Part II: A second look

1. Sketch of full general relativistic calculation
2. Sketch of perturbations in general relativistic cosmology
3. Scalar perturbations – long waves
4. Scalar perturbations – short waves
5. Interpolation, transfer functions
6. Weinberg’s semianalytic result
7. Reading off the cosmological parameters
8. Full calculations with CMBFAST or CAMB
Sketch of full general relativistic calculation

Scalar modes:

\[ C_{TT, \ell}^S = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2 = 16\pi^2 T_0^2 \int_0^\infty dq q^2 \left[ j_\ell(qr_L) F(q) + j'_\ell(qr_L) G(q) \right]^2 \]

is the standard result, but not very transparent for what concerns the dependence of \( C_\ell \) on \( \ell \). In the most interesting case of larger \( \ell \) where the cosmic variance can be neglected we again may use appropriate formula for the spherical Bessel functions in order to make relations more transparent:

\[ j_\ell(z) \rightarrow \left\{ \begin{array}{ll} \cos b \cos \left[ \nu (\tan b - b) - \pi/4 \right] / \nu \sqrt{\sin b} & z > \nu \\ 0 & z < \nu \end{array} \right. \]

where \( \nu \equiv \ell + 1/2 \) and \( \cos b \equiv \nu / z \), with \( 0 \leq b \leq \pi/2 \), valid for \( |\nu^2 - z^2| \gg \nu^{4/3} \). Thus for \( \ell \gg 1 \) this approximation is valid for almost all of the integration range. For

© 2009, F. Jegerlehner
$z > \nu \gg 1$ the phase $\nu (\tan b - b)$ is very strongly increasing function of $z$, such that derivative $j'$ of the spherical Bessel function can be taken to act mainly on this phase:

$$j'_\ell(z) \rightarrow \begin{cases} 
- \cos b \sqrt{\sin b} \sin [\nu (\tan b - b) - \pi/4] / \nu & z > \nu \\
0 & z < \nu 
\end{cases}$$

Replacing $j_\ell$ and $j'_\ell$ by the approximations in the range of validity and changing variables $q \rightarrow b = \cos^{-1}(\nu/qr_L)$ gives

$$C_{TT,\ell}^S = \frac{16\pi^2 \nu}{r_L^3} \int_0^{\pi/2} \frac{db}{\cos^2 b}$$

$$\times \left[ F\left(\frac{\nu}{r_L \cos b}\right) \cos [\nu (\tan b - b) - \pi/4] \\
- \sin b G\left(\frac{\nu}{r_L \cos b}\right) \sin [\nu (\tan b - b) - \pi/4]^2 \right].$$
For $\nu \gg 1$ the circular functions oscillate rapidly and $\cos[\cdots]$, $\sin[\cdots]$, $\cos^2[\cdots]$ and $\sin^2[\cdots]$ average to 1/2 while $\cos[\cdots] \sin[\cdots]$ averages to zero. We may set $\nu \approx \ell$ and change variables to $b \to \beta = 1/\cos b$ and obtain

$$\ell (\ell + 1) C_{TT,\ell}^S = \frac{8\pi^2 \ell^3}{t_L^3} \int_1^\infty \frac{\beta d\beta}{\sqrt{\beta^2 - 1}} \times \left[ F^2 \left( \frac{\ell \beta}{r_L} \right) + \frac{\beta^2 - 1}{\beta^2} G^2 \left( \frac{\ell \beta}{r_L} \right) \right].$$

Note that the oscillatory behavior seen in the plots $\ell (\ell + 1) C^S$ vs. $\ell$ is not originating in the oscillatory behavior of the spherical Bessel functions, which in the $q$ integral is averaged out. The CMB temperature fluctuations imprinted at last scattering in the decoupled photon cloud, and observed as a spherical shell from any point in the universe, are naturally expanded into spherical harmonics, but this “monitoring” is independent of the traces physics has left imprinted in the CMB at the time of last scattering. The oscillations seen in the data (acoustic peaks) in fact are oscillations in the plasma (plasma waves) which existed before recombination. This information is contained in the form factors $F(q)$ and $G(q), \ldots$
which encode the physics at the time of last scattering:

\[
F(q) = \frac{1}{3} \delta_{\gamma q}(t_L) + \frac{S^2(t_L) \psi_q(t_L)}{3 q^2 t_L}
\]

\[
G(q) = -q \delta u_{\gamma q}(t_L) + S(t_L) \psi_q(t_L)/q
\]

where \( \delta T_q/\bar{T} = \delta \rho_{\gamma q}/4\bar{\rho}_\gamma = \delta \gamma q/3 \) has been used. What determines the form factors are the following components: in the period after \( e^+ e^- \)-annihilation, at temperatures \( T \approx 10^9 \, ^\circ\text{K} \), to the time \( t_L \) of last scattering, at \( T \approx 3000 \, ^\circ\text{K} \) the universe was dominated completely by four components: photons, cold dark matter, neutrinos and a baryonic plasma consisting of free electrons, ions and neutral atoms. We will sketch in the following how to get the required form factors.
Sketch of perturbations in general relativistic cosmology

Within general relativity perturbations in energy and matter densities affect the geometry via the metric. One thus has to make sure that the Einstein equations are satisfied, i.e. we are confronted with a metric feedback. The interpretation of the physical meaning of perturbations is obscured by general covariance, the freedom of performing arbitrary smooth coordinate transformations $x^\mu \rightarrow f^\mu(x^0, x^1, x^2, x^3)$. In cosmology, with the Robertson-Walker metric as a constrained metric form, the so called synchronous gauge, seems best adapted for the discussion of perturbations. In the synchronous gauge the complete perturbed metric reads

$$g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = S^2(t) \left[ (1 + A) \delta_{ij} + \frac{\partial^2 B}{\partial_i \partial_j} \right].$$

This gauge in fact does not fix the gauge completely. There is a residual gauge
freedom, which however can be fixed in a natural manner because of the existence of non-relativistic cold dark matter: for which we require
\[ \bar{p}_D = 0 \quad \text{and} \quad \bar{\rho}_D \propto S^{-3}(t) \quad \text{and} \quad \delta p_D = \pi^S_D = 0 \] (see below).

In comparison the Newtonian gauge is fixed by

\[
\begin{align*}
    g_{00} &= -1 - 2\Phi, & g_{0i} &= 0, & g_{ij} &= S^2(t) \delta_{ij} [1 - 2\Psi].
\end{align*}
\]

This gauge is completely fixed, i.e. there are no residual coordinate transformation ambiguities. In this gauge there exists a conserved quantity

\[ R_q = -\Psi_q + H \delta u_q \]

where \( \delta u \) is the velocity potential. The relevance of \( R \) is, that it may be conserved outside the horizon. This is important for having control over the transmission of fluctuations from the very early history of the universe (inflation) to later times like
the recombination era. A theorem by Weinberg infers:

Whatever the content of the universe, there are two independent adiabatic physical scalar solutions of the Newtonian gauge field equations for which the quantity \( R_q \) is time-independent in the limit \( q/S \ll H \).

