Special Relativity in a Nutshell

Basic Postulates: Einstein 1905

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The speed of light \( c \) is limiting velocity. No physical body can move at speed bigger than \( c \). It is not possible to transport information with speed larger than \( c \) (signal velocity \( v \leq c \)). \( \textcircled{1} \) has to be considered as a convention for the synchronization of clocks, which move relative to each other at constant velocity!
Lorentz Transformations

How did Einstein find these postulates?
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Rotations $R$:

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\begin{align*}
\vec{x}' &= R \vec{x} \\
\tau' &= \tau
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\( G \) implements the relative motion; Time \( t \) is absolute and not transformed besides shifts, i.e. **time intervals are invariant.**
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Special Lorentz transformations \( L \):

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with \( v = |\vec{v}|, \gamma = \text{Lorentz factor} \)
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\[
\vec{n} = \frac{\vec{v}}{v} ; \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

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New form of velocity transformation. Not $t$ is invariant but limiting velocity $c$. Time lost absolute meaning, time is relative! $L$ is the prescription which allows us to translate space–time coordinate information between systems having constant relative velocity.
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Note that in Electrodynamics, Maxwell’s unification of electric and magnetic phenomena, the speed of light shows up as a constant

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

which is the speed of propagation of electromagnetic waves. $\varepsilon_0$ permittivity of free space (electric constant), $\mu_0$ permeability of free space (magnetic constant), both are universal constants.
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Lorentz invariance of Electrodynamics is true under the natural assumption that $c$ is a universal constant. Einstein with his covariant reformulation of Electrodynamics, did not attempt to modify Maxwell’s theory, so the universality of the speed of light had to be incorporated in the reformulation of the laws of motion (relativistic kinematics). In fact the universality and finiteness of the speed of light is a key element of special relativity.
Einstein’s covariant reformulation of classical Electrodynamics was triggered by the observation that moving a conducting coil in a magnetic field is described by the Lorentz force law, while moving the magnet relative to the coil at rest is described by the law of induction. Except for the constant relative motion the physics is the same. Therefore physical laws have to be formulated in a manner which does not depend on constant relative motion.
Consequences

1) Physical laws are invariant with respect to Lorentz transformations (generally: Poincaré group)

\[ x' = \gamma (x - vt) \]
\[ y' = y \]
\[ z' = z \]
\[ t' = \gamma \left( t - \frac{vx}{c^2} \right) \]

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Time not absolute anymore, transforms non–trivially

Space-Time
Cosmology

Classical limiting case: $v \ll c$ formally $c \to \infty$ \(L\)-transformation $\rightarrow$ Galilei-transformation
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Indeed: $c \rightarrow \infty$ ; $\beta \rightarrow 0$ ; $\gamma \rightarrow 1$

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relativistic formula of addition of velocities.
Cosmology

\[ c' = c \neq c - v \]
\[ c' = c \neq c + v \]

light flash

60 + 80 = 140?

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Additivity of velocities does not hold for high velocities:

Error at a speed of 100 km/h: \( \frac{v}{c} \approx \frac{100}{300,000} \cdot \frac{1}{60-60} \approx \frac{1}{10,000,000} \)
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In place of cars light flashes: independent of movement of train always \(c\) and not \(c + v_{\text{train}}\)
Cosmology

Limit velocity: $v_2, v_2 < c \rightarrow v < c$ where $v(v_1, v_2)$ is monotonically increasing in $v_1$ and $v_2$ and for $v_1 \rightarrow c \triangleright v \rightarrow c$
Limit velocity: \( v_2, v_2 < c \rightarrow v < c \) where \( v(v_1, v_2) \) is monotonically increasing in \( v_1 \) and \( v_2 \) and for \( v_1 \rightarrow c \rightarrow v \rightarrow c \)
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Exercise: show that two Lorentz transformations along the $x$-axis, $L_1(v_1)$ and $L_2(v_2)$, executed one after the other yield a Lorentz transformation along the $x$-axis $L(v) = L_2(v_2) L_1(v_1)$. Calculate $v$. Of course this can be generalized to arbitrary Lorentz transformations. The simple scalar velocity addition is then replaced by an appropriate vector relation between $\vec{v}$, $\vec{v}_1$ and $\vec{v}_2$. 

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Lect. 4
Exercise: comment the following paradox: the locomotive driver knows his train will not fit into the tunnel the track man is sure it will
3) **Lorentz contraction**

A measuring rod of length $\ell_0$ along the $x$–direction at rest in a IS A appears to be length contracted in a moving IS B (Moving measuring rods appear shortened in the direction they are moving)
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\begin{align*}
t' = 0 & \Rightarrow t = \frac{v}{c^2} x \\
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Transversal to the direction of the movement there is no contraction. This, however, does not mean that we see a moving sphere as an ellipsoid \*\». **Illustration:** a chain of moving dice \*\»

Other: \*\»; Download: theory \*\» (Weiskopf, Kraus, Ruder)
4) **Time dilatation**

Consider a clock with oscillation time $T_0$ at rest at a point $\vec{x} = 0$ in the IS A. Observed from the moving IS B the clock appears to oscillate slower:
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Exercise: In a muon storage ring muons move at the magic energy $E = 3.1$ GeV. Compare their velocity with the one of the light. Calculate their lifetime in the Laboratory frame.
Note that the time dilatation is a real phenomenon. The muons circulating in the storage ring indeed decay sooner or later depending on their speed. Muons at rest would decay on place, while highly relativistic muons circulate a number of times in the ring before they decay. This phenomenon is also well known from cosmic muons, for example. Note that unstable particles are clocks, with an intrinsic lifetime in their rest frame. In the rest frame the lifetime is shortest.
Space-Time: Minkowski Geometry

Event-space: defined by relationship
Event ⇔ Point \( P = P(x^0, x^1, x^2, x^3) \) in event-space
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**Cosmology**

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Motion of a point-particle: continuous series of events $\Rightarrow$ continuous curve in the event-space $\Rightarrow$ World-line of the point-particle
Cosmology

\[ x^0 = ct \]

world line

path
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$$\vec{x} = \vec{n} c t ; \quad |\vec{n}| = 1 , \quad \vec{n} \quad \text{direction/unit/vector}$$
Therefore the world line of a photon in any case is a line on the envelope of a cone: the light cone

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Particle of mass \( m > 0 \) \( \Rightarrow \) \( v < c \)

▷ world line at each point strictly inside light cone.
Measuring Space and Time

Minkowski distance measurement ≡ Minkowski metric
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Theorem: the indefinite quadratic form

\[ s^2 \equiv (x^0)^2 - \vec{x}^2 \]

is invariant under Lorentz transformations:
Measuring Space and Time

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Classification: $s^2 = \text{constant}$ for a plane $x^2 = x^3 = 0$
Events A and B relative to each other lie:

\[ s_{AB}^2 > 0 \quad \text{time like} \quad \text{two–shell hyperboloid} \]
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Cosmology

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Interpretation

1. $s_{AB}^2 > 0$: the events A and B lie time-like to each other. The straight world line between A and B corresponds to the motion of a massive particle at velocity $v < c$. In the rest frame of the particle the events A and B happen at the same place $\vec{x}_A = \vec{x}_B = \vec{x}$ at different times $t_A$ and $t_B$. In other words, there exist an IS in which
Interpretation

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Time measurement with a clock in arbitrary motion

\[ \tilde{v} = \tilde{v}(t) \]

\[ T_{AB} = \int_A^B d\tilde{T} = \int_A^B dt \sqrt{1 - \frac{\tilde{v}^2}{c^2}} \]
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\( ds \) is the Minkowski line element
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Cosmology

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The proper time is a maximum for straight line uniform motion.
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The proper time is a maximum for straight line uniform motion

\[ \int_{\Gamma_1}^{B} ds < \int_{\Gamma_2}^{B} ds = \text{max}. \]
Twin Paradox

Note that time has no longer an absolute meaning, different observers can draw different conclusions concerning non-invariant quantities. The main point in the twin paradox is that one observer stays at rest in the supposed IS, while the other leaves the position and later comes back. The latter observer cannot stay all the time in an IS and in fact acceleration change the ticking of watches as we will see later. If later the twins again sit together in the same IS, they have a different history and there is no reason why this should not distinguish them.
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Minkowski metric (indefinite):

\[
g_{\mu\nu} := \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \end{pmatrix}
\]
Cosmology

To be compared with common 3-d Euclidean space

Metric: quadratic form $d\ell^2 = \sum_{i=1}^{3} \sum_{k=1}^{3} e_{ik} dx^i dx^k$
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To be compared with common 3-d Euclidean space

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Euclidean metric (positive definite)= unit matrix

$$e_{ik} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
Relativistic Kinematics

World line of a point particle: $x^\mu = x^\mu(\lambda)$, $\lambda$ a parameter (parametrization of world line). The natural parameter is the arc length $\lambda = s$, chosen such that $x^0 \geq 0$ for $s \geq 0$, with reference point $x^\mu(0) = (0, \vec{x}(0))$
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We then define: four-velocity $u^\mu = \frac{dx^\mu}{ds}$

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\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (dx^i)^2 = (dx^0)^2 \left\{ 1 - \left( \frac{dx^i}{dx^0} \right)^2 \right\} \]
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Momentum and Energy

Again the requirements are

- relativistic covariance (how to unambiguously transfer information from one IS to another)
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velocity $\vec{v}$  $\Rightarrow$  Four-velocity $u^\mu = \gamma \left( 1, \frac{\vec{v}}{c} \right)$
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By definition $p^\mu$ is a contravariant four-vector.
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Rest energy new term!
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represent the relativistic generalizations of the non-relativistic versions of momentum and energy!
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What is new?
Example 1.: Decay of a neutral Kaon into two identical neutral pions

\[ K \rightarrow \pi^0\pi^0 \]
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Experimentally: $$m_K = 3.7 m_{\pi^0}$$ with $$m_{\pi} c^2 = 134.97 \text{ MeV}$$
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= \left( \frac{E_1}{c} + \frac{E_2}{c}, \vec{0} \right) ; \quad \vec{p}_1 = -\vec{p}_2
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Example 1.: Decay of a neutral Kaon into two identical neutral pions

\[ K \rightarrow \pi^0 \pi^0 \]

Experimentally: \( m_K = 3.7 \, m_{\pi^0} \) with \( m_{\pi^0}c^2 = 134.97 \text{ MeV} \)

In the rest system of the \( K \):

\[
P = (m_Kc, \vec{0}) = p_1 + p_2
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with

\[
\vec{p} = \frac{m \vec{\nu}}{\sqrt{1 - \frac{\vec{\nu}^2}{c^2}}} ; \quad p^0 = \frac{m c}{\sqrt{1 - \frac{\vec{\nu}^2}{c^2}}}
\]
indeed
indeed

\[ p^0 - \vec{p}^2 = m^2 c^2 \quad \Rightarrow \quad p^0 = +\sqrt{m^2 c^2 + \vec{p}^2} \]
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\[ p^0{}^2 - \vec{p}{}^2 = m^2 c^2 \implies p^0 = + \sqrt{m^2 c^2 + \vec{p}{}^2} \]

where we note that the energy must be positive (spectral condition).
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E_1 = E_2 \quad \Leftrightarrow \quad p^0_1 = p^0_2 = p^0 \]

Thus

\[ p = (m_K c, 0) = \left( 2 \frac{E_{\pi^0}}{c}, 0 \right) \]
i.e.
i.e.

