrt{\pi}\left(\frac{1}{2}z\right)^2} K_2(z) + \frac{\Gamma(1+\frac{1}{2})}{\sqrt{\pi}\left(\frac{1}{2}z\right)^1} K_1(z) \\
= & 3\,\frac{K_2(z)}{z^2} + \frac{K_1(z)}{z} = \left(6\,\bar{K}_2(z) + z^2\,\bar{K}_1(z)\right)/z^4\,,
\end{aligned}$$

where $K_\nu(z)$ are modified Bessel functions

$$K_\nu(z) = \frac{\sqrt{\pi}\,\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+\frac{1}{2})} \int_1^\infty \mathrm{d}u\,\left(u^2-1\right)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-zu}\,.$$

We have introduced modified functions $\bar{K}$ normalized to unity at zero:

$$K_\nu(z) \simeq \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} \quad \triangleright \quad \bar{K}_\nu(z) \doteq \frac{2\,K(z)}{\Gamma(\nu)\left(\frac{z}{2}\right)^\nu}\,; \quad \bar{K}_\nu(0) = 1\,.$$

Furthermore, we define

$$M(z) \doteq \bar{K}_2(z) + \frac{z^2}{6}\,\bar{K}_1(z)$$

such that

$$I(z) = 6\,M(z) + \cdots$$

The next term in the expansion we obtain by replacing e^{-zu} by $-\mathrm{e}^{-2zu}$. If we replace the prefactor $z^4 \to \frac{(2z)^4}{16}$ we see that the correction term is $-\frac{1}{16}\,M(2z)$, thus

$$\begin{aligned} I(z) &= 6\left(M(z) - \frac{1}{16}\,M(2z)\right) + \cdots \\ I(0) &= 6\left(1 - \frac{1}{16}\right) + \cdots \simeq 5.625 \end{aligned}$$

The exact value

$$I(0) = \int_0^\infty \mathrm{d}x\, x^3 \frac{1}{\mathrm{e}^x + 1} = \frac{7}{4}\frac{\pi^4}{30} \simeq 5.682...$$

For larger z values the expansion converges better, since $\frac{1}{1+\mathrm{e}^{-zu}}$ gets better approximated by 1.

Utilizing these results, for the electron number density we find

$$
\begin{aligned}
n_e &= \frac{2\,(k_{\rm B}T)^3}{2\pi^2\hbar^3}\int\limits_z^\infty {\rm d}x\,x\,\sqrt{x^2-z^2}\,\frac{1}{{\rm e}^x+1} \\
&= \frac{2\,(k_{\rm B}T)^3}{2\pi^2\hbar^3}\,2\left\{\bar K_2(z)-\frac{1}{8}\,\bar K_2(2z)\right\}+\cdots
\end{aligned}
$$

The expanded integral at $z=0$ yields $2\left(\bar K_2(z)-\frac{1}{8}\,\bar K_2(z)\right)\to 2\left(1-\frac{1}{8}\right)=1.75$ while the exact result is

$$
\int\limits_0^\infty {\rm d}x\frac{x^2}{{\rm e}^x+1}=\frac{3}{4}\cdot 2\cdot\zeta(3)=1.803\ldots\ ;\quad \zeta(3)=1.202\ldots
$$

As mentioned before convergence is worst at $z=0$ and improves for larger z values.

For the pressure we have

$$
\begin{aligned}
p_e &= \frac{2\,(k_{\mathrm{B}}T)^4}{6\pi^2\hbar^3}\int\limits_z^\infty \mathrm{d}x\,\left(x^2-z^2\right)^{3/2}\,\frac{1}{\mathrm{e}^x+1}\\
&= \frac{2\,(k_{\mathrm{B}}T)^3}{6\pi^2\hbar^3}\,6\,\left\{\bar{K}_2(z)-\frac{1}{16}\,\bar{K}_2(2z)\right\}+\cdots
\end{aligned}
$$

and similarly for the energy density

$$
\begin{aligned}
\rho_e &= \frac{2\,(k_{\mathrm{B}}T)^4}{2\pi^2\hbar^3}\int\limits_z^\infty \mathrm{d}x\,x^2\,\sqrt{x^2-z^2}\,\frac{1}{\mathrm{e}^x+1}\\
&= \frac{2\,(k_{\mathrm{B}}T)^3}{2\pi^2\hbar^3}\,6\,\left\{M(z)-\frac{1}{16}\,M(2z)\right\}+\cdots
\end{aligned}
$$

So far we assumed the electron gas to be non-degenerate i.e. $\mu_e \simeq 0$. This approximation can only be good as long as

$$n_{e^-}, n_{e^+} \gg |n_{e^-} - n_{e^+}|$$

In the pair-annihilation regime, however, n_{e^-} and n_{e^+} decrease exponentially ($\propto \mathrm{e}^{-\frac{m_e c^2}{k_\mathrm{B} T}}$). We have

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left\{1 + \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \cdots\right\} \quad \mu \doteq 4\nu^2\,; \quad z = \frac{m_e c^2}{k_\mathrm{B} T}\,.$$

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The exponential suppression is absent when the electron gas is degenerate:

$$\frac{1}{e^{\frac{E}{k_B T}}+1} \quad \Rightarrow \quad \frac{1}{e^{\frac{E-\mu}{k_B T}}+1}$$
$$\|$$
$$\frac{1}{e^{zu}+1} \quad \Rightarrow \quad \frac{1}{e^{zu-\bar{m}}+1}$$

$$E = \left(p^2 - m^2\right)^{1/2} \;\Rightarrow\; E - \mu = \left(p^2 - m^2\right)^{1/2} - \mu \;;\;\; \bar{m} = \frac{\mu}{k_B T} \;.$$

When $\mu_e \neq 0$, one must determine μ_e via n_e. One may assume that matter in average is neutral and consists of p, n and e^- ($n_{\bar{p}}, n_{\bar{n}}, n_{e^+} \simeq 0$), typical for our galaxy. After the pair-annihilation one expects a certain density of non-relativistic electrons, where now

$$n_e \propto S^{-3}(t)$$

The charge neutrality of cosmic matter requires
[3mm]

$$\boxed{n_e = n_p}$$

The proton number density is known from the density of baryons (nucleons), which is usually expressed normalized to the relic black-body photon density as

$$\eta \equiv n_{\rm B}/n_\gamma = (4.7 - 6.5) \times 10^{-10} \text{ (95\% CL))}$$

Here n_γ is fixed from the $2.725\ ^\circ\mathrm{K}$ CMB radiation. The allowed range for the baryon mass density today is

[3mm]

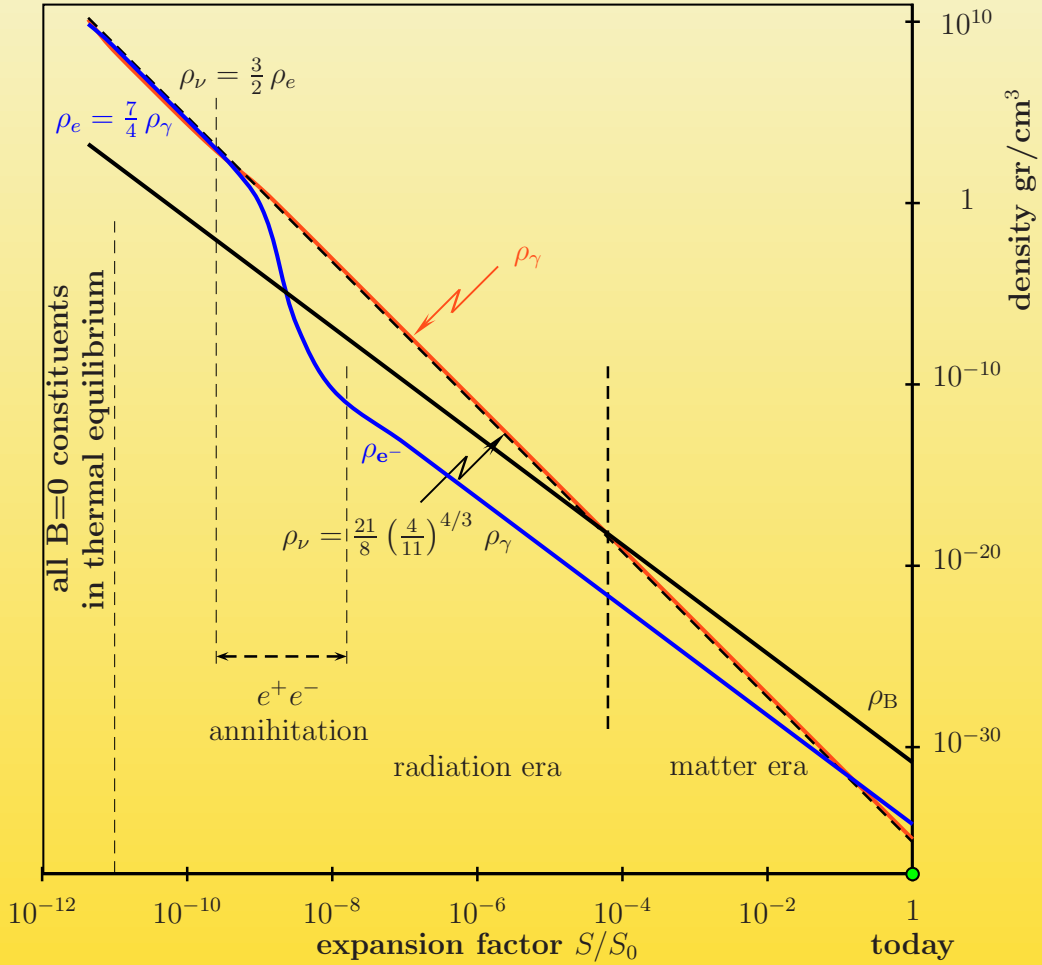
$$\boxed{\rho_{0B} = (3.2 - 4.5) \times 10^{-31}\ \mathrm{gr/cm^3} \text{ or } \Omega_B = \rho_{0B}/\rho_{0,\mathrm{crit}} \simeq (0.017 - 0.024)\, h^{-2}}$$

where $h \equiv H_0/100\ \mathrm{kms^{-1}\,Mpc^{-1}} = 0.72 \pm 0.08$ is the present Hubble parameter.