A gauge invariant definition of \( R \) is

\[
R_Q = A_a/2 + H \delta u_q
\]

and in the synchronous gauge can be expressed uniquely in terms of the gravitational metric variable \( \psi \):

\[
q^2 R_q = - S^2 H \psi_q + 4\pi G S^2 \delta \rho_q + q^2 H \delta u_q
\]

In addition the energy momentum tensor, which we derived for the perfect fluid, has to be generalized by dissipative corrections:
\[
\delta T_{00} = -\bar{\rho} h_{00} + \delta \rho , \\
\delta T_{0i} = \bar{p} h_{0i} - (\bar{\rho} + \bar{p}) \left( \partial_i \delta u + \delta u_i^V \right) , \\
\delta T_{ij} = \bar{p} h_{ij} + S^2 \left[ \delta_{ij} \delta \rho + \partial_i \partial_j \pi^S + \partial_i \pi_j^V + \partial_j \pi_i^V + \pi_{ij}^T \right]
\]

where \( S \) are scalar, \( V \) vector and \( T \) tensor perturbations (satisfying \( \partial_i \pi_i^V = \partial_i \delta u_i^V = 0 \), \( \partial_i \pi_{ij}^T = 0 \), \( \pi_{ii}^T = 0 \)). These forms can be taken as definitions of the fluctuation terms \( \delta \rho \), \( \delta \rho \), \( \delta u_i \equiv \partial_i \delta u + \delta u_i^V \), and the anisotropic inertia terms \( \pi^S \), \( \pi^V \) and \( \pi^T \).
Herewith, the Einstein field equations can be worked out and take the form:

\[-4\pi G S^2 \left[ \delta \rho - \delta p - \nabla^2 \pi^S \right] = \frac{1}{2} \nabla^2 A - \frac{1}{2} S^2 \ddot{A} - 3 S \dot{S} \dot{A} - \frac{1}{2} S \dot{S} \nabla^2 \dot{B},\]

\[-16\pi G S^2 \pi^S = A - S^2 \ddot{B} - 3 S \dot{S} \dot{B},\]

\[8\pi G S (\bar{\rho} + \bar{p}) \delta u = S \dot{A},\]

\[-4\pi G \left( \delta \rho + 3 \delta p + \nabla^2 \pi^S \right) = \frac{2}{3} \ddot{A} + 3 H \dot{A} + \frac{1}{2} \nabla^2 \ddot{B} + H \nabla^2 \dot{B}\]

The equations of energy-momentum conservation now read:

\[\delta p + \nabla^2 \pi^S + \partial_0 \left[ (\bar{\rho} + \bar{p}) \delta u \right] + 3 H (\bar{\rho} + \bar{p}) \delta u = 0 \quad (I)\]

\[\delta \rho + 3 H (\delta \rho + \delta p) + \nabla^2 \left[ \frac{1}{S^2} \left( \bar{\rho} + \bar{p} \delta u + H \pi^S \right) \right] + \frac{1}{2} (\bar{\rho} + \bar{p}) \psi = 0 \quad (II)\]
where

\[ \psi \equiv \frac{1}{2} \left[ 3 \dot{A} + \nabla^2 \dot{B} \right] = \frac{\partial}{\partial t} \left( \frac{h_{ii}}{2S^2} \right) \]

In a perfect or imperfect fluid only the quantity \( \psi \) enters as a feedback of the perturbation of the metric. In fact also the last of Einsteins field equations is dependent on \( \psi \) only:

\[ -4\pi G S^2 \left( \delta \rho + 3 \delta p + \nabla^2 \pi^s \right) = \frac{\partial}{\partial t} \left( S^2 \psi \right) \]  

(III)

In fact not all equations are independent, as Einsteins equations also imply energy momentum conservation. Indeed, equations (I), (II) and (III) plus the equation of state are sufficient to calculate \( \delta \rho \), \( \delta u \) and \( \psi \).

Catch the essentials: take hydrodynamic limit, neglect anisotropic inertia and
assume $\delta u_\gamma = \delta u_B$, and take $p_{\gamma q} = \rho_{\gamma q}/3$ and $p_{\nu q} = \rho_{\nu q}/3$ in the following:

1 Cold dark matter: which is zero pressure and velocity; total density $\bar{\rho}_D(t) + \delta \rho_D(x, t)$, where $\bar{\rho}_D(t) \propto S^{-3}(t)$. Energy conservation implies for the Fourier components: $(\psi_q \equiv (3 \dot{A}_q - q^2 \dot{B}_q)/2)$

$$\delta \dot{\rho}_{Dq} + 3H \delta \rho_{Dq} = -\bar{\rho}_D \psi_q$$

2 Baryonic plasma: due to the Coulomb interaction of electrons with atomic nuclei electrons and baryons act as a single prefect fluid. During the epoch of interest both electrons and baryons are highly non-relativistic, the baryons have negligible pressure and anisotropic inertia and the unperturbed density scales like $\bar{\rho}_B(t) \propto S^{-3}(t)$. The total density is $\bar{\rho}_B(t) + \delta \rho_B(x, t)$, and the perturbation is governed by energy conservation, now with non-zero velocity potential $\delta u_B(x, t)$:

$$\delta \dot{\rho}_{Bq} + 3H \delta \rho_{Bq} - \left(q^2/S^2\right) \bar{\rho}_B \delta u_{Bq} = -\bar{\rho}_B \psi_q$$
In this case Thomson scattering allows the baryonic plasma to exchange momentum with photons, such that momentum conservation couples photons with the plasma:

\[ \delta p_{\gamma q} - q^2 \pi_{\gamma q} + [\partial_0 + 3H][\bar{\rho}_B \delta u_{Bq} + \frac{4}{3} \bar{\rho}_\gamma \delta u_{\gamma q}] = 0 \, , \]

which under the above simplifications reduces to

\[ \frac{d}{dt} \left( \left( \frac{4}{3} \bar{\rho}_\gamma + \bar{\rho}_B \right) \delta u_{\gamma q} \right) + 3H \left( \left( \frac{4}{3} \bar{\rho}_\gamma + \bar{\rho}_B \right) \delta u_{\gamma q} \right) = -(1/3) \delta \rho_{\gamma q} \, . \]

Photons: during recombination photonic processes are pretty complicated. It requires to apply Boltzmann transport theory and usually can only be investigated numerically. Here we consider the early epoch when photons essentially were in thermal equilibrium and a hydrodynamic treatment makes sense. Energy conservation then reads
\[ \delta \dot{\rho}_{\gamma q} + 4 \, H \, \delta \rho_{\gamma q} - \left( 4 q^2 / 3 S^2 \right) \bar{\rho}_\gamma \, \delta u_{\gamma q} = -(4/3) \bar{\rho}_\gamma \psi_q \]