\[ 2E_{\pi^0} = m_K c = 2c \sqrt{m_{\pi^0}^2 c^2 + \vec{p}^2} \]
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This very important result illustrates a novel phenomenon in a relativistic theory: the difference of the squared rest masses

\[ m_K - (2 \, m_\pi)^2 \]

is transmuted to kinetic energy!
Cosmology

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The exist no conservation law for rest masses!
Example 2.: Decay of the neutral $\pi^0$ into two photons $\pi^0 \rightarrow \gamma\gamma$
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$$p^0 = \sqrt{m_\gamma^2 c^2 + \vec{p}^2} \;; \quad m_\gamma = 0 \quad \text{i.e.} \quad p^0 = |\vec{p}| \cdot c$$

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Thus

$$ E_1 = E_2 = |\vec{p}| \cdot c $$

$$ m_\pi c = 2|\vec{p}| \Rightarrow |\vec{p}_\gamma| = \frac{m_\pi c}{2}. $$

As photons have no rest mass, we have the result:

$\| \pi^0$ decays into pure energy of motion $!!!
We first note that

$$p^2 = g_{\mu\nu} p^\mu p^\nu = (p^0)^2 - \vec{p}^2 = m^2 c^2 u^\mu u^\nu g_{\mu\nu}$$
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\[ E = c p^0 = c \sqrt{m^2 c^2 + \vec{p}^2} \approx m c^2 + \frac{\vec{p}^2}{2m} - \frac{1}{8} \frac{\vec{p}^4}{m^3 c^2} + \cdots \]
Tachyons

all physical states

$m = 0 \Rightarrow p^0 = |\vec{p}|$

$E > 0$

$E < 0$

unphysical

$|\vec{p}|$

$\vec{p}$

$p_\mu$

$c$
**Physical Spectral Condition:** all physical states must have $p^0 > 0$ and $p^2 \leq 0$
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Massive particles: $p^2 > 0$; $m > 0$; $p^0 > 0$
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Massive particles: $p^2 > 0$ ; $m > 0$ ; $p^0 > 0$

Massless particles: $p^2 = 0$ ; $m = 0$ ; $p^0 > 0$
Cosmology

for massless particles

\[ \text{i.e. for } s \geq 1 \text{ massless states have a lower number of independent degrees of freedom than massive one's} \]

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for massless particles

\[ p^2 = g_{\mu \nu} p^\mu p^\nu = \left( \frac{E}{c} \right)^2 - \vec{p}^2 = 0 \Rightarrow E = c |\vec{p}| \]

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As \( E = \gamma m c^2 \) and \( \vec{p} = \gamma m \vec{v} \Rightarrow |\vec{v}| = c \).

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Cosmology

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all massless particles propagate with velocity \( c \)!

- each massless state of spin \( s > 0 \) has exactly two possible helicities
- the two helicity states do not mix [two independent degrees of freedom]

\[ \text{\footnotesize i.e. for } s \geq 1 \text{ massless states have a lower number of independent degrees of freedom than massive one's} \]

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Cosmology

for massless particles

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all massless particles propagate with velocity \( c \)!

- each massless state of spin \( s > 0 \) has exactly two possible helicities
- the two helicity states do not mix [two independent degrees of freedom]
- for massive particles \((m > 0)\), the number of states is \( 2s + 1 \)

\[ \text{---} \]

\[ \text{i.e. for } s \geq 1 \text{ massless states have a lower number of independent degrees of freedom than massive one's} \]
Waves and Doppler Shift

Photons in the particle picture carry energy $E$ and momentum $\vec{p}$, which at the same time are electromagnetic waves angular frequency $\omega$ and wave-vector $\vec{k}$ in the wave picture:

$$\omega = 2\pi \nu, \quad |\vec{k}| = \frac{2\pi}{\lambda}$$
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$$

As a wave we have

$$
\vec{E}(\vec{x}, t) = \vec{E}_0 \, e^{i(\vec{k} \cdot \vec{x} - \omega t)} + \text{h.c.}
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Waves and Doppler Shift

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As a wave we have

$$\vec{E}(\vec{x}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} + \text{h.c.}$$

and we may write $\omega t - \vec{k} \cdot \vec{x} = k x = k_{\mu} x^{\mu} = k^0 x^0 - \vec{k} \cdot \vec{x}$
\[ k^\mu = \left( \frac{\omega}{c}, \vec{k} \right) \text{ wave-vector} \]
$k^\mu = \left( \frac{\omega}{c}, \vec{k} \right)$ wave-vector

The particle–wave dualism is characterized by $p^\mu$ or $k^\mu$
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Relativistic Doppler Shift
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**Relativistic Doppler Shift**

We consider the frame dependence of the frequency of an electromagnetic wave. It is given by a Lorentz transformation
Cosmology

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Relativistic Doppler Shift

We consider the frame dependence of the frequency of an electromagnetic wave. It is given by a Lorentz transformation

\[ k'^\mu = \Lambda^\mu_\nu k^\nu . \]
Consider a wave of wave-vector

\[ \vec{k} = |\vec{k}| \ (\cos \theta, -\sin \theta, 0) \]
Consider a wave of wave-vector 

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with \( |\vec{k}| = \frac{2\pi}{\lambda} \).

We now perform a special Lorentz transformation in \( x \)-direction:

\[
\begin{align*}
    k^0' &= \gamma k^0 - \frac{v}{c} \gamma k^1 \\
    \frac{\omega'}{c} &= \gamma \left( \frac{\omega}{c} - \frac{v}{c} \frac{2\pi \cdot \nu}{\lambda \cdot \nu} \cos \theta \right) \\
    \nu' &= \nu \gamma \left( 1 - \frac{v}{c} \cos \theta \right).
\end{align*}
\]

Take \( \Lambda \) to be the Lorentz transformation to the system in which the source is at rest: \( \nu' = \nu_0 \).
We thus get
We thus get

\[ \nu = \frac{\nu_0}{1 - \frac{v}{c} \cos \theta} \sqrt{1 - \frac{v^2}{c^2}} \]
We thus get

\[ \nu = \nu_0 \sqrt{\frac{1 - \frac{\nu^2}{c^2}}{1 - \frac{\nu}{c} \cos \theta}} \]
Exercise 2.: Calculate energy and frequency of the $\gamma$'s in the decay of a $\pi^0$ at rest, as well as the decay in flight for $v_{\pi^0} = 0.9 \, c$.

Exercise 3.: Show that the relativistic action principle of a free particle reads

$$S = -m \, c \, \int_{1}^{2} ds$$

i.e. the action is proportional to the Minkowski arc length. $L(\vec{x}, \dot{\vec{x}}) = -m \, c^2 \sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}$, is the relativistic Lagrangian of a free particle. Formulate the principle of least action in geometrical form.

Exercise 4: Maxwell’s unification of electric and magnetic phenomena resulted in the prediction of electromagnetic waves, propagating at the speed of light and in fact light represents electromagnetic waves. The covariant formulation of Maxwell’s equations reads:
\[ \partial_\mu F^{\mu\nu} = -e j^\nu_{\text{em}} ; \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 , \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the anti-symmetric electromagnetic field strength tensor, with \( A_\mu \) is the electromagnetic four-potential (classical photon field), \( j^\mu_{\text{em}} \) is the electromagnetic current and \( \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) is the dual pseudotensor. Show that in vacuum, where \( j_{\text{em}}^\mu = 0 \), the photon potential \( A_\mu \), in the Lorentz gauge \( \partial_\nu A^\nu = 0 \), satisfies the wave-equation \( \Box A_\mu = 0 \). Contemplate about the role of this fact, the role of light rays in Einstein’s special relativity theory and the meaning of Lorentz invariance in this context.

Remark about the gauge condition needed: note that the field \( A^\mu \) has unphysical degrees of freedom as it has four components while a physical photon only has two physical degrees of freedom. One thus requires some subsidiary conditions to eliminate the unphysical degrees of freedom, i.e. we must fix a gauge. In general the longitudinal component \( \partial_\nu A^\nu(x) \doteq \phi(x) \) is an independent scalar field, which however has nothing to do with the physical photon. The Lorentz condition
\[ \partial_y A^y = 0 \] eliminates this scalar component. In general the unphysical degrees of freedom are sorted out by the requirement of gauge invariance. Indeed, Maxwell’s equations are invariant under photon field transformation \( A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha(x) \), where \( \alpha(x) \) is an arbitrary scalar function.
Appendix: technicalities on Minkowski space and Lorentz transformations

The known simple form of Maxwell’s equations of classical electrodynamics holds for a restricted class of coordinate frames only, the so called inertial frames. The space–time transformations which leave the Maxwell equations invariant form the Lorentz group. Accordingly, Lorentz transformations are transformations between different inertial frames. The invariance group of Maxwell’s equations in this way singles out a particular space-time structure, the Minkowski space. Lorentz invariance is a basic principle which applies for the other fundamental interactions. We briefly sketch the elements which we will need for a discussion of a relativistic theory.

A space-time event is described by a point (contravariant vector)

\[ x'^\mu = (x^0, x^1, x^2, x^3) = (x^0, \vec{x}) ; \quad x^0 = t (= \text{time}) \quad \text{in units where} \quad c = 1 \]
Cosmology

in Minkowski space with metric

\[ g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

The metric defines a scalar product

\[ xy = x^0 y^0 - \vec{x} \cdot \vec{y} = g_{\mu\nu} x^\mu y^\nu \]

invariant under Lorentz transformations (L-invariance) which include

1. rotations

2. special Lorentz transformations (boosts)
As usual we adopt the summation convention: repeated indices are summed over unless stated otherwise. For Lorentz indices $\mu, \cdots = 0, 1, 2, 3$ summation only makes sense (i.e. respects $L$-invariance) between upper (contravariant) and lower (covariant) indices and is called \textbf{contraction}. Thus
\[
y x = g_{\mu \nu} x^\mu y^\nu \equiv \sum_\mu \sum_\nu g_{\mu \nu} x^\mu y^\nu = x^\mu y_\mu \equiv \sum_\mu x^\mu y_\mu
\]
The set of linear transformations
\[
x^\mu \to x'^\mu = \Lambda_\mu ^\nu x^\nu + a^\mu
\]
which leave invariant the \textbf{distance}
\[
(x - y)^2 = g_{\mu \nu} (x^\mu - y^\mu) (x^\nu - y^\nu)
\]
between two events $x$ and $y$ form the \textbf{Poincaré group} $\mathcal{P}$. $\mathcal{P}$ includes the Lorentz transformations and the translations. We denote the group elements by $(\Lambda, a)$. To two transformations $(\Lambda_1, a_1)$ and $(\Lambda_2, a_2)$, applied successively, there corresponds
a transformation \((\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)\) (multiplication law of the group). The Lorentz transformations \((\Lambda, 0)\) by themselves form the Lorentz group. L-invariance of the scalar products implies the invariance condition

\[
\Lambda^\mu_\nu \Lambda^\rho_\sigma g_{\mu\rho} = g_{\nu\sigma}
\]

for the metric. This condition on the matrices \(\Lambda^\mu_\nu\) actually fully determines them and implies \(\det \Lambda = \pm 1\) and \(|\Lambda^0_0| \geq 1\). Transformations with determinant \(\Lambda = +1\) are called proper (+). Such transformations do not change the orientation of frames (no space-reflections). Transformations with the property \(\Lambda^0_0 \geq 1\) are called orthochronous (↑), since they exclude time inversions (no time-reversal).

\(\Lambda^0_i\) \((i = 1, 2, 3)\) are the special Lorentz transformations (=boosts=velocity transformations) and \(\Lambda^i_k\) \((i, k = 1, 2, 3)\) are the rotations. The Lorentz transformations form a group of real \(4 \times 4\) matrices of determinant unity (pseudo-orthogonal transformations).
Special relativity requires physical laws to be invariant under proper orthochronous Poincaré transformations $\mathcal{P}^\uparrow$. Thus $\mathcal{P}^\uparrow$ exhibits the general transformation law between inertial frames.