As we will learn later baryonic matter consists to 98% of Hydrogen and Helium-4, observed in a ratio $H\ :\ {}^4He \sim 3\ :\ 1$ such that about $5/7$ are protons and $2/7$ are neutrons. We thus assume $n_e \sim 5/7 n_B$ such that

$$\boxed{\rho_{0e} \simeq 1.5 \times 10^{-34}\ \mathrm{gr/cm^3}}$$

We thus find the following general picture about the development of the densities after about $10^{11}\ ^\circ\mathrm{K}$:



Possible Neutrino Degeneracy

So far we always have assumed that lepton numbers are essentially zero: $n_M \simeq n_E \simeq 0$. In the hot universe scenario we assumed

$$n_M, n_E, \cdots \ll n_\gamma \; ; \quad n_\gamma \propto \left(\frac{k_B T_\gamma}{\hbar}\right)^3 \; .$$

To high accuracy individual lepton numbers n_T, n_M and n_E are conserved. Charge neutrality $n_Q = 0$ is always assumed. For temperatures $T \lesssim T^* \simeq 10^{12}\,{}^\circ\mathrm{K}$ only e^- and e^+ show up as charged particles (not yet annihilated), the density of relict baryons is negligible. Therefore

$$n_Q = n_{e^+} - n_{e^-} = 0$$

and the chemical potentials must vanish

$$\mu_{e^+} = \mu_{e^-} = 0 \; .$$

Hence,

$$n_E = n_{e^+} - n_{e^-} + n_{\nu_e} - n_{\bar{\nu}_e} = n_{\nu_e} - n_{\bar{\nu}_e} \ ,$$

and similarly for the muon and tau leptons. With,

$$\begin{aligned} n_\nu(E) &= \frac{1}{2\pi^2\hbar^3} \frac{E^2}{e^{\frac{E-\mu_\nu}{k_B T}} + 1} \\ n_{\bar{\nu}}(E) &= \frac{1}{2\pi^2\hbar^3} \frac{E^2}{e^{\frac{E+\mu_{\bar{\nu}}}{k_B T}} + 1} \end{aligned}$$

we obtain

$$n_E = \int \mathrm{d}E \ (n_{\nu_e}(E) - n_{\bar{\nu}_e}(E)) = \frac{1}{2\pi^2} \left(\frac{k_B T}{\hbar}\right)^3 \ h(\frac{\mu_{\nu_e}}{k_B T}) \ ,$$

and corresponding expressions for n_M and n_T, with

$$h(x) = \int_0^\infty \mathrm{d}y\, y^2 \left[(\mathrm{e}^{y-x}+1)^{-1} - (\mathrm{e}^{y+x}+1)^{-1} \right] .$$

Because of the conservation of n_E, n_M, n_T, on the one hand, we must have

$$n_L = n_{0L} \left(\frac{S_0}{S} \right)^3 , \quad L = T, M, E .$$

On the other hand we know that

$$T_\nu = T_{0\nu} \left(\frac{S_0}{S} \right) ,$$

which requires for all temperatures $T \lesssim T^* \simeq 10^{12}\ ^\circ\mathrm{K}$ up to date:

$$\boxed{\frac{\mu_\nu}{k_B T} = \text{constant} \quad \leftrightarrow \quad \mu_\nu = \mu_{0\nu}\left(\frac{S_0}{S}\right) .}$$

Our previous calculation concerning the e^+e^--annihilation era changes only slightly if $\mu_\nu \neq 0$: it amounts to slightly increase the neutrino energy density: by replacing

$$\rho_{\nu,\text{all}} = 6\rho_\nu = \frac{21}{8}\, a\, T_\nu^4$$

by

$$\begin{aligned}
\rho_{\nu,\text{all}} &= \rho_{\nu_e} + \rho_{\bar{\nu}_e} + \rho_{\nu_\mu} + \rho_{\bar{\nu}_\mu} + \rho_{\nu_\tau} + \rho_{\bar{\nu}_\tau} \\
&= \frac{7}{8}\, a\, T_\nu^4 \underbrace{\left(r\left(\frac{\mu_{\nu_e}}{k_B T_\nu}\right) + r\left(\frac{\mu_{\nu_\mu}}{k_B T_\nu}\right) + r\left(\frac{\mu_{\nu_\tau}}{k_B T_\nu}\right)\right)}_{\text{constant in } t} ,
\end{aligned}$$

where

$$r(x) = \frac{60}{7\,\pi^4} \int\limits_0^\infty \mathrm{d}y\, y^3 \left[(\mathrm{e}^{y-x}+1)^{-1} - (\mathrm{e}^{y+x}+1)^{-1} \right] \; ; \quad r(0) = 1 \; .$$

Thus a partial neutrino degeneracy would lead to smaller t values. In the limiting case of complete degeneracy $\left|\frac{\mu_\nu}{k_\mathrm{B}T}\right| \gg 1$, for example, for $\frac{\mu_\nu}{k_\mathrm{B}T} \gg 1$, we get

$$\begin{aligned} n_\nu(E) &\simeq \begin{cases} \frac{1}{2\pi\hbar^3} E^2 & E < \mu_\nu \\ 0 & E > \mu_\nu \end{cases} \\ \text{and} & \\ n_{\bar\nu}(E) &\simeq 0 \; , \end{aligned}$$

and thus

$$r(x) = \frac{60}{7\,\pi^4} \int\limits_0^\infty \mathrm{d}y\, y^3 = \frac{15}{7\pi^4}\, x^4 = \frac{15}{7\pi^4} \left(\frac{\mu_\nu}{k_\mathrm{B} T}\right)^4 \ .$$

With $a = \frac{\pi^2 k_\mathrm{B}^4}{15\hbar^3} = \frac{8\pi^5 k_\mathrm{B}^4}{15\mathrm{h}^3}$ we find

$$\rho_{\nu,\mathrm{all}} = \pi\,\mathrm{h}^{-3} \left(\mu_{\nu_e}^4 + \mu_{\nu_\mu}^4 + \mu_{\nu_\tau}^4\right) \ .$$

This compares to the photon density

$$\rho_\gamma = a\,T_\gamma^4 = \underbrace{\frac{8\,\pi^4}{15}}_{\simeq 52} \pi\,\mathrm{h}^3 \left(k_\mathrm{B} T_\gamma\right)^4 \ll \pi\,\mathrm{h}^{-3} \left(\mu_{\nu_e}^4 + \mu_{\nu_\mu}^4 + \mu_{\nu_\tau}^4\right) \ ,$$

which means that the evolution would be dominated completely by ρ_ν and not be ρ_γ. From the observed value of the deceleration parameter one can derive bounds