**Neutrinos:** are treated in the same spirit as the photons

\[ \delta \dot{\rho}_{\nu q} + 4 \, H \, \delta \rho_{\nu q} - \left( 4 q^2 / 3 S^2 \right) \bar{\rho}_\nu \, \delta u_{\nu q} = -(4/3) \bar{\rho}_\nu \psi_q \]

Momentum conservation for the neutrinos is given by

\[ \frac{d}{dt} \left( \bar{\rho}_\nu \, \delta u_{\nu q} \right) + 3 \, H \, \left( \bar{\rho}_\nu \, \delta u_{\nu q} \right) = -(1/4) \, \delta \rho_{\nu q} . \]

These energy-momentum balance equations go along with the gravitational field equation

\[ \frac{d}{dt} \left( S^2 \psi_q \right) = -4\pi G \, S^2 \left( \delta \rho_{Dq} + \delta \rho_{Bq} + 2 \, \delta \rho_{\gamma q} + 2 \, \delta \rho_{\nu q} \right) . \]
Conveniently one uses fractional perturbations

\[ \delta_{\alpha q} = \frac{\delta\alpha_q}{\bar{\rho}_\alpha + \bar{\rho}_\alpha} ; \quad \alpha = \gamma, D, B, \nu \]

and taking into account that \( S^4 \bar{\rho}_\gamma, S^3 \bar{\rho}_D, S^3 \bar{\rho}_B \) and \( S^4 \bar{\rho}_\nu \) are time-independent the basic set of dynamic equations reads:
\[
\frac{d}{dt} \left( S^2 \psi_q \right) = -4\pi G S^2 \left( \bar{\rho}_D \delta_D q + \bar{\rho}_B \delta_B q + \frac{8}{3} \bar{\rho}_{\gamma} \delta_{\gamma} q + \frac{8}{3} \bar{\rho}_{\nu} \delta_{\nu} q \right),
\]

\[
\dot{\delta}_{\gamma q} - \left( \frac{q^2}{S^2} \right) \delta u_{\gamma q} = -\psi_q,
\]

\[
\dot{\delta}_D q = -\psi_q,
\]

\[
\dot{\delta}_B q - \left( \frac{q^2}{S^2} \right) \delta u_{\gamma q} = -\psi_q,
\]

\[
\dot{\delta}_{\nu q} - \left( \frac{q^2}{S^2} \right) \delta u_{\nu q} = -\psi_q,
\]

\[
\frac{d}{dt} \left( (1 + R) \frac{\delta u_{\gamma q}}{S} \right) = -\frac{1}{3S} \delta_{\gamma q},
\]

\[
\frac{d}{dt} \left( \frac{\delta u_{\gamma q}}{S} \right) = -\frac{1}{3S} \delta_{\nu q},
\]

where \( R = 3\bar{\rho}_B/4\bar{\rho}_{\gamma} \).
In order to find the solutions of the system of DEQs one has to look for the appropriate initial conditions. For this we have to go to sufficiently early times when the universe was radiation dominated $\bar{\rho}_M \ll \bar{\rho}_R$, where

$$\bar{\rho}_M = \bar{\rho}_D + \bar{\rho}_B \quad , \quad \bar{\rho}_R = \bar{\rho}_\gamma + \bar{\rho}_\nu ,$$

such that $S(t) \propto \sqrt{t}$, $8\pi G \bar{\rho}_R/3 = 1/4t^2$ and $R \ll 1$. The basic equation then simplify to:
\[
\frac{d}{dt} (t \psi_q) = -4\pi G t \left( \bar{\rho}_D \delta_D q + \bar{\rho}_B \delta_B q + \frac{8}{3} \bar{\rho}_\gamma \delta_\gamma q + \frac{8}{3} \bar{\rho}_\nu \delta_\nu q \right),
\]

\[
\dot{\delta}_\gamma q = \dot{\delta}_B q = -\psi_q + \left( q^2 / S^2 \right) \delta u_{\gamma q},
\]

\[
\dot{\delta}_v q = -\psi_q + \left( q^2 / S^2 \right) \delta u_{v q},
\]

\[
\dot{\delta}_D q = -\psi_q,
\]

\[
\frac{d}{dt} \left( \frac{(1 + R) \delta u_{\gamma q}}{S} \right) = -\frac{1}{3S} \delta_{\gamma q},
\]

\[
\frac{d}{dt} \left( \frac{\delta u_{v q}}{S} \right) = -\frac{1}{3S} \delta_{v q}.
\]

At early times the perturbations were outside the horizon, in the sense that \( q / S \ll H \), still \( q^2 \) terms have been kept in order to be able to calculate \( R_q \) properly,
and also we allow that matter fluctuations like $\delta_{Bq}$ could be enhanced by $\bar{\rho}_\gamma/\bar{\rho}_B$ relative to $\delta_{\gamma q}$.

The dominant adiabatic (in the sense that all $\delta_{\alpha q}$ are equal at very early time) solution is characterized by the assumption

$$
\delta_{\gamma q} = \delta_{Bq} = \delta_{Dq} = \delta_{vq} \equiv \delta_q, \quad \delta u_{\gamma q} = \delta u_{vq} \equiv \delta u_q
$$

which follows from the natural assumption that the universe was in a state of complete local thermal equilibrium with no non-zero conserved quantities. In this case we can drop the mentioned subleading terms obtaining:
\[ \dot{\delta}_q = -\psi_q , \]
\[ \frac{d}{dt} \left( \frac{\delta u_q}{\sqrt{t}} \right) = -\frac{1}{3 \sqrt{t}} \delta_q , \]
\[ \frac{d}{dt} (t \psi_q) = -\frac{1}{t} \delta_q . \]

Combining the 1st with the 3rd equation we have
\[ \frac{d}{dt} \left( t \frac{d}{dt} \delta_q \right) - \frac{1}{t} \delta_q = 0 , \]
which has solutions \( \delta_q \propto t \) and \( \delta_q \propto 1/t \) and in each case \( \psi_q \) and \( \delta u_q \) is determined by the above system of equations. The mode of interest is the one which grows and thus is the dominant one. This solution reads
\[ \delta_{\gamma q} = \delta_{Bq} = \delta_{\nu q} = \delta_{Dq} = \frac{t^2 q^2 R^o_q}{S^2}, \]
\[ \psi_q = -\frac{t q^2 R^o_q}{S^2}, \]
\[ \delta u_{\gamma q} = \delta u_{\nu q} = -\frac{2 t^3 q^2 R^o_q}{S^2}. \]

The solution is normalized such that the conserved extra quantity which defined \( R \),
\[ q^2 R_q = -S^2 H \psi_q + 4\pi G S^2 \delta \rho_q + q^2 H \delta u_q \]
takes a time-independent value \( q^2 R^o_q \) for \( q/S \ll H \), and the index \( ^o \) indicates “outside the horizon”.