We denote by

$$\partial_\mu = \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

the derivative with respect to $x^\mu = (x^0, \vec{x})$. $\partial_\mu$ transforms as a covariant vector i.e. it has the same transformation property as $x_\mu = g^\mu_\nu x^\nu = (x^0, -\vec{x})$. The invariant D’Alembert operator (four-dimensional Laplace operator) is given by

$$\Box = \partial_\mu \partial^\mu = g^{\mu\nu} \partial_\mu \partial_\nu = \frac{\partial^2}{\partial x^0^2} - \Delta.$$

A contravariant tensor $T^{\mu_1 \mu_2 \cdots \mu_n}$ of rank $n$ is an object which has the same transformation property as the products of $n$ contravariant vectors $x^{\mu_1}_1 x^{\mu_2}_2 \cdots x^{\mu_n}_n$. Covariant or mixed tensors are defined correspondingly.
The Kronecker symbol

\[ \delta^\mu_\nu = g^{\mu\rho} g_{\rho\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

is a 2nd rank mixed tensor. With its help, contracting the invariance condition of the metric with \( g^{\sigma\lambda} \), we may write

\[ \Lambda^\mu_\nu \Lambda_\mu^\lambda = \delta^\lambda_\nu \]

which shows that

\[ \Lambda^\mu_\nu = (\Lambda^{-1})^\nu_\mu \]

is the **transpose of the inverse** of \( \Lambda \). Covariant vectors transform like

\[ x'_\mu = \Lambda^\nu_\mu x_\nu = x_\nu (\Lambda^{-1})^\nu_\mu \]
and the L-invariance of $x^2 = x_\mu x^\mu$ follows immediately.

Keep in mind:
- $x^\mu$, $dx^\mu$ are prototype contravariant four-vectors
  transformation law: $x'^\mu = \Lambda^\mu_\nu x^\nu$ etc.
- $x_\mu = g_{\mu\nu} x^\nu$, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ are prototype covariant vectors
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Thus

$$x'y' = x'^\mu y'_\mu = \Lambda^\mu_\nu \Lambda_\rho^\mu x^\nu x_\rho = \delta_\nu^\rho x^\nu x_\rho = x^\nu x_\nu = xy$$

or, with help of the metric

$$x'y' = g_{\mu\nu}x'^\mu y'^\nu = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho y^\sigma = g_{\rho\sigma} x^\rho y^\sigma = xy.$$
Finally we will need the totally antisymmetric pseudo-tensor\(^2\)

\[
\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 
+1 & (\mu\nu\rho\sigma) \text{ even permutation of (0123)} \\
-1 & (\mu\nu\rho\sigma) \text{ odd permutation of (0123)} \\
0 & \text{otherwise}
\end{cases}
\]

With the help of this tensor the determinant of any 4x4 matrix \(A\) is given by

\[
\varepsilon^{\mu\nu\rho\sigma} A^\alpha_\mu A^\beta_\nu A^\gamma_\rho A^\delta_\sigma = \det A \; \varepsilon^{\alpha\beta\gamma\delta}.
\]

\(^2\)One easily checks that it transforms as a rank 4 tensor and that it is numerically invariant (identically the same in any inertial frame). Useful relations are

\[
\begin{align*}
\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} &= -24 \\
\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma'} &= -6 \delta_{\sigma'}^\sigma \\
\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho'\sigma'} &= -2 \delta_{\rho'}^\rho \delta_{\sigma'}^\sigma + 2 \delta_{\sigma'}^\rho \delta_{\rho'}^\sigma \\
\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu'\rho'\sigma'} &= -\delta_{\rho'}^{\nu'} \delta_{\sigma'}^{\rho} \delta_{\sigma}^{\nu} + \delta_{\rho'}^{\nu'} \delta_{\sigma'}^{\rho} \delta_{\sigma}^{\nu} + \delta_{\rho'}^{\nu'} \delta_{\sigma'}^{\rho} \delta_{\sigma}^{\nu} - \delta_{\rho'}^{\nu'} \delta_{\sigma'}^{\rho} \delta_{\sigma}^{\nu} - \delta_{\rho'}^{\nu'} \delta_{\sigma'}^{\rho} \delta_{\sigma}^{\nu} + \delta_{\rho'}^{\nu'} \delta_{\sigma'}^{\rho} \delta_{\sigma}^{\nu}
\end{align*}
\]
Cosmology

The 4–dimensional volume element is invariant under proper Lorentz transformations which satisfy $\det \Lambda = 1$:

$$d^4 x \rightarrow \det \Lambda \, d^4 x = d^4 x$$

A 3–dimensional hyper-surface element is defined by a covariant vector

$$dS_\mu = \varepsilon_{\mu \nu \rho \sigma} \, dx^\nu \, dx^\rho \, dx^\sigma$$

and by partial integration we obtain

$$\int_V d^4 x \, \partial_\mu f(x) = \int_\Sigma dS_\mu f(x)$$

where $\Sigma = \partial V$ is the boundary of $V$. The Gauss law takes the form

$$\int_V d^4 x \, \partial_\mu f^\mu(x) = \int_\Sigma dS_{\mu} f^\mu(x).$$
Cosmology

For an infinite volume

\[ \int_V \ldots \to \int d^4x \ldots = \int_{-\infty}^{+\infty} dx^0 \int_{-\infty}^{+\infty} dx^1 \int_{-\infty}^{+\infty} dx^2 \int_{-\infty}^{+\infty} dx^3 \ldots \]

the surface terms vanish if the function falls off sufficiently fast in all directions. In this case a component–wise partial integration yields

\[ \int d^4x \ g(x) \partial_\mu f(x) = - \int d^4x \ (\partial_\mu g(x)) \ f(x) \]

and the integral of a divergence is vanishing

\[ \int d^4x \partial_\mu f(x) = 0 . \]
Postulates of Special Relativity

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\[ d = \sqrt{(\vec{x}_1 - \vec{x}_2)^2}; \quad E_1: (\vec{x}_1, t), \quad E_2: (\vec{x}_2, t) \]
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average rest is an IS! Ernst Mach) → Connection to Cosmology!

Note that the key point of relativity is not relativity but invariance! The invariance of
physics with respect to Poincaré transformations, in particular special Lorentz
transformations (velocity transformations). The proper (\( \det \Lambda = +1 \) i.e. no
space-time reflections), orthochronous (\( \Lambda_0^0 \geq 1 \), i.e. no time-reflections) Poincaré
group \( \mathcal{P}^+ \) is a symmetry group for the class of inertial systems.
1. Acceleration in Special Relativity
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In an inertial system (IS) a body which is not subject to any forces is freely falling. Its world line is

\[ x^\mu = u^\mu \cdot s \]

where \( u^\mu \) is the four-velocity, \( u^2 = 1 \), and \( s \) the proper time.
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General Relativity

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Coordinates in the IS: \( x^\mu \); arbitrary curved coordinates: \( x'^\mu = f^\mu(x^0, x^1, x^2, x^3) \). The change in coordinates \( x'^\mu \) under a change of the original coordinates \( x^\mu \) is
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\[ \frac{\partial x'^\mu}{\partial x^\alpha} \]

we assume to be non–singular (non-degenerate) and the differentials change according to
\[ dx^\mu = \frac{\partial x^\mu}{\partial x'^\alpha} \, dx'^\alpha \]
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\[ ds^2 = \delta_{\mu\nu} \, dx^\mu dx^\nu = g_{\mu\nu}(x') \, dx'^\mu dx'^\nu \]
\[ dx^\mu = \frac{\partial x^\mu}{\partial x'^\alpha} \, dx'^\alpha \]

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Thus in the non–IS we see the metric
\[ dx^\mu = \frac{\partial x^\mu}{\partial x'^\alpha} \, dx'^\alpha \]

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\[ g_{\mu\nu}(x') = \circ \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \]
\[ \mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x'^\alpha} \mathrm{d}x'^\alpha \]

\[ \mathrm{d}s^2 = \delta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}(x') \mathrm{d}x'^\mu \mathrm{d}x'^\nu \]

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\[ g_{\mu\nu}(x') = \delta_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \]

Note that \( g_{\mu\nu}(x') \) can be arbitrarily complicated, while \( \delta_{\mu\nu} \) is simple, the Minkowski metric:
\[ dx^\mu = \frac{\partial x^\mu}{\partial x'^\alpha} \, dx'^\alpha \]

\[ ds^2 = \overset{\circ}{g}_{\mu\nu} \, dx^\mu \, dx^\nu = g_{\mu\nu}(x') \, dx'^\mu \, dx'^\nu \]

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\[
\overset{\circ}{g}_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
In case the system $x'^\mu$ is an IS $x^\mu \to x'^\mu$ is a Lorentz transformation and thus $g_{\mu\nu}(x') = g_{\mu\nu}$. In all other cases the deviation
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describes acceleration.
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Note that in the new non-IS with world-line

$$x'^\mu = x'^\mu(\lambda) ; \quad \dot{x}'^\mu = \frac{dx'^\mu}{d\lambda},$$
In case the system $x'{}^\mu$ is an IS $x{}^\mu \rightarrow x'{}^\mu$ is a Lorentz transformation and thus $g_{\mu\nu}(x') = \hat{g}_{\mu\nu}$. In all other cases the deviation $g_{\mu\nu} - \hat{g}_{\mu\nu}$ describes \textit{acceleration}.

Note that in the new non-IS with world-line

\[ x'{}^\mu = x'{}^\mu(\lambda) ; \quad \dot{x}'{}^\mu = \frac{d x'{}^\mu}{d \lambda} , \]

the distance between events A and B is

\[ s_{AB} = \int_A^B ds = \int_A^B \sqrt{g_{\mu\nu}(x') \dot{x}'{}^\mu \dot{x}'{}^\nu} \, d \lambda . \]
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2. Gravity
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2. Gravity

Gravitation is the force which lets fall the apple from the tree and which lets circle the moon around the earth. Newton was the first who realized that the same force is responsible for both phenomena. Gravitation rules the motion of celestial bodies: planets, the Sun, double stars, galaxies, clusters of galaxies etc. and the universe as a whole.
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Gravitation acts universally on all bodies in the same manner. In contrast, for example, to the electromagnetic force, which only acts on charged bodies. In addition there is no neutralization, there is no screening of gravitation, no repulsion only attraction.
Newton’s classical form
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\[ \Delta \phi(x) = 4\pi G_N \mu(x) \]
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is not compatible with the special relativity principle. Naive relativistic generalizations do not take into account of the true nature of gravity. Einstein’s conclusion: the principle of relativity must be modified!
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Key: The equivalence principle!
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Key: The equivalence principle!

3. The Equivalence Principle

Gravitation is a very weak universally attractive interaction. It manifests itself essentially only in the presence of large masses and it cannot be screened by any means.
According to Newton’s theory of gravity

Force:

\[ F = \gamma G \frac{M_g m_g}{r^2}; \quad m_g \text{ additive} \]
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Acceleration: \[ a = \frac{F}{m_i} = \frac{m_g}{m_i} \gamma G \frac{M_g}{r^2} \]
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Experimental fact: \( m_g \equiv m_i \) for an appropriate choice of the units (definition of \( \gamma G \)). This has been verified with very high accuracy!
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describes the motion of arbitrary bodies in a gravitational field.
This means that bodies fall completely independent of any properties at the same speed. Therefore freely falling bodies are gravity free. In the framework of Newton’s theory this fact remains accidental.
Cosmology

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Einstein: $m_g = m_i$ has fundamental meaning!
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\[ m_g = m_i \] has fundamental meaning!