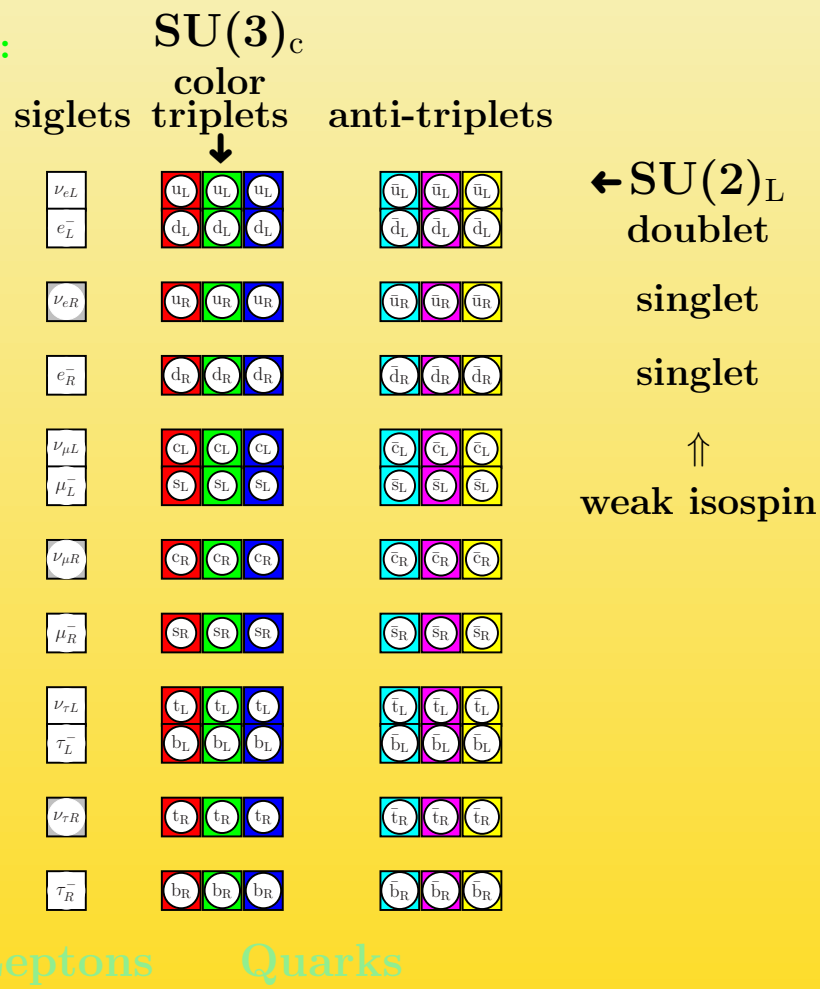
on chemical potential contribution $\mu^4_{\nu_e} + \mu^4_{\nu_\mu} + \mu^4_{\nu_\tau}$. Such degenerate neutrino background would also affect laboratory experiments like Tritium β-decay ${}^3H \to {}^3He + e^- + \bar{\nu}_e$ which shows that $|\mu_e|$ can be at most a few eV. Much more stringent constraints come from cosmological nucleosynthesis.

Summary so far: we have worked out the thermal history of our universe back to a state when electrons, positrons, neutrinos and photons were in thermal equilibrium at temperature of about $3 \times 10^{11}\ {}^\circ\mathrm{K}$.

Exactly the same reasoning allow us to go further back, the only thing we need to know are the particles and their thresholds, which can be created and annihilated at higher energies. In fact we can go back as much as we know the elementary particle content, which is what we have in the Standard Model of elementary particle physics. Sufficiently high above the thresholds (e.g. beyond the electroweak scale $v = 1/\sqrt{\sqrt{2}G_\mu} = 246.22$ GeV) we can take all particles as highly relativistic, and what matters then is just the number of relativistic degrees of freedom.

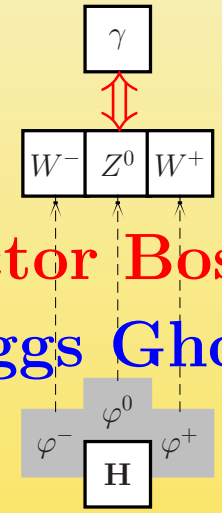
The Standard Model

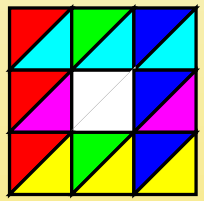
Constituents of matter:
Spin 1/2 fermions



Carrier of Forces:
Spin 1 Gauge Bosons

Photon





Vector Bosons

Higgs Ghosts

Octet of Gluons

"Cooper Pairs":
Spin 0 Higgs Boson

Higgs Particle

Higgs mechanism: breaking weak isospin spontaneously

weak bosons W, Z and all fermions get masses

$m_i \propto y_i\, v$; $v = \langle H \rangle > \simeq 246.21$ **GeV**; $G_{\text{Fermi}} = \frac{1}{\sqrt{2}v^2}$ **(radioactivity)**

Dynamics: strongly interacting quarks and gluons

$$\begin{aligned}\mathscr{L}_{\rm QCD} &= -\tfrac{1}{4}F^{(a)}_{\mu\nu}F^{(a)\,\mu\nu} + i\sum_q \bar\psi^i_q\gamma^\mu\left(D_\mu\right)_{ij}\psi^j_q \\ &\quad -\sum_q m_q\bar\psi^i_q\psi_{qi} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm FP}\end{aligned}$$

$$\begin{aligned}F^{(a)}_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc}A^b_\mu A^c_\nu \\ \left(D_\mu\right)_{ij} &= \delta_{ij}\partial_\mu - ig_s\sum_a \frac{\lambda^a_{ij}}{2}A^a_\mu\end{aligned}$$

- Quarks: $q = u, d, s, c, b, t$ (Dirac fields)
- Gluons: color-octet A_a, $a = 1, 2, \ldots, 8$ (Yang–Mills fields)

SM particle content:

Quarks	$t, \bar{t}$	171.3 ± 1.6 GeV	spin= $\frac{1}{2}$	$g = 2 \cdot 2 \cdot 3 = 12$	
	$b, \bar{b}$	$4.20^{+0.17}_{-0.07}$ GeV	3 colors		
	$c, \bar{c}$	$1.27^{+0.07}_{-0.11}$ GeV			
	$s, \bar{s}$	105^{+25}_{-35} MeV			
	$d, \bar{d}$	$3.5 - 6.0$ MeV			
	$u, \bar{u}$	$1.5 - 3.3$ MeV			
					72
Gluons		8 massless bosons	spin= 1	$g = 2 \cdot 8 = 16$	
					16
Leptons	τ^-, τ^+	1776.84 ± 0.171 MeV	spin= $\frac{1}{2}$	$g = 2 \cdot 2 = 4$	
	μ^-, μ^+	105.658 MeV			
	e^-, e^+	0.510999 MeV			
					12
Neutrinos	$\nu_\tau, \bar{\nu}_\tau$	< 18.2 MeV	spin= $\frac{1}{2}$	$g = 2$	
	$\nu_\mu, \bar{\nu}_\mu$	< 190 keV			
	$\nu_e, \bar{\nu}_e$	< 2 eV			
					6

Weak bosons	$W^\pm$	80.403 ± 0.029 GeV	spin= 1	$g = 3$	
	Z	91.1876 ± 0.0021 GeV			
Photon	γ	$0\ (< 6 \times 10^{-17}$ eV$)$		$g = 2$	11
Higgs	H	> 114.4 GeV	spin= 0	$g = 1$	1

$$g_f = 72 + 12 + 6 = 90$$
$$g_B = 16 + 11 + 1 = 28$$

Thus the total energy density

$$\rho(T) = \sum \rho_i(T) = \frac{\pi^2}{30} g_*(T)\, T^4$$

which in general defines an effective number of degrees of freedom $g_*(T)$.

If all particles are ultra relativistic

$$g_*(T) = g_B(t) + \frac{7}{8}\, g_f(T)$$

with $g_B = \sum_i g_i$ is the sum over relativistic bosons and $g_f = \sum_i g_i$ the sum over the relativistic fermions.