Also from the basic set of equations we have
\[ \frac{d}{dt} \left( \delta_{Bq} - \delta_{\gamma q} \right) = 0 \]

and for the adiabatic solution which satisfies \( \delta_{Bq} = \delta_{\gamma q} \) at early times satisfies

\[ \delta_{Bq} = \delta_{\gamma q} \]

at all times, which simplifies the basic system further. Even this idealized system cannot be solved analytically. While there is no principal objection in solving the problem numerically, the advantage of analytic results is that they help to understand physics at work in a more transparent way. There are two regimes where the system of equations can be solved analytically: the long wavelength regime \( q \ll q_{\text{EQ}} \) and the short wavelength regime \( q \gg q_{\text{EQ}} \), where \( q_{\text{EQ}} \) is the wave number for which \( q/S = H \) at radiation-matter equality of densities. After a possible inflation era i.e. in standard Friedmann cosmologies \( q/S \) decreases more slowly than \( H \) [shown above] such that:

- for long wavelength \( \bar{\rho}_M = \bar{\rho}_R \) for \( q/S \ll H \)
for short wave length $q/S \rightarrow H$ for $\bar{\rho}_M \ll \bar{\rho}_R$

**Calculation of $q_{\text{EQ}}$:**

what we need is the redshift

$$1 + z_{\text{EQ}} = \Omega_M/\Omega_R = \Omega_M h^2 / 4.15 \times 10^{-5},$$

and the Hubble rate. The latter in the radiation dominated era is

$$H = H_0 \sqrt{\Omega_R T^4/T_{0,\gamma}^4} = 2.1 \times 10^{-20} \left(\frac{T}{T_{0,\gamma}}\right)^2 \text{s}^{-1} = 2.1 \times 10^{-20} (1 + z)^2 \text{s}^{-1}.$$  

At matter radiation equality the energy density is twice that of the radiation thus $H$ is larger by a factor $\sqrt{2}$. Fluctuations that just enter the horizon at matter radiation equality have physical wave number = Hubble rate, thus

$$q_{\text{EQ}}/S_{\text{EQ}} = H_{\text{EQ}} = \sqrt{2} \Omega_R H_0 d_A (1 + z_L)$$

$$= \sqrt{2} \times 2.1 \times 10^{-20} (1 + z_{\text{EQ}})^2 \text{s}^{-1} = 1.72 \times 10^{-11} (\Omega_M h^2)^2 \text{s}^{-1}$$
This gives the critical physical wave length at present as

$$\lambda_0 = \frac{2\pi}{q_{EQ}/S_0} = \frac{2\pi (1 + z_{EQ})}{q_{EQ}/S_{EQ}} = 85 (\Omega_M h^2)^{-1} \text{ Mpc}.$$  

This compares to the distance from earth to the Virgo cluster which is about 18 Mpc.

Perturbations which are now observed to extend over distances which are larger or smaller than $\lambda_{EQ}$ came within the horizon before or after matter-radiation equality, respectively.

How much matter do we “see”?

Present mass density $\rho_{0M} \rightarrow$ mass now contained in a sphere of diameter $\lambda_0$

$$M_{\text{observable}} \approx \frac{\pi}{6} \rho_{0M} \lambda_0^3 = 0.9 \times 10^{17} (\Omega_M h^2)^{-2} M_\odot$$
This compares to a large galaxy of mass about $M_{\text{galaxy}} \sim 10^{12} M_\odot$.

Thus any perturbation relevant for the formation of galaxies or even clusters of galaxies must have been well within the horizon at the time of radiation-matter equality.

**What is the critical multipole $\ell_{\text{EQ}}$?**

that we see in the anisotropy of the CMB. The integral over $q$ which yields the multipole coefficient $C_\ell$ is dominated by physical wave numbers $k_L = q/S(t_L) \approx \ell/r_L$, which means that the integral is dominated by wave numbers of order $q_{\text{EQ}}$ that just come into the horizon at matter-radiation equality if $\ell_{\text{EQ}} = q_{\text{EQ}} t_L$

$$\ell_{\text{EQ}} = \left(\frac{q_{\text{EQ}}}{S_{\text{EQ}}/S_0}\right) \left(\frac{S_0}{S_L}\right) S_L r_L = \frac{H_{\text{EQ}} (1 + z_L)}{(1 + z_{\text{EQ}})} S_L r_L .$$
With $S_L r_L = d_A$ and $1 + z_{EQ} = \Omega_M / \Omega_R$ we find

$$\ell_{EQ} = \Omega_M \sqrt{2/\Omega_R} H_0 d_A (1 + z_L).$$

In numbers: $\Omega_M = 0.26$, $\Omega_\Lambda = 0.74$ yields $d_A = 3.38 H_0^{-1} (1 + z_L)^{-1}$, with $\Omega_R = 8.01 \times 10^{-5}$ ($T_0 = 2.725 \, ^\circ\text{K}$ and $H_0 = 72 \, \text{km sec}^{-1} \, \text{Mpc}^{-1}$) yields

$$\ell_{EQ} \approx 140$$

Thus multipoles $\ell > \ell_{EQ}$ only can arise from the radiation dominated era.

In the following a dimensionless rescaled wave number $\kappa$ will be used:

$$\kappa \equiv \sqrt{2} \frac{q}{q_{EQ}} = \frac{(q/S_0) \sqrt{\Omega_R}}{H_0 \Omega_M} = \frac{q/S_0}{0.052 \Omega_M h^2 \text{Mpc}^{-1}}$$

Long and short wavelength correspond to $\kappa \ll 1$ (long) and $\kappa \gg 1$ (short), respectively.
In the following we only consider the dominant scalar perturbations in various regimes, which at the end will have to be patched together.
Scalar perturbations – long waves

Very long wavelength are outside horizon at the time $t_{EQ}$, they contribute to low $\ell < \ell_{EQ}$.

- In radiation-dominated era $q\bar{\rho}_R/S H \bar{\rho}_M = \text{const}$ [as $S \propto t^{1/2}$ ; $H \propto S^{-2}$] and since small at $\bar{\rho}_M = \bar{\rho}_R$ then $q/SH \ll \bar{\rho}_M/\bar{\rho}_R$ throughout radiation-dominated era.

- During matter-dominated era $q^2 \bar{\rho}_R/S^2 H^2 \bar{\rho}_M = \text{const}$ [as $S \propto t^{2/3}$ ; $H \propto S^{-3/2}$] and since small at $\bar{\rho}_M = \bar{\rho}_R$ then $(q/SH)^2 \ll \bar{\rho}_M/\bar{\rho}_R$ throughout matter-dominated era.