The following “Gedankenexperiment” illustrates the point: an observer in a sufficiently small box
free body
inertial force

$a = -g$

engine

free body
gravity force

gravity
Left: box in gravity free space accelerated (I). Right: box in gravitational field non-accelerated (II).
Left: box in gravity free space accelerated (I). Right: box in gravitational field non-accelerated (II).

If the box is sufficiently small, such that the gravitational field \( \vec{g} \) = constant within the box we can say:
There exists no experiment by means of which the observer in the box can tell in which situation (I) or (II) he is. Therefore the equivalence principle must hold:
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To each event in an accelerated reference frame (RF) there corresponds the same event in a non-accelerated RF, which is subject to a gravitational field. In other words: a no–IS is equivalent to a gravitational field.
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- inertial forces and gravity are unified (they are locally indistinguishable) and distinguishing inertial and gravitational mass is not possible as a matter of principle

- freely falling boxes (IS’s) are force–free; in them all bodies move uniformly on straight lines, i.e. freely falling systems represent **local** inertial systems (IS’s)
The equivalence principle may be formulated as a generalized relativity principle:
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For freely falling observers special relativity holds
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Fact: all gravitational fields observed in nature are inhomogeneous at large. They result from mass concentrations like stars etc. and vanish at infinity. It is at least natural to assign such behavior to true gravitational fields.
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- in the presence of gravitational fields there only exist local, i.e. in a limited region of space–time, freely falling Lorentz frames. In contrast, if gravitational fields are absent, a global Lorentz system would exist.
The equivalence principle may by formulated as a generalized relativity principle:

For freely falling observers special relativity holds
(existence of local Lorentz systems)

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- in the presence of gravitational fields the relationship between different IS’s in general is very complicated.
the difference between inertial fields and “true” gravitational fields is a matter of the behavior of the fields at infinity.
The Role of the Metric $g_{\mu\nu}(x)$

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obviously must describe the local gravitational field also if the field is a real gravitational field.
Real gravitational fields: according to the equivalence principle they must be described by the metric
Cosmology

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Cosmology

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Consequently: in the presence of gravity space-time is curved!
Cosmology

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Consequently: in the presence of gravity space-time is curved!

\[ \implies \text{Riemann Geometry!} \]
The metric $g_{\mu\nu}(x)$ in any case is determined only modulo arbitrary coordinate transformations (four arbitrary non-singular differentiable functions).
Cosmology

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Illustration: accelerated rocket and the behavior of light as a consequence of the equivalence principle.
Cosmology

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Illustration: accelerated rocket and the behavior of light as a consequence of the equivalence principle.

An accelerated box acts like a gravitational field $\Rightarrow$ light rays must be bent in a gravitational field!
Indeed: solar eclipse 1919 starlight is bent by the gravitational field of the sun!
Curved Spaces

Familiar examples:

A) Positively curved space
Curved Spaces

Familiar examples:

A) Positively curved space

Example: Neighborhood of the Sun
Curved Spaces

Familiar examples:

A) Positively curved space

Example: Neighborhood of the Sun

\[
\alpha + \beta + \gamma > 180^\circ
\]

path of a light ray
A) Negatively curved space
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Example: Carousel (merry-go-round)
A) Negatively curved space

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Result: locally, i.e., in sufficiently small regions
Cosmology

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- Accelerated systems are equivalent to systems subject to gravity
- Clocks go slower
- Scales get shorter
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Shortest distance: straight line ⇔ geodesic line
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- Clocks go slower, scales get shorter.

**Shortest distance:** straight line $\Leftrightarrow$ geodesic line

The shortest distance between two points or the path of a light signal do not coincide any longer with the straight line between the points.
Result: locally, i.e., in sufficiently small regions

- accelerated systems are equivalent to systems subject to gravity
- clocks go slower
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Shortest distance: straight line $\Leftrightarrow$ geodesic line

The shortest distance between two points or the path of a light signal do not coincide any longer with the straight line between the points.

“Straight Line” must be replaced by “Geodesic Line”, the path a freely falling body takes between the two points.
Notions like straightest, directest, shortest path may have different meaning now. For example: the cheapest way to get from New York to Paris is not taking the straight line between the two cities but the great circle:

\[ \text{New York} \rightarrow \text{Paris} \]

not the easiest way to get from New York to Paris

\text{geodesic on the sphere cut through earth center}

\text{Einstein: geometrical model}

Instead of saying that light rays or shortest distances are bent, one may say \textit{space is curved}!
The New View of the World

On a bumpy square some people play with bowls. One observes that some spots are avoided others preferred.

An observer on place says: the bowls are rolling according to the curvature of the terrain (Einstein’s point of view).

A distant observer, to whom the square looks plain, would conclude that certain (unexplained) forces act on the bowls (Newton’s point of view).
① “Space tells matter how to move”

② “Matter tells space how to curve”
Non-Euclidean Geometry

Gauss 1817, Lobachevsky 1826, Bolyai 1832, Riemann 1854

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Cosmology

- Positively curved, finite, unbounded (no boundary), prototype sphere

  curvature type: $k = 1$
  circles (radius $r$): $A < \pi r^2$, $C < 2\pi r$
  triangles: $\alpha + \beta + \gamma > 180^\circ$

- Not curved, infinite, Minkowski space = pseudo Euclidean is also flat!
  prototype plain

  curvature type: $k = 0$
  circles (radius $r$): $A = \pi r^2$, $C = 2\pi r$
  triangles: $\alpha + \beta + \gamma = 180^\circ$

- Negatively curved, infinite, prototype saddle

  curvature type: $k = -1$
  circles (radius $r$): $A > \pi r^2$, $C > 2\pi r$
  triangles: $\alpha + \beta + \gamma < 180^\circ$
Example: 4D surface of a sphere

We consider a sphere of radius $R$ in 5D flat space. The corresponding 4D surface is the curved space we want to look at. The 4D surface appears “embedded” in 5D flat space.
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We denote the coordinates on the 4D sphere by $x_i$ ($i = 1, 2, 3, 4$), $w$ is the 5th coordinate in 5D space and $R$ the radius of the sphere:
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$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + w^2 = R^2 = \text{constant}$$
The metric follows from an infinitesimal variation of the coordinates

\[ x_i \rightarrow x_i + dx_i \]
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\[ 2x_1 \text{d}x_1 + 2x_2 \text{d}x_2 + 2x_3 \text{d}x_3 + 2x_4 \text{d}x_4 + 2w \text{d}w = 0 , \]
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Cosmology

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How can we know whether we live in a curved space? For example, suppose space-time is a 4–dimensional surface of the sphere: if we proceed in a given direction we come back to the starting point.
Locally curvature is characterized by vector parallel transport.

Examples:
General: $R_{\mu,\rho \sigma}^\nu$ Riemann’s curvature tensor

Parallel transport of a vector $A^\mu$ along a closed square determined by the shifts $(\Delta u, \Delta v)$ in the parameters which parametrize the world sheet: $x^\mu = x^\mu(u, v)$. The mismatch $\Delta A^\nu$ of the parallel transported vector $A^\mu$ is a local measure of the curvature, and is proportional to the vector and to the area of the square. The curvature tensor is the corresponding tensor coefficient.

\[
\frac{\Delta A^\nu}{\Delta u \Delta v} = R_{\mu,\rho \sigma}^\nu A^\mu \frac{\partial x^\rho}{\partial u} \frac{\partial x^\sigma}{\partial v}
\]

$\Delta u, \Delta v \to 0$
Einstein’s Equations

**Newton:**  
\[ K = \gamma \frac{m_1 m_2}{r^2} = m_2 a_2 = m_1 a_1 \]

force \( \propto \frac{(\text{mass})_1 (\text{mass})_2}{(\text{distance})^2} \) = (inertial mass) \( \times \) acceleration

where mass=gravitational mass = inertial mass
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Einstein: 
\[
(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R) = \kappa T_{\mu\nu}
\]

(geometric curvature of space) \( \propto \) (matter distribution in space)
Einstein’s Equations

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\end{pmatrix}
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\end{pmatrix}
\]

Riemann: metric \( g_{\mu\nu} \rightarrow R_{\mu\nu,\rho\sigma} \) curvature tensor
\[ g_{\rho\sigma} R^\rho_{\mu\nu} = R_{\mu\nu} \; ; \; g_{\mu\nu} R^{\mu\nu} = R \]

\( R_{\mu\nu} \) Ricci tensor, \( R \) scalar curvature, \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) Einstein tensor

The Einstein’s equation is a partial differential equation of 2nd order for the metric \( g_{\mu\nu} \) for given \( T_{\mu\nu} \) (sources)

The geometrical volume element is an important ingredient in constructing the
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\[ d V_{\text{invariant}} = d^4 x \sqrt{- \det g} . \]

Exercise: show that \( d V_{\text{invariant}} \) is invariant under general coordinate transformations.

The geometrical volume element is an important ingredient in constructing the
invariant action for gravity:
Cosmology

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$$S_G = \frac{c^4}{16\pi G_N} \int d^4 x \sqrt{-g} R$$
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Application to the Sun (stars):
Schwarzschild Solution

Prediction of GR for spherically symmetric mass distributions:
Schwarzschild Solution

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A spherically symmetric system in a framework of general coordinate invariance means: there exists a coordinate system in which the solution looks spherically symmetric.
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A spherically symmetric system in a framework of general coordinate invariance means: there exists a coordinate system in which the solution looks spherically symmetric

\[ ds^2 = e^a c^2 dt^2 - e^b dr^2 - r^2 d\Omega^2 \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \)
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where \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2 \)

The last term is fixed by rotational invariance and the asymptotic flatness condition, i.e., for \( r \rightarrow \infty \) the last term dominates and must be Euclidean.
Einstein’s equation in external space, where \( T_{\mu\nu} = 0 \) yields

\[
e^a = e^b = 1 - \frac{r_0}{r} \quad \text{for} \quad r \gg r_0 \rightarrow 1 + \frac{2\phi}{c^2}
\]

where

\[
\phi = -\frac{G_N M}{r}
\]

is Newton’s potential.
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the **Schwarzschild radius**.

Typical numbers for $r_0$ follow:
Sun: \( M_\odot \approx 2 \times 10^{33} \text{ gr} \)
\[ r_{0\odot} \approx \left( 2 G_N M_\odot \right)/c^2 \approx 3 \text{ km} \]
\[ R_\odot \approx 7 \times 10^5 \text{ km} \]

Earth: \( M_\oplus \approx 6 \times 10^{27} \text{ gr} \)
\[ r_{0\oplus} \approx \left( 2 G_N M_\oplus \right)/c^2 \approx 0.9 \text{ cm} \]
\[ R_\oplus \approx 6.4 \times 10^3 \text{ km} \]

Planet closest to the Sun: distance to Sun \( r \approx 0.5 \times 10^8 \text{ km} \)

Mercury: \( M_\oplus \approx 3 \times 10^{26} \text{ gr} \)
\[ r_{0\oplus} \approx \left( 2 G_N M_\oplus \right)/c^2 \approx 0.45 \text{ mm} \]
\[ R_\oplus \approx 2.4 \times 10^3 \text{ km} \]

Note that \( r_{0\odot}/r \approx 6 \times 10^{-8} \ll 1 \) which is Newton’s regime
similarly, \( m_\oplus/M_\odot \approx 1.5 \times 10^{-7} \ll 1 \) which tells us that mercury is a test particle.
Black Hole (BH): \[ R_{BH} < r_{0BH} = (2 \, G_N \, M_{BH}) / c^2 \]
\[ \rightarrow \] very massive object (see below)

A black hole is obtained if the size of the “star” is smaller than the Schwarzschild radius. The latter is a coordinate singularity and for a far away observer photons emitted get infinite red shift at \( r_0 \), which means photons emitted by the BH (inside \( r_0 \)) get trapped by the strong gravitational field and can never reach a distant observer. \( r_0 \) in this case is a horizon. A traveler approaching the BH reaches the horizon in a finite proper time and does not see a singularity in his local coordinate frame, i.e., the singularity is not physical, still communication with a very distant observer (i.e. \( r \rightarrow \infty \)) breaks down.
Geometrical Interpretation

1. Spatial metric

\[ d\ell^2 = \frac{1}{1 - \frac{r_0}{r}} \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \]

\( d\ell \) is the frame independent proper length. We denote \( \beta = 1 - \frac{r_0}{r} \geq 1 \).