If all SM are relativistically excited we have

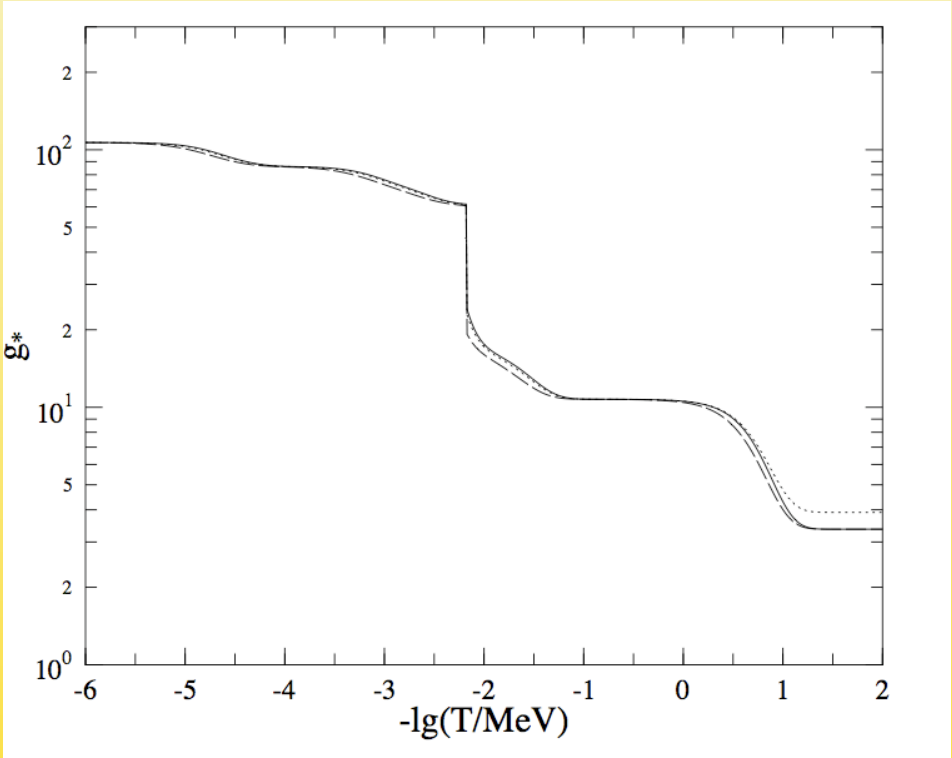
$$\boxed{g_*(T) = 106.75}$$

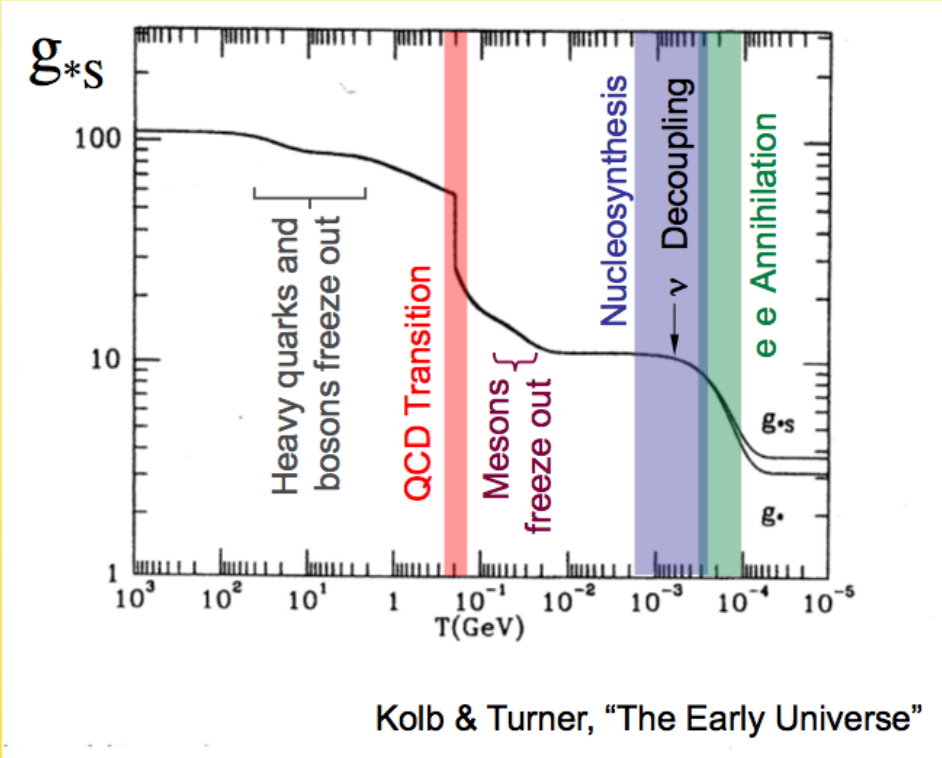
Mass effects can be taken into account as done above for electrons and positrons in the discussion of the e^+e^--annihilation. Mass effects manifest themselves slightly different in energy density ρ, pressure p and entropy density s. Therefore

$$\rho(T) = \frac{\pi^2}{30}\, g_*(T)\, T^4 \;; \quad p(T) = \frac{\pi^2}{90}\, g_{*p}(T)\, T^4 \;; \quad s(T) = \frac{2\,\pi^2}{45}\, g_{*s}(T)\, T^3$$

define slightly different effective numbers of degrees of freedom when masses

play a role. Numerical results are shown in the following figure.





The functions $g_*(T)$) (solid), $g_{*p}(T)$ (dashed), and $g_{*s}(T)$ (dotted) calculated for the SM particle content.

In the following we will often adopt the convention to express temperatures in

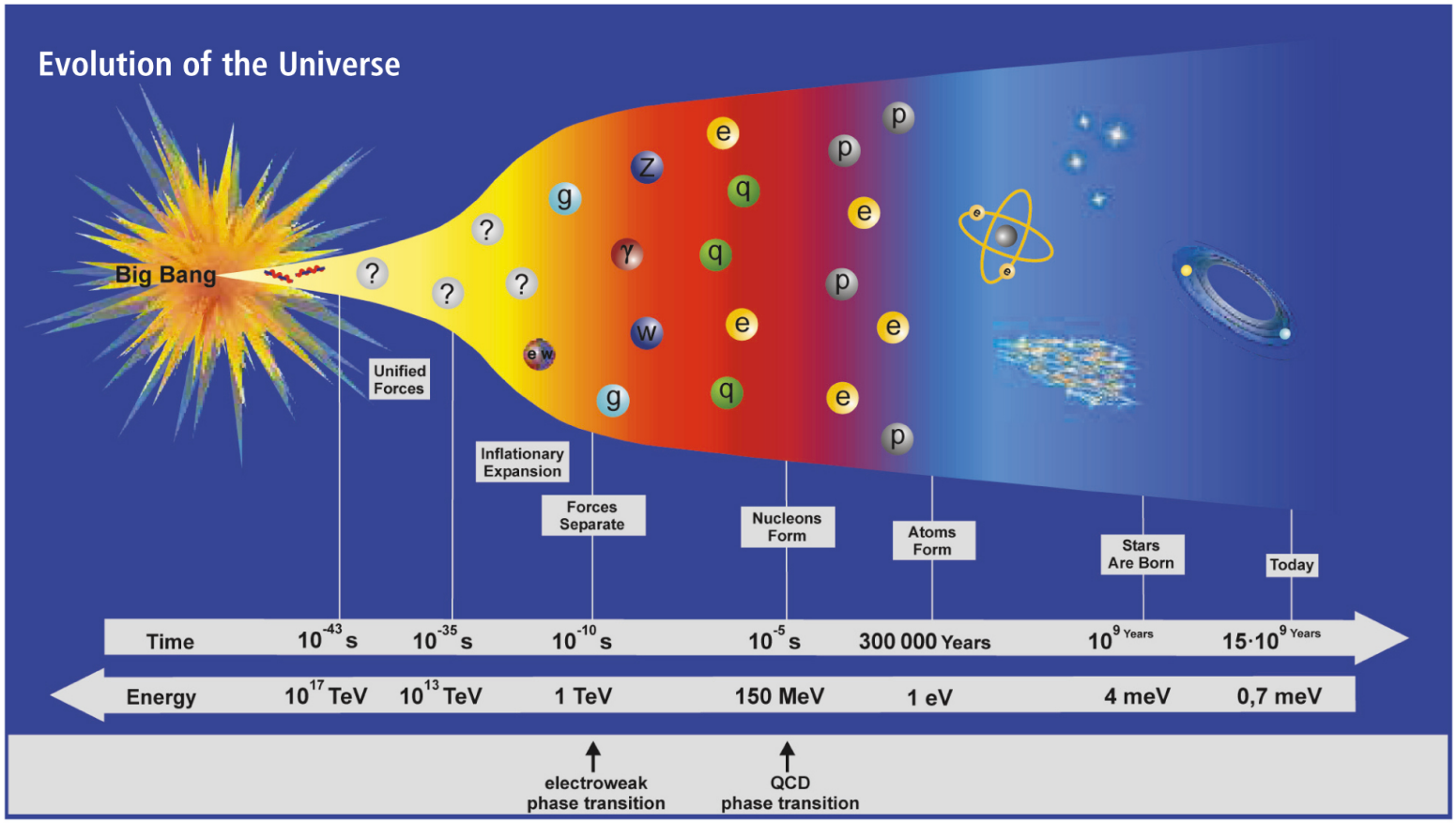
energy units, as $E = k_B\, T$ (hence $1\ ^\circ\mathrm{K}\ \sim 8.6 \times 10^{-5}\ \mathrm{eV}$) is a universal relation and as particle physicist we are more familiar with energy units.

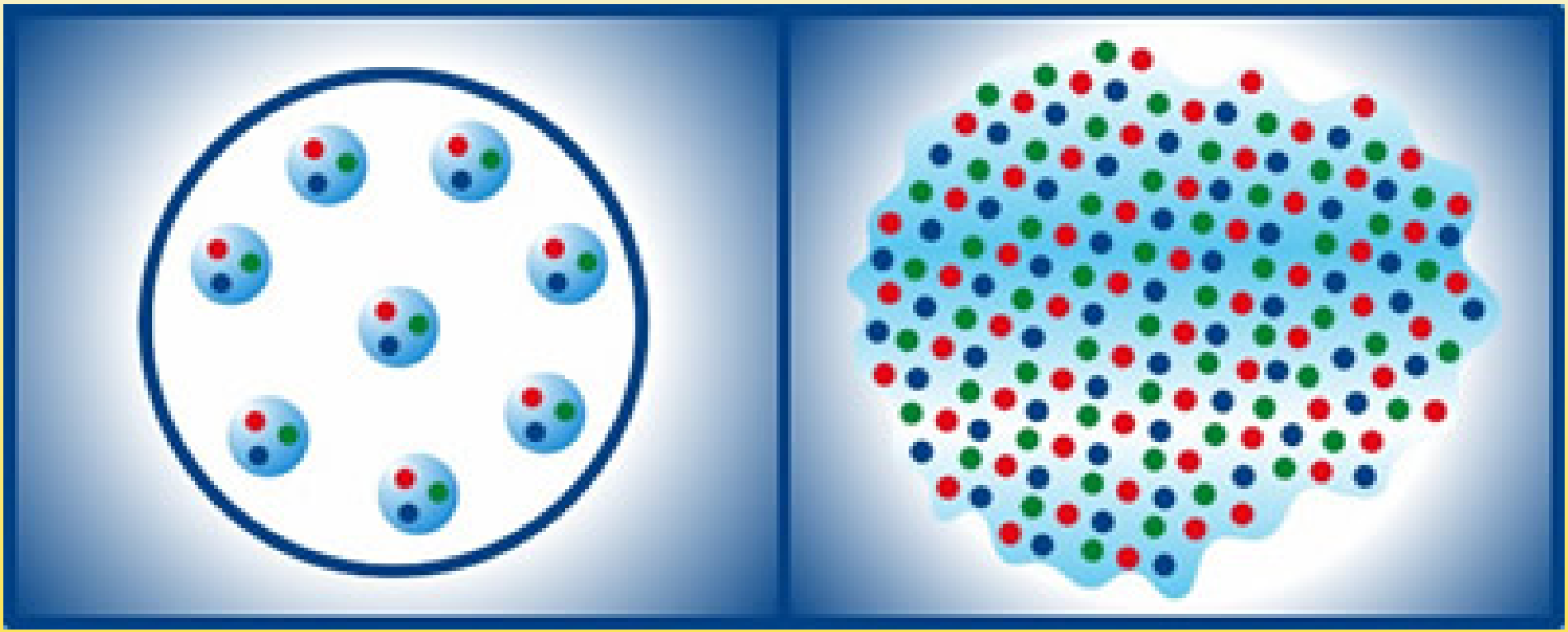
The thermal history of the universe as a scan of the SM:

T	event	$g_*(T)$	comment
~ 200 GeV	all states relativistic	106.75	
~ 100 GeV	EW phase transition	106.75	
< 170 GeV	top annihilation	96.25	
< 80 GeV	$W^\pm, Z, H$ annihilation	86.25	
< 4 GeV	bottom annihilation	75.75	
< 1 GeV	charm, τ annihilation	61.75	
~ 150 MeV	QCD phase transition	17.25	$(u, d, g \to \pi^{\pm,0},\ 37 \to 3)$
< 100 MeV	$\pi^\pm, \pi^0, \mu$ annihilate	10.75	$e^\pm, \nu, \bar{\nu}, \gamma$ left
< 500 keV	$e^\pm$ annihilation	7.25	

The role of the QCD phase transition

Quark-Gluon Plasma: The Stuff of the Early Universe



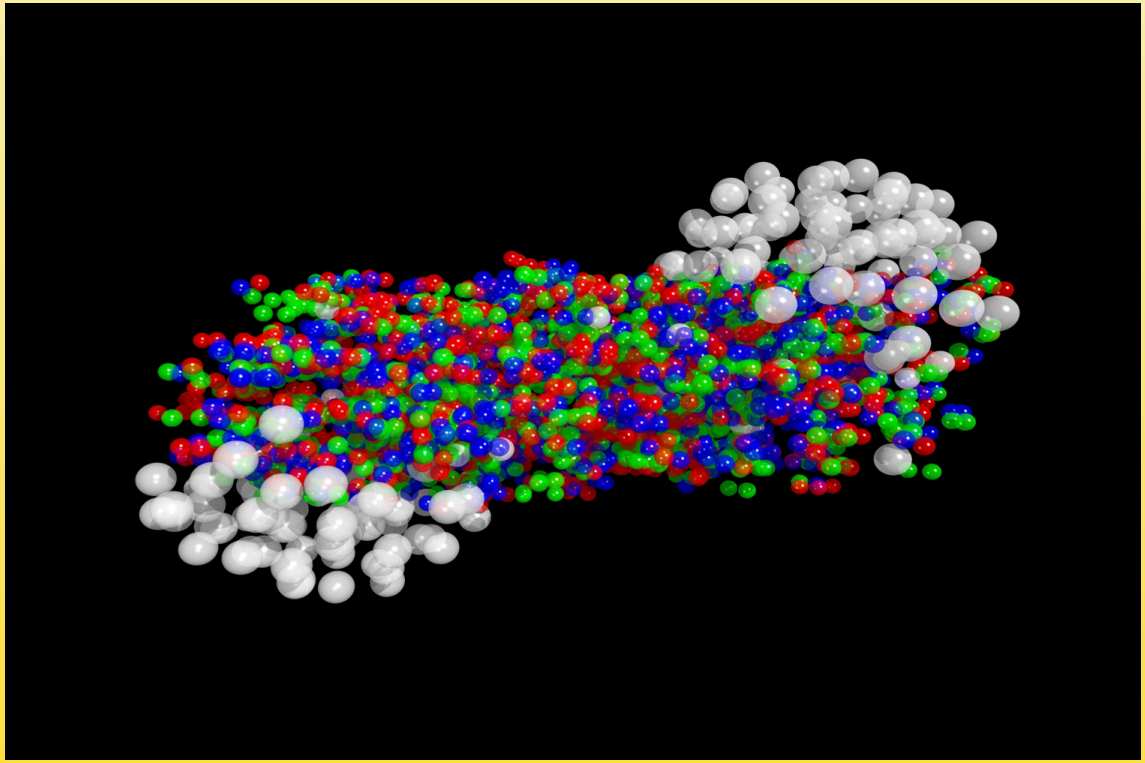


QCD under extreme conditions:

- High temperature matter: early universe
- High density matter: in neuron stars and other supernova remnants

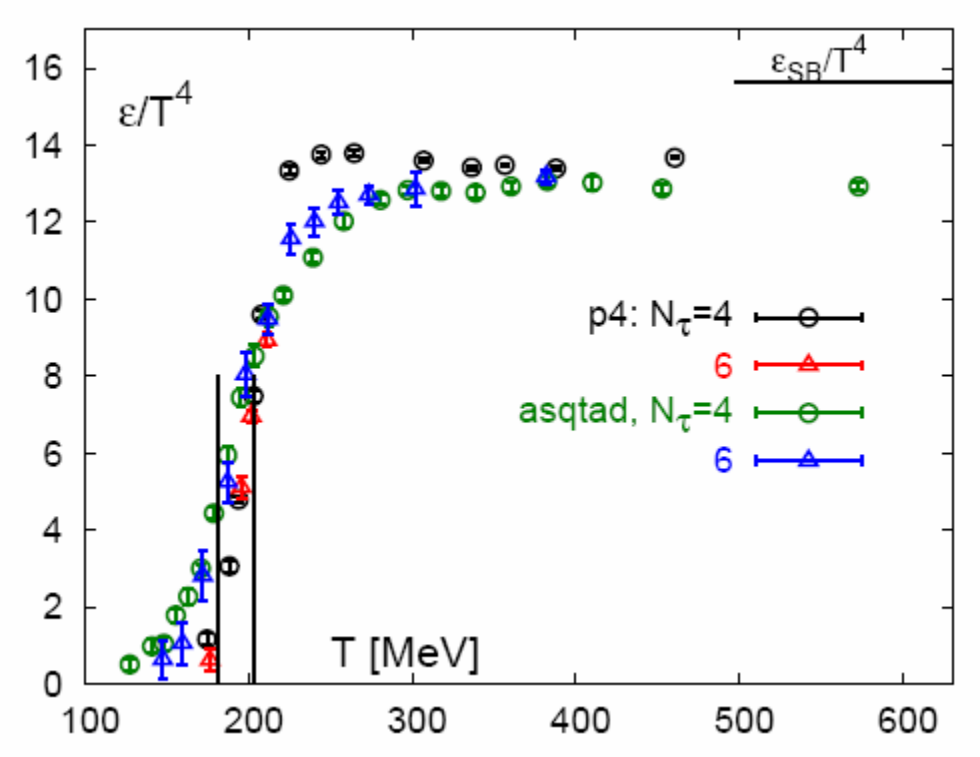
❒ Laboratory test to come: heavy ion collisions RICH/Brookhaven, LHC/Geneva

Nuclei in collision:



QCD phase transition: non-perturbative regime ➠ only known approach

lattice QCD [field theory in a heat bath at finite temperature]



Physics of the early universe:

- Entropy per co-moving volume is conserved

- All entropy is in relativistic species
  Expansion covers many decades in T, so typically either $T \gg m$ (relativistic) or $T \ll m$ (frozen out)