Outside the horizon: Basic equations with $\delta_{Bq} = \delta_{yq}$ for $q/S \ll H$ and adiabatic boundary condition reads
\[ \frac{d}{dt} (S^2 \psi_q) = -4\pi G S^2 \left[ \bar{\rho}_M + \frac{8}{3} \bar{\rho}_R \right] \delta_q , \]

\[ \dot{\delta}_q = -\psi_q , \]

\[ \delta_q \equiv \delta_{\gamma q} = \delta_{\nu q} = \delta_{Bq} = \delta_{Dq} . \]

Combining the 1st with the 2nd equation we have

\[ \frac{d}{dt} \left( S^2 \frac{d}{dt} \delta_q \right) = 4\pi G S^2 \left[ \bar{\rho}_M + \frac{8}{3} \bar{\rho}_R \right] \delta_q . \]

A change of variable \( t \rightarrow y \equiv S(t)/S_{EQ} = \bar{\rho}_M/\bar{\rho}_R \) yields \( \bar{\rho}_M = \rho_{EQ}/y^3 \) and
\( \bar{\rho}_R = \rho_{EQ}/y^4 \) and the Friedmann equation reads

\[
\frac{d}{dt} = \frac{H_{EQ}}{\sqrt{2}} \frac{\sqrt{1 + y}}{y} \frac{d}{dy}
\]

and the DEQ in \( y \) reads

\[
y \sqrt{1 + y} \frac{d}{dy} \left( y \sqrt{1 + y} \frac{d}{dy} \delta_q \right) - \frac{3}{2} \left( y + \frac{8}{3} \right) \delta_q = 0
\]

with solutions \( \delta_1 = \left( 16 + 8y - 2y^2 + y^3 \right)/y^2 \) and \( \delta_2 = \sqrt{1 + y}/y^2 \) and the one with the correct initial condition, worked out above to behave as \( t \propto y^2 \) for \( t \to 0 \), is \( \delta_1 - 16 \delta_2 \). The other results follow from this solution and the other equations:
\[ \delta_q = \frac{4q^2 R_q^o}{5H_{EQ}^2 S_{EQ}^2 y^2} \left(16 + 8y - 2y^2 + y^3 - 16 \sqrt{1+y}\right), \]

\[ \delta u_{\gamma q} = -\frac{\sqrt{2}y}{3H_{EQ} (1+R)} \int_0^y \frac{dy'}{\sqrt{1+y'}} \delta_q(y') \; ; \; \delta u_{\nu q} = (1+R) \delta u_{\gamma q}, \]

\[ \psi_q = \frac{\sqrt{2}q^2 R_q^o}{5H_{EQ} S_{EQ}^2 y^4} \left(2 \sqrt{1+y} \left(32 + 8y - y^3\right) - 64 - 48y\right). \]

The normalization is such that \( R_q \) takes the time-independent value \( R_q^o \) outside the horizon.

\[ \textbf{2 The matter-dominated era:} \] Here \( \bar{\rho}_M \gg \bar{\rho}_R \) and we also take \( \bar{\rho}_B \ll \bar{\rho}_D \) as in reality.
Here no assumption about the wavelength is made. The solutions overlap with the previous ones, which serve as boundary condition. This yields

\[
\begin{align*}
\frac{d}{dt} \left( t^{4/3} \psi_q \right) &= -\frac{2}{3} t^{2/3} \delta_{Dq}, \\
\dot{\gamma}_q - (q^2 / S^2) \delta_{u_{\gamma q}} &= -\psi_q, \\
\dot{\delta}_{Dq} &= -\psi_q, \\
\frac{d}{dt} \left( t^{-2/3} (1 + R) \delta_{u_{\gamma q}} \right) &= -\frac{1}{3} t^{-2/3} \delta_{\gamma q}, \\
\dot{\gamma}_q - (q^2 / S^2) \delta_{u_{\nu q}} &= -\psi_q, \\
\frac{d}{dt} \left( t^{-2/3} \delta_{u_{\nu q}} \right) &= -\frac{1}{3} t^{-2/3} \delta_{\nu q}.
\end{align*}
\]
\[
\delta_{Dq} = \frac{4q^2 y R_q^o}{5H_{EQ}^2 S_{EQ}^2} = \frac{9q^2 t^2 R_q^o}{10S^2}, \\
\psi_q = \frac{-2 \sqrt{2} q^2 R_q^o}{5H_{EQ} S_{EQ}^2 \sqrt{y}} = -\frac{3q^2 t R_q^o}{5S^2}, \\
\delta_{\gamma q} = \delta_{Bq} = \frac{3R_q^o}{5} \left[ 1 + 3R - (1 + R)^{-1/4} \cos \phi \right], \\
\delta u_{\gamma q} = \frac{3 t R_q^o}{5} \left[ -1 + \frac{S}{\sqrt{3} q t (1 + R)^{3/4}} \sin \phi \right].
\]
with

\[ \phi \equiv \int_0^t \frac{q \, dt'}{S(t') \sqrt{3 (1 + R(t'))}} = \frac{\sqrt{3} q \, t}{S \sqrt{R}} \ln \left( \sqrt{R} + \sqrt{1 + R} \right). \]


\[ \text{digression: calculation of } \delta_{\gamma q} \]

With the simple solution found for \( \psi_q \) one find a simple particular solution

\[ \delta_{\gamma q}^{(1)} = \frac{3t^2 q^2 (1 + 3R) R_q^o}{5S^2(t^2 q^2 / S^2 + 2R)} ; \quad \delta u_{\gamma q}^{(1)} = \frac{3t^3 q^2 R_q^o}{5S^2(t^2 q^2 / S^2 + 2R)} \]

to which suitable solutions of the corresponding homogeneous equations

\[ \dot{\delta}_{\gamma q}^{(2)} = \left( q^2 / S^2 \right) \delta u_{\gamma q}^{(2)} ; \quad \frac{d}{dt} \left( t^{-2/3} (1 + R) \delta u_{\gamma q}^{(2)} \right) = \frac{1}{3} t^{-2/3} \delta_{\gamma q}^{(2)}, \]
must be added. Eliminating the velocity potential yields

\[
\frac{d}{dt} \left( t^{-2/3} (1 + R) S^2 \frac{d}{dt} \delta^{(2)}_{\gamma q} \right) + \frac{q^2}{3} t^{-2/3} \delta^{(2)}_{\gamma q} = 0
\]

with the coefficients chosen such that \( \delta^{(1)}_{\gamma q} + \delta^{(2)}_{\gamma q} \) satisfies the earlier found solution outside the horizon for \( q^2 t^2 / S^2 \ll 1 \). The solutions are linear combinations of Gauss hypergeometric functions

\[
\mathcal{F}_1(\frac{1}{4} - \frac{1}{4} x, \frac{1}{4} + \frac{1}{4} x, \frac{1}{2}, -R), \quad \sqrt{R} \mathcal{F}_1(\frac{3}{4} - \frac{1}{4} x, \frac{3}{4} + \frac{1}{4} x, \frac{3}{2}, -R)
\]

with \( x = \sqrt{1 - 16\eta} \); \( \eta \equiv \frac{3t^2 q^2}{4S^2 R} \) where \( \eta \) is time-independent during the matter-dominated era. This solution is not very intuitive. Under the condition

\[
R \ll t^2 q^2 / S^2 \ll \bar{\rho}_M / \bar{\rho}_R
\]

one can get simpler solutions of the form \( \delta^{(2\pm)}_{\gamma q} \propto (1 + R)^{-1/4} \exp(\pm i \phi) \), where \( \phi \) was defined above. Together with the appropriate boundary conditions one obtains the results given before.