For \( r \gg r_0, \beta \rightarrow 1 \): Euclidean and we have the proper normalization.
Geometry around a Black Hole

Horizon
Black Hole and Space Warps

Horizon

Curvature visualized (embedding diagram)

\[ dr \]

\[ r \]
Radial behavior: $\theta, \varphi = \text{constant}$

For equal $\Delta r$

\[
\Delta s_1 = \frac{1}{\sqrt{1 - \frac{r_0}{r_1}}} \Delta r
\]
\[
\Delta s_2 = \frac{1}{\sqrt{1 - \frac{r_0}{r_2}}} \Delta r
\]
Thus proper length of rods shorten the more the closer to \( r_0 \):

\[
\frac{\Delta s_1}{\Delta s_2} = \frac{\sqrt{1 - \frac{r_0}{r_2}}}{\sqrt{1 - \frac{r_0}{r_1}}}
\]

This compares to the Lorentz contraction between moving IS’s in Special Relativity.

Circumference of circle:

\[
\Delta C = 2\pi \Delta r \\
\Delta s = \sqrt{\beta} \Delta r
\]

This yields:
and for the Sun with $r_0/R \approx 0.43 \times 10^{-5}$ we get

$$\frac{\Delta C}{\Delta s} \approx 2\pi \left(1 - \frac{r_0}{2R}\right)$$

Radial behavior, time dependent

$$ds^2 = \frac{1}{\beta} c^2 dt^2 - \beta dr^2$$
Cosmology

Clock in Schwarzschild field at rest: $dr = 0$

\[
\begin{align*}
ds & = c \, dT = c \frac{1}{\sqrt{\beta}} \, dt \\
\end{align*}
\]

(clock go slower!)

Compared to time dilatation in Special relativity.

This also directly implies a red shift.

On observer far away observes a red shifted object:

\[
\Delta T_\infty = \Delta x^0 / c \; ; \; \Delta T_R = \frac{1}{\sqrt{\beta}} \Delta x^0 / c
\]

This implies e.g. $\Delta \nu \simeq \phi_\oplus - \phi_\odot \simeq 2.12 \times 10^{-6}$ ($\phi$ Newton potential).
General covariance and Riemannian geometry

In presents of gravitational fields physical laws must be written in a general covariant way, which does not depend of the particular Coordinate System (CS).

The notion of CS has a fundamentally different meaning than in special relativity, where a class of ISs exist for which physical laws are particularly simple. The equivalence principle implies that the principle of special relativity can only hold locally, for freely falling bodies/observers.

Of course, also in flat space, where global ISs exist, we may formulate physical laws in arbitrary coordinates. This happens by substitution by general covariant expressions:
In special RT different ISs are related by *linear* transformations (pseudo orthogonal) and physical laws are linearly equivalent. In general RT CSs are related by *non-linear* transformations and one cannot claim that all CSs are physically equivalent. The concrete form of motions depends on the CS.

Most importantly, in the presence of gravitational fields the *metric* takes a *dynamics* role and is promoted to a *physical quantity*. General covariance is the key tool to separate internal geometrical properties of space-time from coordinate effects (inertial effects).

The formal consequence of the above discussion is that in the presence of...
The event space is assumed to be a topological Hausdorff space (distinct points have disjoint neighborhoods).

For freely falling bodies/observers Minkowskian geometry holds (existence of local maps), i.e., there exist local CSs and we can perform coordinate transformations.

The event space can be covered by countable many local CSs (existence of an atlas of maps) and is assumed to be connected (one universe).

There exists a covariant 2nd rank tensor field, the space-time metric \( g_{\mu\nu}(x) \) of signature \((+ - - -)\) and

\[
ds^2 = g_{\mu\nu}(x) \, dx^\mu \, dx^\nu
\]

is the fundamental metric form. The metric \( g_{\mu\nu}(x) \) provides the scale lengths, the clock times and exhibits the gravitational field.

Up to singular points continuous differentiability is assumed.
Formally, the metric $g_{\mu\nu}(x)$ comprises the entire structure of the Riemannian space. In particular the metric determines the geometrical arc length, geodesic curves (minimum or maximum length), parallel transport of vectors and tensors (Christoffel symbols or the affine connection) as well as the curvature (curvature tensor) of a Riemannian manifold. More details are given in the following and in an Appendix below.
Cosmology

GR Formal Tools

Space-time equipped with metric, events:

\[ x \in M^4 ; \ g_{\mu\nu} ; \ g = \det g_{\mu\nu} < 0 ; \ \text{signature} \ +---\ -\ -\ - \]

Tetrad (natural “4-bein”): \( e_\mu \doteq \frac{\partial}{\partial x^\mu} \) basis of tangent–space at point \( x \).

Relationship between neighboring tetrads of infinitesimally close points:

\[ de_\mu = \Gamma^\alpha_{\mu\nu} e_\alpha \, dx^\nu \]

Christoffel symbols: = affine connection (gauge potential in gauge theories)

\[ \Gamma^\rho_{\mu\nu} \doteq \frac{1}{2} g^{\rho\sigma} \left\{ \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right\} \]
**Cosmology**

**Covariant derivative:** \( D_\mu \varphi(x) = \partial_\mu \varphi(x) \); \( D_\mu g_{\rho\sigma}(x) = 0 \)

\[
D_\mu A^\nu(x) = \partial_\mu A^\nu(x) + \Gamma^\nu_{\rho\mu} A^\rho(x)
\]

\[
D_\mu B_\nu(x) = \partial_\mu B_\nu(x) - \Gamma^\rho_{\nu\mu} B_\rho(x) \text{ etc.}
\]

**Parallelism:** let \( x^\mu = x^\mu(\lambda) \) be a path \( C \)

\( A^\nu \) is parallel along \( C \) if \( DA^\nu = D_\mu A^\nu x^\mu \ d\lambda = 0 \)

\[
\frac{DA^\nu}{d\lambda} = \lim_{\varepsilon \to 0} \frac{A^\nu_{\parallel}(P_\varepsilon) - A^\nu(P_0)}{\varepsilon}
\]
Curvature Tensor: \[ (D_\mu D_\nu - D_\nu D_\mu) \varphi(x) = 0 \]

\[ (D_\mu D_\nu - D_\nu D_\mu) A^\rho(x) = R^{\rho}_{\sigma, \nu \mu} A^\sigma(x) \]

\[ (D_\mu D_\nu - D_\nu D_\mu) B_\rho(x) = -R^{\sigma}_{\rho, \nu \mu} B_\sigma(x) \text{ etc.} \]

Area \( A \): \( x^\mu = x^\mu(u, v) \)

Surface element:

\[ d\Sigma^{\rho \sigma} \equiv \frac{\partial x^\rho}{\partial u} \Delta u \frac{\partial x^\sigma}{\partial v} \Delta v - \frac{\partial x^\sigma}{\partial u} \Delta u \frac{\partial x^\rho}{\partial v} \Delta v \]

Shift of vector in parallel transport along infinitesimal closed path:

\[ \Delta A^\nu = A^\nu_{\parallel}(C) - A^\nu \]
measure for the curvature of space!

**Ricci Tensor:** \( R_{\mu \nu}(x) = R_{\mu, \nu \rho}^\rho(x) \)

**Scalar Curvature:** \( R(x) = g^{\mu \nu} R_{\mu \nu}(x) \)

**Einstein Tensor:** \( G_{\mu \nu}(x) = R_{\mu \nu}(x) - \frac{1}{2} g_{\mu \nu}(x) R(x) \)

**Bianchi Identity:** \( D_\mu G^\mu_{\nu} \equiv 0 \)
The Riemann tensor can be written in terms of the Christoffel symbols:

\[ R^\rho_{\sigma,\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \]

as it may be worked out from the definition by parallel transport around infinitesimal “parallelograms”.
Symmetry properties: \( g_{\mu \nu}(x) = g_{\nu \mu}(x) \)

\[
\Gamma_{\mu,\rho\sigma} \triangleq g_{\mu \nu} \Gamma_{\rho\sigma} = \frac{1}{2} \left( \partial_\rho g_{\sigma \mu} + \partial_\sigma g_{\rho \mu} - \partial_\mu g_{\rho \sigma} \right)
\]

\[
\Gamma_{\mu,\rho\sigma} = \Gamma_{\mu,\sigma\rho}; \quad \Gamma_{\mu,\rho\sigma} + \Gamma_{\sigma,\rho\mu} = \partial_\rho g_{\mu \sigma}
\]

\[
R_{\mu\nu,\rho\sigma} \triangleq g_{\mu \alpha} R^\alpha_{\nu,\rho\sigma}
\]

\[
R_{\mu\nu,\rho\sigma} = -R_{\mu\nu,\sigma\rho}; \quad R_{\mu\nu,\rho\sigma} = -R_{\nu\mu,\rho\sigma}
\]

\[
R_{\mu\nu,\rho\sigma} = R_{\rho\sigma,\mu\nu}
\]

\[
R_{\mu[\nu,\rho\sigma]} = 0; \quad [\ldots]: \sum_{\text{cycl.}}
\]

\[
D_\lambda R_{\mu\nu,\rho\sigma} \overset{\text{def}}{=} R_{\mu\nu,\rho\sigma;\lambda}; \quad R_{\mu\nu,[\rho\sigma;\lambda]} = 0 \quad \text{Bianchi – Identity}
\]

\[
R_{\mu\nu} = R_{\nu\mu}; \quad G_{\mu\nu} = G_{\nu\mu}
\]
Geometrical arc length:

\[ s_{AB} = \int_{A}^{B} ds ; \quad ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \]

fundamental metric form.

Geodesic lines: \( s_{AB} = \int_{A}^{B} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\lambda \)

Geodesic lines: \( \int ds \) extremal

\[
\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}
\]
**Theorem 1:** In flat space (∃ CS, such that $g_{\mu\nu} = \delta_{\mu\nu}$ globally) $\Leftrightarrow R^0_{\sigma,\mu\nu} \equiv 0$

Note: $g_{\mu\nu}(x') = g_{\rho\sigma \text{null}} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$ is system of differential equations, coordinates are globally determinable, but is not necessarily Minkowskian, except iff $R^0_{\sigma,\mu\nu} = 0$ (integrability condition).

On the one hand

|| the space corresponding to a homogeneous gravitational field is flat.

However, a global homogeneous gravitational field is unphysical!