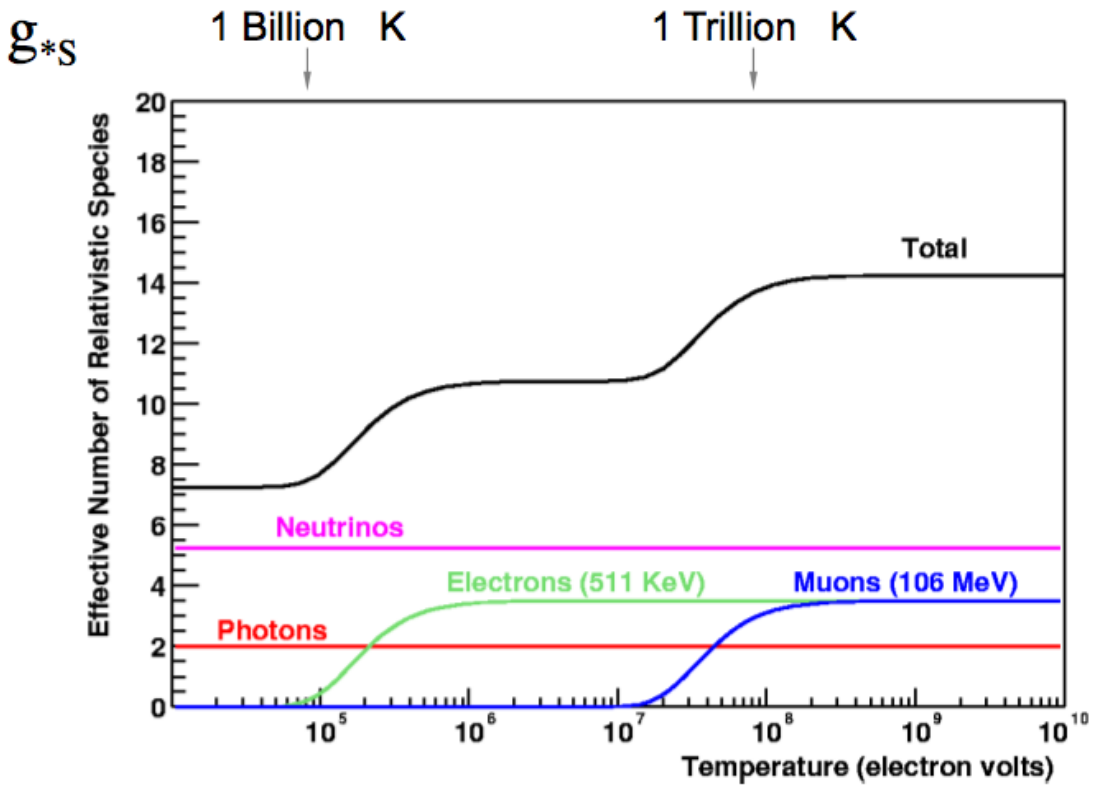
- All chemical potentials are negligible

Entropy S in co-moving volume $(D_c)^3$ preserved

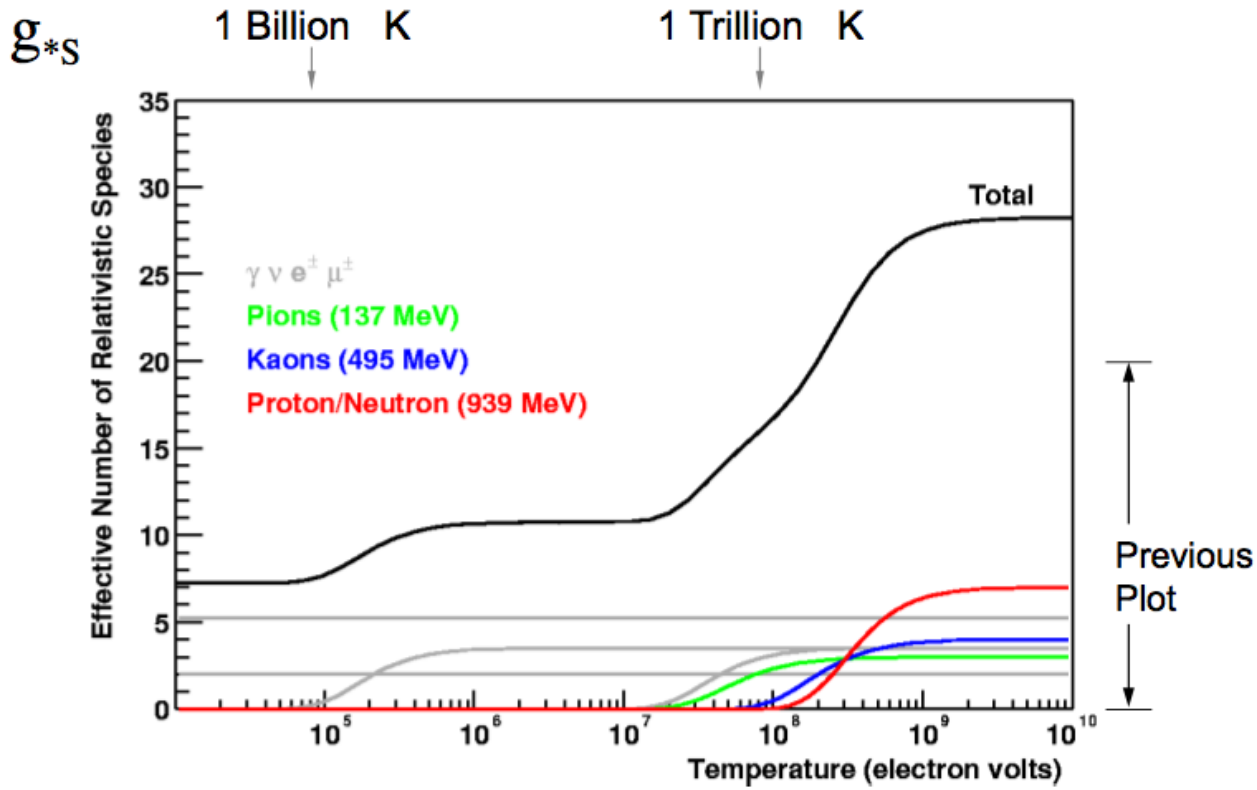
g_{*S} effective number of relativistic species

Entropy density $\frac{S}{V} = \frac{S}{D_c^3}\frac{1}{S(t)^3} = \frac{2p^2}{45} g_S^* \, T^3$

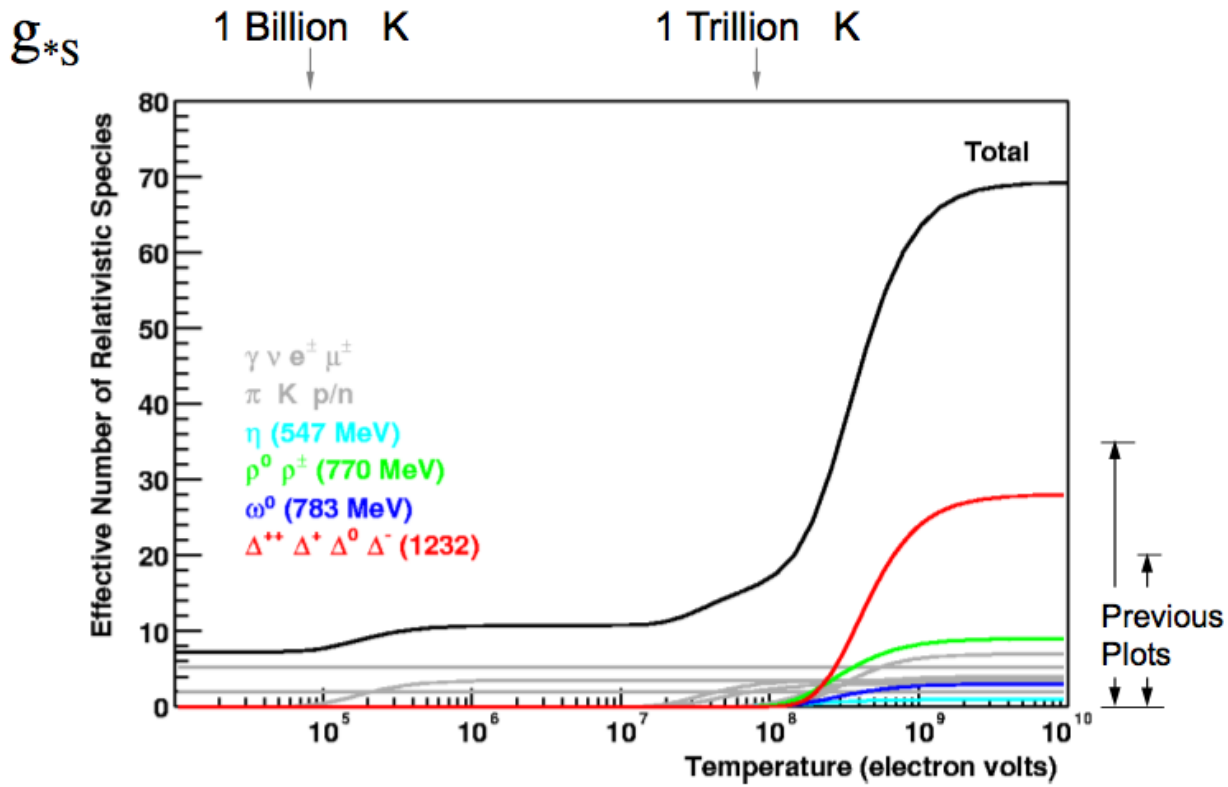
- $T = (g_{*S})^{-1/3} \, \frac{1}{S(t)}$