end digression

The results for \( \delta_{\gamma q} = \delta_{Bq} \) and \( \delta u_{\gamma q} \) apply for moderately long wavelength only up to
the time of last scattering, while the one for \( \delta_{Dq} \) and \( \psi_q \) are unaffected by the decoupling of the baryon plasma to the extent that cold dark matter is dominating and until spatial curvature or the cosmological constant term come into play.
Interpretation of the “outside the horizon” term $R^o_q$

If we assume that the fluctuation of the Newtonian gravitational potential $\phi(x, t)$ around the time of last scattering is dominated by fluctuations in the cold dark matter density, in Fourier space the Poisson equation says

$$\delta\phi(q, t) = -4\pi G \left( S(t)^2 / q^2 \right) \delta\rho_D(q, t) = -4\pi G \left( S^2(t) / q^2 \right) \bar{\rho}_D(t) \delta_Dq(t),$$

and with the solution for $\delta_Dq$ given above and the Friedmann equation $H^2 = (2/3t)^2 = 8\pi G \bar{\rho}_D/3$ we see that $\delta\phi(q, t)$ takes a time-independent value

$$\delta\phi(q) = -\frac{3}{5} \alpha(q) R^o_q.$$
Earlier, in Lect. 12 Subsect. [2], we defined the correlation \( \langle \delta \phi(q) \delta \phi(q') \rangle = P_\phi(q) \delta^{(3)}(q + q') \) we find

\[
P_\phi(q) = \frac{9}{25} |R_q^0|^2.
\]

In the first part of this lecture we found \( P_\phi(q) = N_\phi^2 q^{-3} \) to yield a temperature fluctuation multipole coefficient \( C_\ell = 8\pi N_\phi^2/9\ell(\ell + 1) \), which corresponds to the assumption that

\[
|R_q^0|^2 = N^2 q^{-3} \text{ with } N_\phi^2 = 9N^2/25.
\]
Scalar perturbations – short waves

In this case all perturbations are inside the horizon. They are responsible for CMB anisotropies with $\ell \gg \ell_{\text{EQ}} \sim 140$. For these wavelength $q/SH \gg \frac{\rho_M}{\rho_R}$ in the radiation-dominated era and $(q/SH)^2 \nu \gg \frac{\rho_M}{\rho_R}$ for the matter-dominated era. Relatively simple analytic solutions may be found for two overlapping regimes, discussed in the following.

The radiation-dominated era: The simplified set of basic equations here reads
We are interested in the adiabatic solution with equal initial perturbations for all components. As the DEQs are the same for neutrinos and photons thus
\[ \delta_{\nu q} = \delta_{\gamma q} ; \quad \delta u_{\nu q} = \delta u_{\gamma q} . \]

and thus

\[
\frac{d}{dt}(t\psi_q) = -\frac{32\pi G \bar{\rho}_R t}{3} \delta_{\gamma q} = -\frac{1}{t} \delta_{\gamma q}
\]

For the leading adiabatic solution for \( q/S H \ll 1 \) we must have

\[
\delta_{\gamma q} = \delta_{B q} = \delta_{\nu q} \to \frac{q^2 t^2 R_q^o}{S^2} , \quad \delta_{D q} \to \frac{q^2 t^2 R_q^o}{S^2} ,
\]

\[
\psi_q \to -\frac{q^2 t R_q^o}{S^2} , \quad \delta u_{\gamma q} = \delta u_{\nu q} \to \frac{3q^2 t T R_q^o}{9S^2} .
\]

The solution then reads...
\[
\begin{align*}
\delta_{\gamma q} &= \delta_{Bq} = \delta_{vq} = 3R_q^o \left( \frac{2}{\Theta} \sin \Theta - \left(1 - \frac{2}{\Theta^2}\right) \cos \Theta - \frac{2}{\Theta^2} \right), \\
\psi_q &= R_q^o t^{-1} \left( \frac{2}{\Theta} \sin \Theta + \frac{2}{\Theta^2} \cos \Theta - \frac{2}{\Theta^2} - 1 \right), \\
\delta_{Dq} &= -6R_q^o \int_0^\Theta \left( \frac{2}{\theta^3} \sin \theta + \frac{2}{\theta^4} \cos \theta - \frac{2}{\theta^4} - \frac{1}{\theta^2} \right) \theta d\theta, \\
\delta u_{\gamma q} &= \delta u_{vq} = 4tR_q^o \left( \frac{\sin \Theta}{2\Theta} - \frac{1 - \cos \Theta}{\Theta^2} \right),
\end{align*}
\]

where

\[\Theta \equiv \frac{2qt}{\sqrt{3}S}.\]

We thus observe oscillations of circular frequency \(\omega = \frac{2}{\sqrt{3}} k\), where \(k = q/S\) is the
physical wave number.

Deep inside the horizon:

1. Fast modes.

Fast modes come from deep inside the horizon when \( \frac{q}{S} \gg H \). For the rapidly oscillating fast modes neutrinos are not involved because of the large mean free path and we may neglect them. The metric is affected essentially only by the cold dark matter, which is also not participating in the photon-electron-baryon plasma.
oscillations. The equations for these effects therefore simplify to

\[
\dot{\delta}_{\gamma q} = \left( \frac{q^2}{S^2} \right) \delta u_{\gamma q} ,
\]

\[
\frac{d}{dt} \left( \frac{(1 + R) \delta u_{\gamma q}}{S} \right) = -\frac{1}{3S} \delta_{\gamma q} ,
\]

\[
\frac{d}{dt} \left( S^2 \psi_q \right) = -\frac{16\pi G S^2}{3} \bar{\rho}_\gamma (R) + 2 \delta_{\gamma q} ,
\]

\[
\dot{\delta}_{Dq} = -\psi_q ,
\]

and eliminating the velocity fluctuation potential \( \delta u_{\gamma q} \) one gets the key equation for \( \delta_{\gamma q} \)

\[
\frac{d}{dt} \left( S \left( 1 + R \right) \frac{d}{dt} \delta_{\gamma q} \right) + \frac{q^2}{2S} \delta_{\gamma q} = 0 ,
\]