On the other hand

$R^0_{\sigma,\mu\nu} \neq 0$ (space curved) is a measure for existing gravitational fields

**Theorem 2:** In flat space (∃ CS, such that $\Gamma^0_{\mu\nu} = 0$ globally) $\Leftrightarrow \frac{d^2 x^\mu}{ds^2} = 0$

i.e. $x^\mu = u^\mu s + x_0^\mu \triangleright$ geodesics are straight lines.
Einstein Equation

In case of Maxwell’s equations one has to construct a vector valued function of the gauge four-potential $A^\nu$, at most of second order in the derivatives, such that

$$f^\mu(\partial_\alpha \partial_\beta A^\nu) = \mu_0 j^\mu(x)$$

where $j^\mu(x)$ is the Hermitian conserved covariant electromagnetic four-current: $\partial_\mu j^\mu = 0$. This fixes

$$f^\mu(\partial_\alpha \partial_\beta A^\nu) = \partial_\mu \partial_\nu A^\nu - \Box A^\mu(x) = \partial_\nu F^{\mu\nu}.$$

In gravity, matter is described by the energy-momentum tensor $T_{\mu\nu}$ in place of the current. Explicit examples will be discussed later. Here we assume $T_{\mu\nu}$ to exist and to have the properties

- it is covariantly conserved: $D_\mu T_{\mu\nu} = 0$,
Cosmology

- it is symmetric: $T_{\mu\nu} = T_{\nu\mu}$,
- it is Hermitian.

Up to a constant, which is fixed by the Newtonian limit, the r.h.s. of the field equation we are looking for is

$$\frac{8\pi G_N}{c^2} T_{\mu\nu}$$

while the l.h.s. should be a function of $g_{\mu\nu}$ and derivatives of it up to second order,

$$f^{\mu\nu}(g_{\alpha\beta}, \partial_{\rho} g_{\alpha\beta}, \partial_{\sigma} \partial_{\rho} g_{\alpha\beta}) = \frac{8\pi G_N}{c^2} T_{\mu\nu}$$

By the properties of $T_{\mu\nu}$ we require

$$D_\mu f^{\mu\nu} = 0 ; \quad f^{\mu\nu} = f^{\nu\mu}.$$
Here one may use a Theorem of Cartan: If we require $f^{\mu\nu}$ to be linear in the second derivatives, then $f^{\mu\nu}$ must have the form

$$f^{\mu\nu} = a R^{\mu\nu} + b g^{\mu\nu} R + c g^{\mu\nu},$$

which on the one hand says that first derivatives of $g^{\mu\nu}$ give no covariant tensor, because $D^\rho g^{\mu\nu} \equiv 0$. On the other hand second derivatives give a tensor which has to be related to the curvature tensor.

We then have

$$D^\mu f^{\mu\nu} = a D^\mu R^{\mu\nu} + b D_\nu R,$$

which together with the Bianchi identity $D^\mu R^{\mu\nu} = \frac{1}{2} D_\nu R$ and the condition that $T^{\mu\nu}$ being conserved yields

$$D^\mu f^{\mu\nu} = \left(\frac{a}{2} + b\right) D_\nu R = 0,$$
and thus

\[ b = -\frac{a}{2}. \]

The constant multiplying \( T_{\mu\nu} \) on the r.h.s. is chosen such that we may set \( a = 1 \) (classical limit) and we have the result

\[
\left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + c g_{\mu\nu} = \kappa T_{\mu\nu}
\]

\[
\kappa = \frac{8\pi G_N}{c^2} = 1.86 \times 10^{-27} \text{ cm/gr}
\]

Einstein (Mach) concept: empty space \( T_{\mu\nu} \equiv 0 \Leftrightarrow \exists \text{ Coordinate System: } g_{\mu\nu} = \tilde{g}_{\mu\nu} \) requires \( c = 0 \)
Cosmology

If $c = 0$

- empty space ⇔ flat space

- equivalent matter-geometry duality

  matter ⇔ curved geometry

Einstein’s equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$
Experimental Tests of GR

see also: ≫≫ and references there

1 Newton limit

GR yields all results of Newton’s gravity theory in weak gravitational fields and for slowly moving bodies. This is true for smallest details of planetary motion etc.

2 Mercury perihelion shift

According to Newton planetary orbits are exact ellipses (discarding usually tiny perturbations by third party bodies). Einstein predicts that ellipses in fact are to be replaced by rosette trajectories. The effect is largest for the planet closest to the sun, mercury.
The shift predicted for the rotation of the perihelion (the closest point of the trajectory to the sun), $L$ the angular momentum

$$
\delta \varphi \simeq 2 \pi \frac{3 G_N^2 M_\odot}{c^4 L^2}
$$

Per 100 years predicted: 42.98” (arc seconds)

observed: (42.98 ± 0.043)”

Test: \( \frac{\text{Experiment}}{\text{Theory}} = 1.000 \pm 0.001 \)
Observations are optical and radar, accuracy 0.1%

The perihelion shift has also been observed for Venus: predicted 8.6 seconds of arc per century, and 8.4 seconds has been observed.

In 1975, binary pulsar PSR 1913+16 was discovered. The two massive sources, in very tight orbit, display a perihelion advance of 4.23 degrees per year!

Light deflection by gravitation

Light passing at distance $R_\odot$ from the Sun gets deflected by
which is twice to value one get in Newton’s theory $\delta_{\text{Newton}} = \frac{1}{2} \delta$ using $m = E/c^2$ as an effective photon mass, $E$ the photon energy.

Recent measurements Very Long Baseline Interferometry (VLBI), observation of radio sources and quasars.

Light passing Sun
predicted: $1.75''$ (arc seconds)
observed: $(2.2 \pm 1.5)''$ (1919)

Radio waves
observed: $(1.73 \pm 0.05)''$ (1972)
Cosmology

Test: $\frac{\text{Experiment}}{\text{Theory}} = 1.000 \pm 0.001$

4 Gravitational red shift

Photon is heavy: $m_{\gamma,\text{eff}} = \frac{E_\gamma}{c^2}$, light from the sun requires energy to leave gravitational field of the sun. The corresponding red shift is

$$z = \frac{\delta \lambda}{\lambda} \simeq \frac{M_\odot}{R_\odot} \sim 10^{-6}$$

which is too small in comparison with other perturbations.

For a back hole: $z = \infty$ photons cannot escape any longer!

Well known: laboratory experiments (Mössbauer effect), earth gravitational effect

Pound & Rebka 1960:
For $h = 22.6 \text{ m}$; $\frac{\Delta \nu}{\nu} = 2.5 \times 10^{-16} \pm 10\%$

Today: accuracy 1%, Voyager 1 in the field of Saturn

5 Radar echo delay (Shapiro)

The idea is to compare radar echo measurement of close to the sun planets and satellites:
COSMOLOGY

a) beam passes close to sum
b) beam passes very far from sum

Signal delay:

\[ \delta t \sim 4 \, m \left( \log \frac{4R_1R_2}{R_0^2} + 1 \right) \]

Viking mission: two mirrors on Mars, two mirrors in orbit

Predictions confirmed at 0.1% accuracy!

Cassini probe: more recently gives a 0.002% test of general relativity
Gravitational Lenses

Unlike optical lenses gravitational lenses have a focal line in place of a focal point, which in the ideal case of exactly aligned optical axes gives raise to the so called Einstein ring (quasar 1938). Theory see \(\star\).

The picture shows an Einstein cross, a quasar (2237) behind a galaxy (in the center) is seen four times due to collimation of light be the galaxy \(\star\). In fact
gravitational lenses are defocusing rather than focusing the light from a source!
For Einstein-Cross calculations see: \(\ast\)
Gravitational Lenses
PRC95-43 · ST Scr OPO · October 18, 1995 · K. Ratnatunga (JHU), NASA
NASA Hubble Space Telescope image of the rich galaxy cluster, Abell 2218, is a spectacular example of gravitational lensing. The arc-like pattern spread across the picture like a spider web is an illusion caused by the gravitational field of the cluster. **
Evidence for the Existence of Black Holes


Velocity field reveals black hole of mass $2.45 \pm 0.4 \times 10^6 \, M_\odot$,

New key tool: Infrared/Submillimeter Astronomy
In normal optical astronomy center of Milky Way hidden by stars and dust. In infrared motion of stars in galactic center observable: flux of a faint source increased by a factor of 5-6 and fainted again after about 30 min (flares)

Profile of proper motion reveals unambiguously the existence of a black hole of a few $10^6 \, M_\odot$
More recently (2002) $4.31 \pm 0.38 \times 10^6 \; M_\odot$ has been determined for the Sagittarius A complex in the galactic center.
Flares:
Look at animations here: ♫♫
Black Holes and Galactic Centers

There are strong indications that many (most?) galaxies host a black hole:

Center: Black Hole $\sim 1.2 \times 10^9 M_\odot$?
Do all galaxies have black holes at their centers? Although not even a single galaxy has yet been proven to have a central black hole, the list of candidates is growing. Recent results from the Hubble Space Telescope now indicate that most - and possibly even all - large galaxies may harbor a black hole. In all the galaxies studied, star speeds continue to increase closer to the very center. This in itself
Cosmology indicates a center millions of times more massive than our Sun is needed to contain the stars. This mass when combined with the limiting size make the case for the black holes.
Geodesic effect and Lense-Thirring effect

NASA’s Gravity Probe B by means of extremely precise gyroscopes for the first time has been able to evidence two effects predicted by Einstein’s GRT in the gravitational field of the earth:

- The first is the **geodetic effect**, which is the warping of spacetime around a gravitational body, such as a planet.

- The second effect of gravity is **frame dragging** (Lense-Thirring effect), which is the amount that a spinning object pulls the fabric of spacetime along with it.
An artist’s concept of Gravity Probe B orbiting Earth, which is warping spacetime.
Gravitational Waves

GR predicts gravitational waves, they are extremely hard to detect, usually background effects are much bigger than the expected signal.

A new generation of gravitational wave antennas based on Laser Interferometry are presently under construction or are taking data: LIGO (photo), VIRGO, GEO600, LISA and TAMA 300. They serve to detect violent processes in space like gamma ray burst. So far no signal established.
Gravitational waves were indirectly confirmed to exist when observations were made of the binary pulsar PSR 1913+16, for which the Nobel Prize was awarded to Hulse and Taylor in 1993*

Explained by energy loss due to gravitational wave emission!
Perspectives

- Solar system weak fields at $10^{-3}$ precision
- Binary Pulsars strong fields at $10^{-3}$ accuracy
- Equivalence Principle: STEP (Satellite Test of the Equivalence Principle) experiment by NASA+ESA $10^{-17}$ (at present $10^{-12}$)
- Constancy of gravitational constant $G_N$:

$$\frac{1}{G_N} \frac{\Delta G_N}{\Delta t} < 10^{-12}/\text{year}$$

Monitoring the position of the Moon with lasers, Viking radar, binary pulsar,...

- planned: NASA space-gyroscopes, studying weak field effect at $10^{-5}$
- Gravitational wave searches; laser interferometry (Michelson experiment with Fabry Pérot interferometer): LIGO, VIRGO, GEO600, LISA, TAMA300
Expect to observe violent events in galaxies (gamma-ray bursts etc)
For more science fiction see ♠️
Tests beyond GR

Above we have discussed predictions of GR and their confrontation with experiments. There is another approach, so to say the inverse problem: which attempts to determine the metric by measurements of as many observables as possible. Experiments are guided here by the general approach assuming an arbitrary “Metric Theory of Gravity” where the metric is parametrized according to the “parametrized post-Newtonian (PPN) formalism”:

\[ g_{\mu\nu} = g_{\mu\nu}^{\text{GR}} + \delta g_{\mu\nu}^{\text{PPN}}, \]

where the perturbation \( \delta g_{\mu\nu}^{\text{PPN}} \) in leading order depends on 3 physical parameters \( \bar{\alpha}, \bar{\beta}, \bar{\gamma} \). Given that GR describes observations successfully the PPN parameters are small perturbations. For a recent review see \( \star \).
Appendix: geodesic lines and covariant derivatives

A) Geodesic Curves

For simplicity we assume a Riemannian manifold with positive definite metric here. The basic features carry over to what we need in GR.