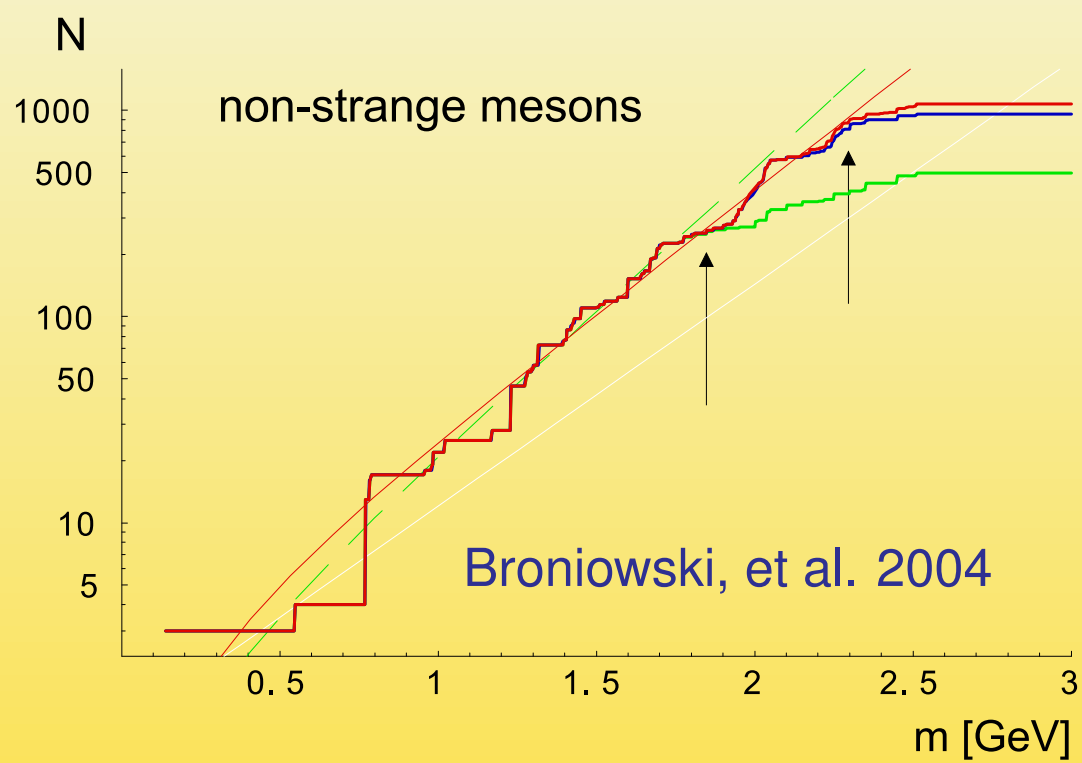
Start with light particles, no strong nuclear force



Now add *hadrons* = feel strong nuclear force



Keep adding more hadrons….



How many hadrons?

Density of hadron mass states dN/dM increases exponentially:

$$\frac{dN}{dM} \sim M^a \exp M/T_{\rm H} \; (T_{\rm H} \sim 2 \times 10^{12}\,{}^\circ{\rm K} \; = 170 \text{ MeV})$$

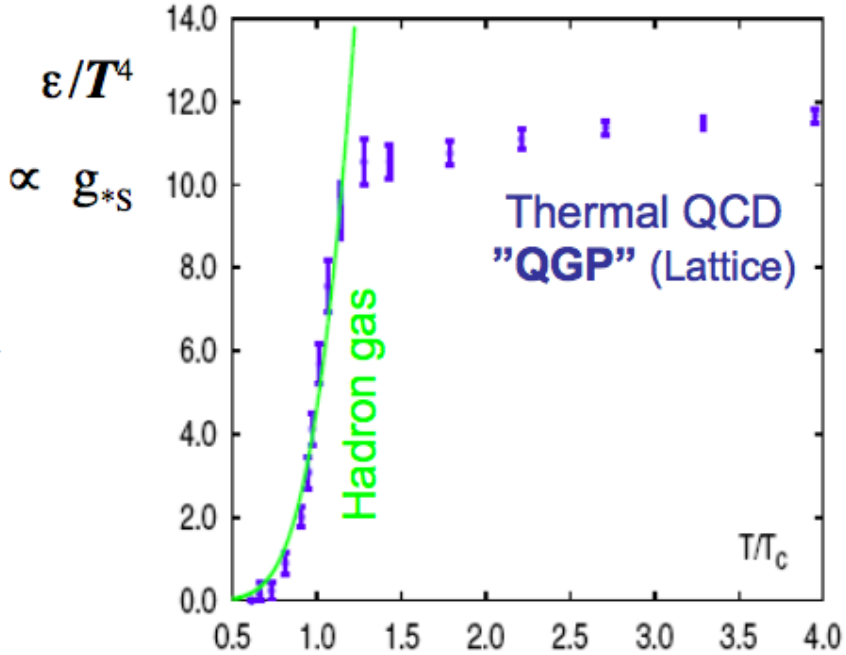
QCD to the rescue!

Replace Hadrons (messy and numerous)

by Quarks and Gluons (simple and few)

D. Gross
H.D. Politzer
F. Wilczek
American

QCD Asymptotic Freedom (1973)

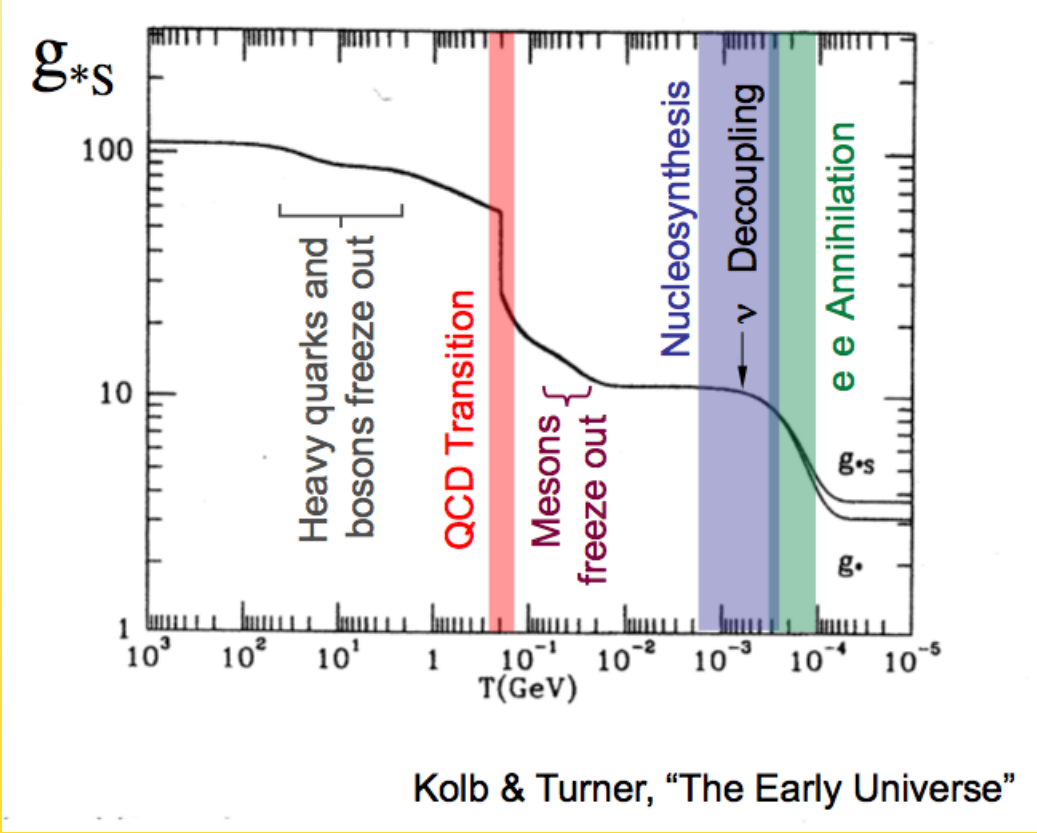


Karsch, Redlich, Tawfik, Eur.Phys.J.**C29**:549-556,2003

"In 1972 the early universe seemed hopelessly opaque…conditions of ultrahigh temperatures…produce a theoretically intractable mess. But asymptotic freedom renders ultrahigh temperatures friendly…"
Frank Wilczek, Nobel Lecture (RMP 05)

from P. Stankus, APS 09

Before [QCD] we could not go back further than 200,000 years after the Big Bang. Today since QCD simplifies at high energy, we can extrapolate to very early times when nucleons melted to form a quark-gluon plasma. David Gross, Nobel Lecture (RMP 05)



Kolb & Turner, "The Early Universe"

Cosmological phase transition....

...when the universe cools down below 175 MeV

10^{-5} seconds after the big bang...

Quarks and gluons form baryons and mesons

before: simply not enough volume per particle available

Cosmological phase transitions

QCD phase transition T=175 MeV Electroweak phase transition T= 150 GeV
baryogenesis? others earlier?

Electroweak phase transition ?