which describes imprinting the acoustic peaks into the photon density i.e. the CMB. If \( S \) and \( R \) would be constant this is just a wave equation for sound waves.
with physical wave number $q/S$ and velocity

$$v_s = 1/\sqrt{3 (1 + R)}.$$  

The condition of constant entropy gives

$$d\rho_B/\bar{\rho}_B = d\rho_\gamma/ (\bar{\rho}_\gamma + \bar{p}_\gamma) = (3/4) d\rho_\gamma/\bar{\rho}_\gamma$$

and thus

$$v_s^2 = dp/d\rho = d\rho_\gamma/3(d\rho_\gamma + d\rho_B) = 1/3 (1 + R).$$

For $q/S \gg H$ taking advantage of the fact that $S$ and $R$ are slowly varying, one can solve the equation in the so called WKB approximation. What happens is that the time dependence of $S$ and $R$ lead to a damping of the oscillatory behavior, so called Silk damping. The fast modes then are given by

$$\delta_{\gamma q}^{\pm} = \frac{1}{(1+R)^{1/4}} \exp \left[ \pm i q \int_0^t \frac{dt}{S(t)} \frac{dt}{\sqrt{3(1+R(t))}} - \int_0^t \Gamma \, dt \right]$$

With this solution also the other fast mode solutions $\delta u_{\gamma q}^{\pm}$, $\psi_q^{\pm}$ and $\delta_{Dq}^{\pm}$ are determined by the system of DEQs.
The Silk damping is characterized by a characteristic decay time $1/\Gamma$ where

$$\Gamma = \frac{q^2 t_\gamma}{6 S^2 (1 + R)} \left\{ \frac{16}{15} + \frac{R^2}{1 + R} \right\} .$$

In general

$$\Gamma = \frac{k^2}{2(\rho + p)} \left\{ \zeta + \frac{4}{3} + \chi \left( \frac{\partial \rho}{\partial T} \right)_n \times \left[ \rho + p - 2T \left( \frac{\partial p}{\partial T} \right)_n + v_s^2 T \left( \frac{\partial \rho}{\partial T} \right)_n - \frac{n}{v_s^2} \left( \frac{\partial p}{\partial n} \right)_T \right] \right\}$$

where, $\eta = \text{shear viscosity}$, $\chi = \text{heat conduction}$, and $\zeta = \text{bulk viscosity}$, $v_s = \text{sound speed}$ [$1/v_s^2 = \sqrt{3(1 + R)}$] and $n = \text{number density}$. Here we have $\rho = \rho_B + \rho_\gamma$, $p = \rho_\gamma/3$ with $\rho_B \propto n$ and $\rho_\gamma \propto T^4$. Setting $k = q/S$ we have

$$\Gamma = \frac{2q^2}{8S^2 \bar{\rho}_\gamma (1 + R)} \left\{ \zeta + \frac{4}{3} \eta + \frac{\chi T R^2}{3(1 + R)} \right\}$$

The viscosity and the heat conduction coefficients for photons interacting with non-relativistic plasma with mean free time $t_\gamma = 1/\sigma_T n_e$ are

$$\eta = \frac{16}{45} \bar{\rho}_\gamma t_\gamma , \quad \chi T = \frac{4}{3} \bar{\rho}_\gamma t_\gamma , \quad \zeta = 0 .$$

© 2009, F. Jegerlehner
Here $R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$ is determined by the time-dependent baryon to photon density ratio, and $ct_\gamma = 1/\sigma_T n_e$ is the mean free path, $\sigma_T$ the Thomson cross section and $n_e$ the electron density.

2. Slow modes.

“Slow modes” have suppressed time derivatives. In addition we are in the regime $q/S \ll H$. Thus “slow modes” are those for which time derivatives yield factors of order $H$, while the “fast modes” were those for which time derivatives yield a factor $q$. As $q/S \ll H$ while $H = O(1/t)$, these are solutions for which fractional rates change $\propto H$: $\dot{\delta}_{aq} \propto H \delta_{aq}$. Eqs. (4) (Subsect. 2) then imply $\dot{\delta}_{Dq} \propto H \delta_{Dq} = -\psi_q$ and hence $\delta_{Dq} = O(\psi_q/H)$, while $\delta_{\gamma q} = O(H\delta u_{\gamma q})$ and $\delta_{\nu q} = O(H\delta u_{\nu q})$ (last two eqs.). Hence, $\dot{\delta}_{\gamma q} = O(H^2\delta u_{\gamma q})$ and $\dot{\delta}_{\nu q} = O(H^2\delta u_{\nu q})$ such that they can be dropped such that

$$(q^2/S^2) \delta u_{\gamma q} = (q^2/S^2) \delta u_{\nu q} = \psi_q.$$
We then find (last two eqs.) $\delta_{\gamma q}$ and $\delta_{\nu q}$ are of order $O((HS^2/q^2)\psi_q)$. In the first eq. we thus have

$$\frac{\rho_\gamma + \rho_\nu}{\rho_D} = O\left\{\left(\frac{\bar{\rho}_B + 8/3 \bar{\rho}_R}{\bar{\rho}_D}\right)\left(\frac{S^2H}{q^2}\right)\right\}$$

Further simplification requires that

$$\frac{\bar{\rho}_R}{\bar{\rho}_D} < \left(\frac{\bar{\rho}_R}{\bar{\rho}_D}\right)_{\text{crit}} = \left(\frac{3}{8}\right)^{1/3} \left(\frac{q}{SH \bar{\rho}_D}\right)$$

and in fact this happens still before radiation-matter quality. We then get

$$\frac{d}{dt}\left(S^2 \psi_q\right) = -4\pi G \bar{\rho}_D S^2 \delta_{Dq}$$

with the last three Eqs. (●) remain unchanged. Eliminating $\psi_q$ we get a second
order differential equation for $\delta_{Dq}$

$$\frac{d}{dt} \left( S^2 \frac{d}{dt} \delta_{Dq} \right) = 4\pi G S^2 \bar{\rho}_D \delta_{Dq}.$$ 

A transformation of variables $t \rightarrow y = S/S_{EQ} = \bar{\rho}_M/\bar{\rho}_R$ with the help of Friedmann’s equation

$$\dot{y}^2/y^2 = \frac{8\pi G}{3} (\bar{\rho}_M + \bar{\rho}_R) = \frac{8\pi G \rho_{EQ}}{3} \left(1/y^3 + 1/y^4\right)$$

the yields Mészáros equation

$$y \left(1 + y\right) \frac{d^2 \delta_{Dq}}{dy^2} + \left(1 + \frac{3}{2} y\right) \frac{d\delta_{Dq}}{dy} - \frac{2}{3} (1 - \beta) \delta_{Dq} = 0,$$
where $\beta \equiv \bar{\rho}_B/\bar{\rho}_M = \Omega_B/\Omega_M$. For $\beta = 0$

$$\delta_{Dq}^{(1)} = 1 + \frac{3}{2} y ; \quad \delta_{Dq}^{(2)} = \left(1 + \frac{3}{2} y\right) \ln \frac{\sqrt{1 + y + 1}}{\sqrt{1 + y - 1}} - 3 \sqrt{1 + y} .$$