Curve Length

Let $x^\lambda(\lambda), \lambda_0 \leq \lambda \leq \lambda_1$ be a parametrized curve between points $P_0$ and $P_1$. Then

$$\int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

i) is invariant under coordinate transformations,

ii) and invariant under reparametrization,

and defines the length of the curve from $P_0$ to $P_1$. 
– i) follows from
\[ g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu = g_{\rho\sigma}(x') \dot{x}'^\rho \dot{x}'^\sigma \]
which is easily verified.

– ii) follows considering a change of the parameter \( \lambda = \lambda(\lambda') \) which we take as a monotonic and smooth function with \( \lambda_0 = \lambda(\lambda'_0) \) and \( \lambda_1 = \lambda(\lambda'_1) \). Then

\[ \int_{\lambda_0}^{\lambda_1} d\lambda \left( g_{\mu\nu} \frac{d\lambda'}{d\lambda} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} = \int_{\lambda_0}^{\lambda_1} d\lambda \left( g_{\mu\nu} \frac{d\lambda'}{d\lambda} \frac{dx^\mu}{d\lambda'} \frac{dx^\nu}{d\lambda'} \right)^{1/2} \]

\[ = \int_{\lambda_0}^{\lambda_1} d\lambda \left( g_{\mu\nu} \frac{d\lambda'}{d\lambda} \frac{dx^\mu}{d\lambda'} \right)^{1/2} \frac{d\lambda'}{d\lambda} = \int_{\lambda'_0}^{\lambda'_1} d\lambda' \left( g_{\mu\nu} \frac{dx^\mu}{d\lambda'} \frac{dx^\nu}{d\lambda'} \right)^{1/2} \]

q.e.d. \( \square \)
**Cosmology**

Arc length from $P_0$ to a variable point $P$ is given by

$$s(\lambda) = \int_{\lambda_0}^{\lambda} d\lambda' \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}; \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda'},$$

or equivalently

$$\dot{s} = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu},$$

respectively,

$$(\dot{s})^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \iff ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

which is the fundamental metric from of the Riemannian manifold $M$. Since $s(\lambda)$ is a monotonically increasing function of $\lambda$ the arc length $s$ may be utilized as a curve-parameter, where

$$\left(\frac{ds}{ds}\right)^2 = 1 = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds},$$

which means that the corresponding four-velocity $u^\mu = \frac{dx^\mu}{ds}$ is a vector of modulus unity.
For positive definite metric spaces on defines:

Def: Distance $d(P_0, P_1)$ between two points $P_0$ and $P_1$ \(\equiv\) infimum of the lengths of piecewise smooth curves between $P_0$ and $P_1$. For Minkowski type metric spaces (as relevant for special and general relativity) the corresponding generalized distance notions have been discussed earlier.

### Geodesic Lines

Determining the shortest distance between two points on a manifold in general is a difficult problem. A much simpler problem is to find a system of differential equations for the more general class of curves of stationary length which includes the shortest ones.

1. **Formulation of the problem**

Let $x^\mu(\lambda) ; \lambda_0 \leq \lambda \leq \lambda_1$ describe a smooth curve $K_0$ between two fixed points $P_0$ and $P_1$. 
and \( P_1 \). Each possible neighboring curve \( K_1 \) may be represented in the form

\[
K_1 : \quad z_1^\mu(\lambda) = x^\mu(\lambda) + \varepsilon y^\mu(\lambda) \quad \text{with} \quad \varepsilon = 1 ,
\]

with the help of a smooth function \( y^\mu(\lambda) ; \quad \lambda_0 \leq \lambda \leq \lambda_1 \) with \( y^\mu(\lambda_0) = y^\mu(\lambda_1) = 0 \).

With

\[
K_\varepsilon : \quad z_\varepsilon^\mu(\lambda) = x^\mu(\lambda) + \varepsilon y^\mu(\lambda)
\]
we obtain a family $K_\varepsilon \ (−1 \leq \varepsilon \leq +1)$ of neighboring curves of $K_0$ of length $L(\varepsilon)$. Def: $K_0$ is called curve of stationary length or geodesic line, if

$$\frac{dL}{d\varepsilon}(0) = 0$$

for any choice of the function $y^\mu(\lambda)$ which satisfies $y^\mu(\lambda_0) = y^\mu(\lambda_1) = 0$. Formally, we may consider $K_0$ as an extremal curve of the variational problem:

$$\delta L = \delta \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = 0.$$

Analytic implementation

Given

$$L(\varepsilon) = \int_{\lambda_0}^{\lambda_1} d\lambda \left\{ g_{\mu\nu}(x + \varepsilon y)(\dot{x}^\mu(\lambda) + \varepsilon \dot{y}^\mu(\lambda))(\dot{x}^\nu(\lambda) + \varepsilon \dot{y}^\nu(\lambda)) \right\}^{1/2}$$
we obtain

$$\frac{dL}{d\varepsilon}(0) = \int_{\lambda_0}^{\lambda_1} d\lambda \frac{1}{2\sqrt{g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^\alpha} y^\alpha \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu} \ddot{y}^\mu \ddot{y}^\nu + g_{\mu\nu} \dot{x}^\mu \ddot{y}^\nu \right\} .$$

A partial integration yields

$$\int_{\lambda_0}^{\lambda_1} d\lambda \frac{g_{\mu\nu} \dot{x}^\mu}{2\sqrt{g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} \ddot{y}^\nu = \left. \frac{g_{\mu\nu} \dot{x}^\mu}{2\sqrt{g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} y^\nu \right|_{\lambda_0}^{\lambda_1} - \int_{\lambda_0}^{\lambda_1} d\lambda \frac{d}{d\lambda} \left( \frac{g_{\mu\nu} \dot{x}^\mu}{2\sqrt{g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} \right) y^\nu$$

where the first term vanishes because of the boundary condition \(y(\lambda_0) = y(\lambda_1) = 0\). Therefore,

$$\frac{dL}{d\varepsilon}(0) = \int_{\lambda_0}^{\lambda_1} d\lambda \ y^\alpha \left\{ \frac{1}{2\sqrt{g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu - \frac{d}{d\lambda} \left[ \frac{g_{\alpha\mu} \dot{x}^\mu + g_{\alpha\nu} \dot{x}^\nu}{2\sqrt{g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} \right] \right\} .$$
Necessary and sufficient condition that \( \frac{dL}{d\varepsilon}(0) = 0 \) for any choice of \( y^\mu(\lambda) \) satisfying \( y(\lambda_0) = y(\lambda_1) = 0 \) is that \( \{\cdots\} = 0 \) in the integrand. If we choose the arc length \( s \) as a parameter on \( K_0 \) we have \( g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 1 \) and the geodesic line condition reads

\[
\left\{ \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu - \frac{d}{ds} g_{\alpha\nu} \dot{x}^\nu - \frac{d}{ds} g_{\alpha\mu} \dot{x}^\mu \right\} = 0
\]

or (eventually renaming summed over indices)

\[
\frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu - \frac{\partial g_{\alpha\nu}}{\partial x^\mu} \dot{x}^\mu \dot{x}^\nu - \frac{\partial g_{\alpha\mu}}{\partial x^\nu} \dot{x}^\mu \dot{x}^\nu - 2 g_{\alpha\nu} \ddot{x}^\nu = 0.
\]

Thus

\[
g_{\alpha\rho} \ddot{x}^\rho = -\frac{1}{2} \left( \frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \dot{x}^\mu \dot{x}^\nu
\]
and contracting with \( g^{\alpha \beta} \) we arrive at

\[
\ddot{x}^\beta = -\frac{1}{2} g^{\beta \alpha} \left( \frac{\partial g_{\alpha \nu}}{\partial x^\mu} + \frac{\partial g_{\alpha \mu}}{\partial x^\nu} - \frac{\partial g_{\mu \nu}}{\partial x^\alpha} \right) \dot{x}^\mu \dot{x}^\nu.
\]

This gives rise to the definition of the Christoffel symbols:

**Def:** Christoffel symbols

\[
\Gamma^\alpha_{\mu \nu} \equiv \frac{1}{2} \left( \frac{\partial g_{\nu \alpha}}{\partial x^\mu} + \frac{\partial g_{\mu \alpha}}{\partial x^\nu} - \frac{\partial g_{\mu \nu}}{\partial x^\alpha} \right) ; \quad \Gamma^\beta_{\mu \nu} \equiv g^{\beta \alpha} \Gamma^\alpha_{\mu \nu}.
\]

Symmetry of \( g_{\mu \nu} \) implies the symmetry relations

\[
\Gamma^\alpha_{\mu \nu} = \Gamma^\alpha_{\nu \mu} ; \quad \Gamma^\beta_{\mu \nu} = \Gamma^\beta_{\nu \mu} ; \quad \Gamma^\alpha_{\mu \nu} + \Gamma^\nu_{\alpha \mu} = \frac{\partial g_{\nu \alpha}}{\partial x^\mu}.
\]

As a result the system of differential equations of geodesic lines takes the form
This system is of the form

\[
\begin{align*}
\dot{y}^p &= F(x^1, \cdots, x^n, y^1, \cdots y^n); \quad p = 1, \cdots, n \\
\dot{x}^p &= y^p
\end{align*}
\]

and in the theory of differential equations is proven to have a unique solution \( x^\mu(\lambda) \) to given initial values \( x^\mu(0) \) and \( \dot{x}^\mu(0) \) for \( \lambda \in (-\varepsilon, \varepsilon) \). As a consequence we have

**Theorem 1.** *Through each point of a Riemannian manifold there exists exactly one geodesic line in a given direction.*

Furthermore, it follows that to each point \( P \) of a Riemannian manifold there exists a neighborhood \( U(P) \) such that for any point \( Q \in U(P) \) there is exactly one geodesic line between \( P \) and \( Q \) within the given neighborhood. For the positive metric case this is also the shortest connection and hence the distance between \( P \) and \( Q \).
Another important fact is that the Christoffel symbols do not represent a tensor! While the metric is a second rank tensor, the ordinary derivatives of the metric which define the $\Gamma_{\rho,\mu\nu}$ are not tensors, as one easily checks by direct calculation: with

$$g_{\mu'\nu'} = X_{\mu'}^\mu X_{\nu'}^\nu g_{\mu\nu}$$

and

$$X_{\mu'}^\mu_{\alpha'} = \frac{\partial}{\partial x^{\alpha'}} X_{\mu'}^\mu = \frac{\partial}{\partial x^{\alpha'}} \frac{\partial x^\mu}{\partial x^{\mu'}} = \frac{\partial}{\partial x^{\alpha'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} = X_{\alpha'}^\mu_{\mu'}$$

we obtain

$$\frac{\partial g_{\mu'\nu'}}{\partial x^{\alpha'}} = \left( X_{\mu'}^\mu_{\alpha'} X_{\nu'}^\nu + X_{\mu'}^\mu X_{\nu'}^\nu_{\alpha'} \right) g_{\mu\nu} + X_{\mu'}^\mu X_{\nu'}^\nu \frac{\partial g_{\mu\nu}}{\partial x^{\alpha'}} X_{\alpha'}^\alpha.$$

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For the Christoffel symbol the three terms combine to

\[ \Gamma_{\nu' \alpha'}^{\beta'} = X_{\rho}^{\beta'} X_{\nu' \alpha'}^{\rho} + X_{\nu' \alpha'}^{\nu} X_{\alpha' \beta}^{\rho} \Gamma_{\nu \alpha}^{\beta} \]

which more explicitly reads

\[
\Gamma_{\nu' \alpha'}^{\beta'} = \frac{\partial x^\nu}{\partial x'^{\nu'}} \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^\beta} \Gamma_{\nu \alpha}^{\beta} + \frac{\partial^2 x^{\rho}}{\partial x'^{\nu} \partial x'^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^\rho} \]

everything at the same point. The \( \Gamma_{\rho \mu \nu} \)'s thus have inhomogeneous transformation properties and they play the same role as gauge potential in local gauge theories. Like the gauge potentials, the Christoffel symbols are the basic ingredients, called affine connections, needed for the construction covariant derivatives.