10^{-12} seconds after big bang

fermions, W and Z bosons get mass

Standard model: cross over transition baryogenesis if 1st order bubble formation "out of vacuum"

QCD at High temperature, at high T less order more symmetry (magnets crystals)

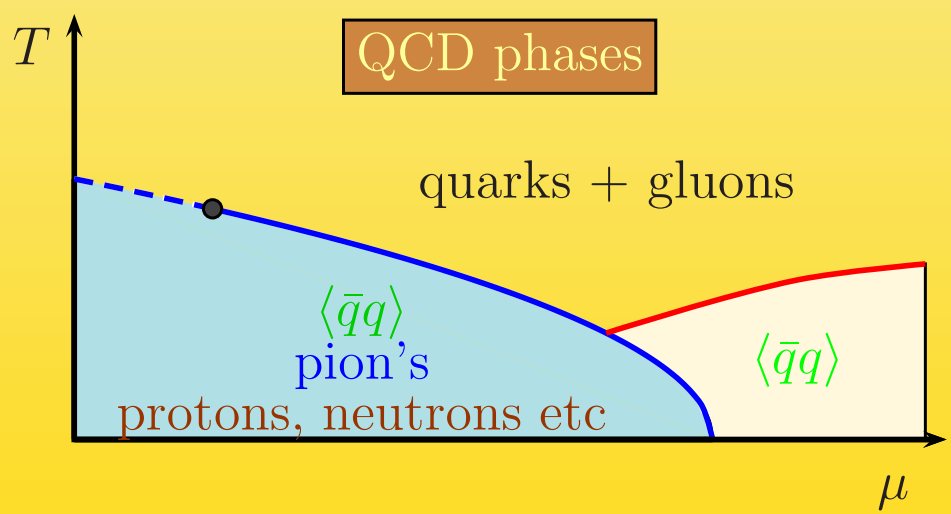
- Quark-gluon plasma
- Chiral symmetry restored (no pions, no quark condensates)
- "Deconfinement" (no linear heavy quark potential at large distances)
- Lattice QCD simulations: both effects happen at the same temperature

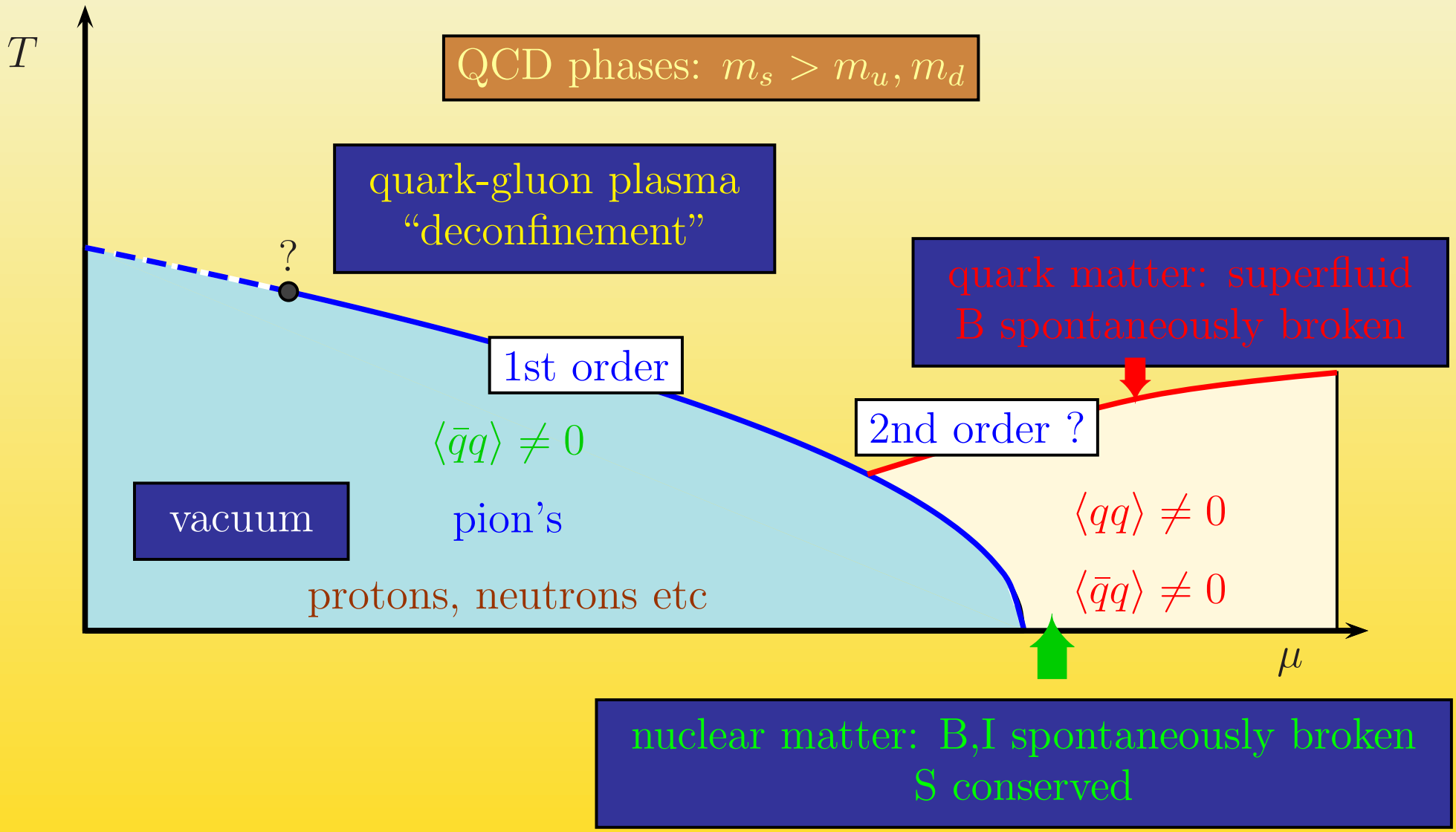
QCD - phase transition

Quark-gluon plasma		Hadron gas	
❖ Gluons:	$8 \times 2 = 16$	❖ Light mesons:	8
❖ Quarks:	$9 \times 7/22 = 12.5$	❖ (Pions:	3)
❖ DOF:	28.5	❖ DOF:	8
Chiral symmetry		Chiral symmetry broken	

Large difference in number of degrees of freedom !

Strong increase of density and energy density at T_c !




T
QCD phases: $m_s > m_u, m_d$
quark-gluon plasma
"deconfinement"
?
1st order
quark matter: superfluid
B spontaneously broken
2nd order ?
$\langle \bar{q}q \rangle \neq 0$
vacuum
pion's
$\langle qq \rangle \neq 0$
$\langle \bar{q}q \rangle \neq 0$
protons, neutrons etc
μ
nuclear matter: B,I spontaneously broken
S conserved

Exercise pages

Estimate the collision rate for the NC process

$$e^+e^- \leftrightharpoons \bar{\nu}_i\nu_i$$

at $T \sim 10^{11}\ {}^\circ\mathrm{K}$ and compare it with the charged current (CC) muon capture

$$e^- + \mu^+ \leftrightarrow \nu_e + \bar{\nu}_\mu$$

(inverse muon decay) process rate.

Exercise: Reproduce the Table T versus t given above by numerically integrating the corresponding integral equation.

Exercise: In calculating the cosmic neutrino background we assumed neutrinos to be massless. Discuss what modifications we have to expect due to the fact that observed neutrino oscillations require neutrino masses at the meV scale.

Exercise: For large enough temperatures, when $T \gg |\mu|$ and $T \gg m$, we may use the approximations $\mu = 0$ and $m = 0$. Show that in this ultrarelativistic limit the following results are found (here we use units such that $k_{\rm B} = 1$, i.e. temperatures in energy units):

$$n = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi q^2\,\mathrm{d}q}{\mathrm{e}^{q/T} \pm 1} = \begin{cases} \frac{3}{4\pi^2}\,\zeta(3)\,gT^3 & \text{fermions} \\ \frac{1}{\pi^2}\,\zeta(3)\,gT^3 & \text{bosons} \end{cases}$$

$$\rho = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi q^3\,\mathrm{d}q}{\mathrm{e}^{q/T} \pm 1} = \begin{cases} \frac{7}{8}\frac{\pi^2}{30}\,gT^4 & \text{fermions} \\ \frac{\pi^2}{30}\,gT^4 & \text{bosons} \end{cases}$$

$$p = \frac{g}{(2\pi)^3} \int_0^\infty \frac{\frac{4}{3}\pi q^3\,\mathrm{d}q}{\mathrm{e}^{q/T} \pm 1} = \frac{1}{3}\rho$$

Above $\zeta(3) \equiv \sum_{n=1}^{\infty}(1/n^3) = 1.20206...$ where $\zeta(n)$ is Riemann's zeta function. Furthermore, calculate the average particle energy

$$\langle E \rangle = \frac{\rho}{n}$$

Exercise: In the non-relativistic limit $T \ll m$ and $T \ll m - \mu$, the typical kinetic energies are much below the mass m and $E = m + q^2/2m$. The second condition leads to small occupation numbers (dilute gas) and is usually satisfied in cosmology if the first one is satisfied. We can the use the approximation

$$\mathrm{e}^{(E-\mu)/T} \pm 1 \simeq \mathrm{e}^{(E-\mu)/T}$$

such that bosons and fermions have the same distributions, the non-relativistic

Maxwell-Boltzmann one. Show that the following relations hold in this case:

$$
\begin{aligned}
n &= g\left(\frac{mT}{2\pi}\right)^{3/2} \mathrm{e}^{-\frac{m-\mu}{T}} \\
\rho &= n\left(m+\frac{3T}{2}\right) \\
p &= nT \ll \rho \\
\langle E\rangle &= m+\frac{3T}{2} \\
n-\bar{n} &= 2g\left(\frac{mT}{2\pi}\right)^{3/2} \mathrm{e}^{-\frac{m}{T}} \sinh\frac{\mu}{T}
\end{aligned}
$$

Previous ⋘| , next |⋙ lecture.