The fact that the \( \Gamma \)'s are not tensors actually has an important consequence. Locally they can be transformed away: in each point \( P \) of a manifold there exist a coordinate system in which \( \Gamma_{\rho \mu \nu}^{\mu}(P) = 0 \); this particular CS is called geodetic coordinate system (GCS) defined by
This is a system where geodesic lines are chosen as coordinate axes. GCSs are often a useful tool for proofs of tensor relations, like covariant derivatives of sums, products, tensor contractions etc.

Note that in contrast to the non-covariant Christoffel symbols, if a tensor vanishes in one system it vanishes in any other system, because of the homogeneous transformation law tensors obey.

**B) Covariant derivatives**

In order to construct a covariant derivative, we observe that the disturbing second derivative of the coordinates may be expressed in terms of the Christoffel
symbols:

$$\frac{\partial^2 x^\rho}{\partial x^\nu' \partial x^\alpha'} = \frac{\partial x^\rho}{\partial x^{\beta'}} \Gamma_{\nu'\alpha'}^{\beta'} - \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \Gamma_{\nu\alpha}^\rho \quad \text{or} \quad X_{\nu'\alpha'}^\rho = X_{\alpha'}^\rho \Gamma_{\nu'\alpha'}^{\beta'} - X_{\nu'}^\nu X_{\alpha'}^\alpha \Gamma_{\nu'\alpha'}^\rho .$$

A general mixed tensor field $S_{\mu'\ldots\alpha'\ldots}$ has the transformation law

$$S_{\mu'\ldots\alpha'\ldots}^\mu'\ldots X_{\mu'}^\mu \ldots X_{\alpha'}^\alpha \ldots S_{\alpha'\ldots}^\mu\ldots .$$

Using the abbreviations/relations

$$X_{\nu'\alpha'}^\mu = \frac{\partial}{\partial x_{\nu'}} X_{\nu'}^\mu = \frac{\partial^2 x^\mu}{\partial x^\alpha' \partial x^\nu'} \quad \Rightarrow \quad X_{\nu'\alpha'}^\mu = X_{\alpha'}^\mu \quad X_{\nu'}^\nu' \\
X_{\nu'\alpha}^\mu = \frac{\partial}{\partial x_{\alpha'}} X_{\nu'}^\mu = \frac{\partial^2 x^\mu}{\partial x^\alpha' \partial x^\nu'} = \left( \frac{\partial}{\partial x_{\alpha'}} X_{\nu'}^\mu \right) X_{\alpha'}^\alpha' \quad \Rightarrow \quad X_{\nu'\alpha}^\mu = X_{\nu'}^\nu' X_{\alpha'}^\alpha'$$

we try to structure the expression we obtain by looking at
the derivative of the transformed tensor:

\[
\frac{\partial S^\mu'^\alpha'\ldots}{\partial x^{\nu'}} = X^{\mu'}_{\rho\nu'} X^\alpha\ldots S^{\rho\ldots\alpha\ldots} + \ldots + X^{\mu'}_{\nu'} X^{\rho\ldots\alpha\ldots\nu'} + \ldots
\]

\[
+ X^{\mu'}_{\mu'} \ldots X^\alpha\ldots \frac{\partial S^{\mu\ldots\alpha\ldots}}{\partial x^{\nu'}}
\]

\[
= \left( X^{\mu'}_{\mu'} \Gamma^\mu_{\rho\nu} - X^{\rho'}_{\rho\nu} X^{\sigma'}_{\sigma\nu} \Gamma^\mu_{\rho\sigma'} \right) X^{\nu'}_{\nu'} X^\alpha\ldots S^{\rho\ldots\rho\ldots\alpha\ldots} + \ldots
\]

\[
+ X^{\mu'}_{\mu'} \ldots \left( X^{\rho'}_{\rho\nu} \Gamma^\rho_{\alpha\nu'} - X^{\alpha'}_{\alpha\nu} X^{\nu'}_{\nu} \Gamma^\rho_{\alpha\nu} \right) \ldots S^{\mu\ldots\rho\ldots} + \ldots
\]

\[
+ X^{\mu'}_{\mu'} \ldots X^{\alpha'}_{\alpha} \ldots X^{\nu'}_{\nu} \frac{\partial S^{\mu\ldots\alpha\ldots}}{\partial x^{\nu'}}
\]
\[
\frac{\partial S^{\mu'\ldots}_{\alpha'\ldots}}{\partial x^{\nu'}} = X^{\mu'}_\mu \cdots X^{\alpha'}_{\alpha} \cdots X^{\nu'}_{\nu} \left\{ \frac{\partial S^{\mu'\ldots}_{\alpha'\ldots}}{\partial x^{\nu'}} + \Gamma^{\mu}_{\rho\nu} S^{\rho'\ldots}_{\alpha'\ldots} + \cdots - \Gamma^{\rho}_{\alpha\nu} S^{\mu'\ldots}_{\rho'\ldots} - \cdots \right\}
\]

\[
- X^{\rho'}_\rho \underbrace{X^{\sigma'}_{\gamma} X^{\nu'}_{\nu'}}_{\delta^{\sigma'}_{\gamma'}} \Gamma^{\mu'}_{\rho'\sigma'} \cdots X^{\alpha'}_{\alpha} \cdots S^{\rho'\ldots}_{\alpha'\ldots} - \cdots
\]

\[
+ X^{\mu'}_\mu \cdots X^{\rho'}_{\rho} \Gamma^{\rho'}_{\alpha'\nu'} \cdots S^{\mu'\ldots}_{\rho'\ldots} + \cdots
\]

\[
= X^{\mu'}_\mu \cdots X^{\alpha'}_{\alpha} \cdots X^{\nu'}_{\nu} \left\{ \frac{\partial S^{\mu'\ldots}_{\alpha'\ldots}}{\partial x^{\nu'}} + \Gamma^{\mu}_{\rho\nu} S^{\rho'\ldots}_{\alpha'\ldots} + \cdots - \Gamma^{\rho}_{\alpha\nu} S^{\mu'\ldots}_{\rho'\ldots} - \cdots \right\}
\]

\[
- \Gamma^{\mu'}_{\rho'\nu'} S^{\rho'\ldots}_{\alpha'\ldots} - \cdots
\]

\[
+ \Gamma^{\rho'}_{\alpha'\nu'} S^{\mu'\ldots}_{\rho'\ldots} + \cdots
\]

Note that in the bracket on the first line only unprimed objects have been included. If we move the primed extra terms to the l.h.s of the equation, we indeed obtain an
object which transforms as a tensor:

\[
\left\{ \frac{\partial S_{\mu'\cdots}^{\alpha'\cdots}}{\partial x^\nu'} + \Gamma_{\rho'\nu'}^{\mu'} S_{\rho'\cdots}^{\rho\cdots} + \cdots - \Gamma_{\alpha'\nu'}^{\rho'} S_{\mu'\cdots}^{\rho'\cdots} - \cdots \right\}
\]

\[
= X_\mu' \cdots X_\alpha' \cdots X_\nu' \left\{ \frac{\partial S_{\mu'\cdots}^{\alpha'\cdots}}{\partial x^\nu'} + \Gamma_{\rho\nu}^{\mu} S_{\rho\cdots}^{\rho\cdots} + \cdots - \Gamma_{\alpha\nu}^{\rho} S_{\mu\cdots}^{\rho\cdots} - \cdots \right\}
\]

We may formulate this result as a theorem:

**Theorem 2.** If \( S_{\mu\nu\cdots}^{\alpha\beta\cdots} \) is a tensor field of type \((p, q)\), then

\[
D_\lambda S_{\mu\nu\cdots}^{\alpha\beta\cdots} \overset{\text{def}}{=} \partial_\lambda S_{\mu\nu\cdots}^{\alpha\beta\cdots} + \Gamma_{\rho\lambda}^{\mu} S_{\rho\nu\cdots}^{\alpha\beta\cdots} + \Gamma_{\rho\nu}^{\mu} S_{\rho\alpha\cdots}^{\mu\beta\cdots} + \cdots - \Gamma_{\alpha\lambda}^{\rho} S_{\mu\nu\cdots}^{\rho\beta\cdots} - \Gamma_{\beta\lambda}^{\rho} S_{\mu\nu\cdots}^{\alpha\rho\cdots} - \cdots
\]

is a tensor field of type \((p, q+1)\), which defines the **covariant derivative** of the tensor field \( S_{\mu\nu\cdots}^{\alpha\beta\cdots} \).
Often one uses the notation \( D_\lambda S^{\mu\nu\ldots}_{\alpha\beta\ldots} = S^{\mu\nu\ldots}_{\alpha\beta\ldots;\lambda} \).

Exercise: prove the following fundamental properties: 1) \( D_\lambda g_{\mu\nu} = 0 \), 2) \( D_\lambda \delta^\mu_\nu = 0 \), 3) \( D_\lambda \varphi(x) = \partial_\lambda \varphi(x) \), the gradient of \( \varphi \) for scalar fields \( \varphi(x) \).

The covariant derivative allows us to define a coordinate independent covariant derivative of a tensor field along a curve: for a given curve \( x^\mu(\lambda) \) we define

\[
\frac{D}{d\lambda} S^{\mu\nu\ldots}_{\alpha\beta\ldots} \overset{\text{def}}{=} \left( D_\alpha S^{\mu\nu\ldots}_{\alpha\beta\ldots} \right) \dot{x}^\alpha.
\]

This in turn is basic for defining parallelism: a contravariant vector field \( A^\mu(x) \) (as an example) is parallel along the given curve if its covariant derivative vanishes along the curve:

\[
\frac{D}{d\lambda} A^\mu = (D_\alpha A^\mu) \dot{x}^\alpha = \left( \frac{\partial A^\mu}{\partial x^\alpha} + \Gamma^\mu_{\beta\alpha} A^\beta \right) \dot{x}^\alpha = 0.
\]
Furthermore, the **curvature tensor** now may be obtained from the commutator of two covariant derivatives:

\[
[D]_{\rho\sigma} A^\mu \equiv \left( D_\rho D_\sigma - D_\sigma D_\rho \right) A^\mu = R^\mu_{\alpha,\rho\sigma} A^\alpha.
\]

Exercise: work out \( R^\mu_{\alpha,\rho\sigma} \) in terms of the Christoffel symbols by calculating the covariant commutator.

Exercise: show that

1) \[
\Gamma^\rho_{\mu\rho} = \frac{1}{2} g^{\rho\sigma} \frac{\partial g_{\rho\sigma}}{\partial x^\mu} = \frac{1}{2g} \frac{\partial g}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \ln \sqrt{|g|}
\]

with \( g = \det g_{\mu\nu} \).
2) \[
\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\sigma} \left( \sqrt{|g|} g^{\rho\sigma} \right) + g^\sigma{}_{\lambda} \Gamma^\rho{}_{\sigma\lambda} = 0 .
\]

Exercise: using the above results to verify the following relations:

| Gradient     | (Grad \( \varphi \))_\mu \equiv D_\mu \varphi = \partial_\mu \varphi , |
| Rotation     | (Rot A)_{\mu\nu} \equiv D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu |
| Divergence   | (Div A) \equiv D_\mu A^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} A^\mu \right) |
| Laplacian    | \Delta \varphi \equiv \text{Div Grad} \varphi = \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right) |

Previous \( \ll \), next \( \gg \) lecture.