For general $\beta$ Hu and Sugiyama find $\delta_{Dq}$ as a linear combination of the two independent solutions

$$(1 + y)^{-\alpha_{\pm}} \, _2F_1(\alpha_{\pm}, \alpha_{\pm} + \frac{1}{2}, 2\alpha_{\pm} + \frac{1}{2}; \frac{1}{1 + y})$$

with $\, _2F_1$ the Gauss hypergeometric function and $\alpha_{\pm} = \frac{1 \pm \sqrt{1 + 24\beta}}{4}$. We note that $\beta = \bar{\rho}_B/\bar{\rho}_M \approx 1/6$ so that one may take $\beta \approx 0$. With $\delta_{Dq}$ given the other fractional perturbations follow from the remaining DEQs.

3. Fast-slow matching
The solutions obtained for the radiation dominated era are valid for fast and slow modes, and the corresponding expansions yields:

<table>
<thead>
<tr>
<th>solution</th>
<th>fast</th>
<th>slow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{\gamma q} = \delta_{Bq} = \delta_{\nu q}$</td>
<td>$-3R_q^o \cos \Theta$</td>
<td>$-\frac{6R_q^o}{\Theta^2}$</td>
</tr>
<tr>
<td>$\psi_q$</td>
<td>$\frac{6R_q^o}{t\Theta} \sin \Theta$</td>
<td>$-\frac{3R_q^o}{t}$</td>
</tr>
<tr>
<td>$\delta_{Dq}$</td>
<td>$\frac{12R_q^o}{\Theta^2} \cos \Theta$</td>
<td>$6R_q^o \left( -\frac{1}{2} + \gamma + \ln \Theta \right)$</td>
</tr>
<tr>
<td>$\delta u_{\gamma q} = \delta u_{\nu q}$</td>
<td>$\frac{2tR_q^o}{\Theta} \sin \Theta$</td>
<td>$-\frac{4tR_q^o}{\Theta^2}$</td>
</tr>
</tbody>
</table>

where $\gamma = 0.5772 \ldots$ is Euler’s constant and $\Theta = \frac{2qt}{\sqrt{3}S}$

- Matching radiation dominated region with fast modes deep inside the horizon:
  In the radiation dominated era $R \ll 1$ and damping is negligible (short mean free path) the argument of circular functions:

$$q \int_0^t \frac{dt}{S \sqrt{3(1 + R)}} \equiv \hat{\Theta} \rightarrow \frac{2qt}{\sqrt{3S}} = \Theta$$
such that the physical linear combination of $\delta^{\pm}_{\gamma q}$ found for the fast modes deep inside the horizon which fits to the radiation dominated region solution is:

$$\delta_{\gamma q}^{\text{fast}} = -\frac{3}{(1+R)^{1/4}} \hat{\mathcal{R}}_q \cos \hat{\Theta}.$$

Using the DEQs and $q/S \gg H$

$$\delta_{u_{\gamma q}}^{\text{fast}} = \frac{S \sqrt{3}}{q (1 + R)^{3/4}} \hat{\mathcal{R}}_q \sin \hat{\Theta}$$

$$\psi_{q}^{\text{fast}} = 16 \sqrt{3} \pi G \bar{\rho}_\gamma (2 + R) (1 + R)^{1/4} (S/q) \hat{\mathcal{R}}_q \sin \hat{\Theta}$$

$$\delta_{Dq}^{\text{fast}} = 48\pi G \bar{\rho}_\gamma (2 + R) (1 + R)^{3/4} (S/q)^2 \hat{\mathcal{R}}_q \cos \hat{\Theta}$$

where the effective $\mathcal{R}_q^o$ defined by

$$\hat{\mathcal{R}}_q = \mathcal{R}_q^o e^{-\int_0^t \Gamma dt}$$
now includes the damping. Actually, the results for $\psi^\text{fast}_q$ and $\delta^\text{fast}_{Dq}$ differ by a factor $\bar{\rho}_\gamma/\bar{\rho}_R$ from the result obtained earlier, because neutrinos above where treated as a perfect fluid which is not valid throughout the radiation dominated era.

Matching radiation dominated region with slow modes deep inside the horizon: in the radiation-dominated era we neglected baryon and dark matter densities in calculating the gravitational potential, while in calculating the slow modes deep inside the horizon we found that dark matter became the dominating source for the gravitational potential. Actually, the dark matter component allows for the smooth matching: The two Mészáros solutions meet the radiation-dominated era for $y \ll 1$ where $\delta^{(1)}_{Dq} \to 1$ and $\delta^{(2)}_{Dq} \to -\ln(y/4) - 3$. The linear combination which fits smoothly with $\delta^\text{slow}_{Dq}$ from the radiation-dominated era is

$$
\delta^\text{slow}_{Dq} = 6\mathcal{R}^0_q \left\{ \left[ -\frac{7}{2} + \gamma + \ln \left( \frac{4q \sqrt{2}}{\sqrt{3}H_{\text{EQ}}S_{\text{EQ}}} \right) \right] \delta^{(1)}_{Dq} - \delta^{(2)}_{Dq} \right\},
$$

where, here $t = 1/2H = y^2/\sqrt{2}H_{\text{EQ}}$. Using the slow mode DEQs for the given $\delta^\text{slow}_{Dq}$
we have $\psi_{q}^\text{slow} = -\delta_{Dq}^\text{slow}$, $\delta u_{\gamma q}^\text{slow} = -(S^2/q^2) \dot{\delta}_{Dq}^\text{slow}$ and $\delta_{\gamma q}^\text{slow} = \frac{3S}{q^2} \frac{d}{dt} \left( S (1 + R) \dot{\delta}_{Dq}^\text{slow} \right)$.

In the matter-dominated era where $y \gg 1$ ($\delta_{Dq}^{(1)} \to 3y/2$ and $\delta_{Dq}^{(2)} \to 4/15y^{3/2}$) we obtain

\[
\delta_{Dq}^\text{slow} \to \frac{9S R_{q}^o}{S_{E Q}} A(\kappa)
\]

and with

\[
A(\kappa) \equiv -\frac{7}{2} + \gamma + \ln \left( \frac{4\kappa}{\sqrt{3}} \right)
\]
Furthermore,

\[ \psi_q^{\text{slow}} \rightarrow -\frac{6S^2R_q^o}{S_{EQ}t} \mathcal{A}(\kappa) \]

\[ \delta u_{\gamma q}^{\text{slow}} \rightarrow -\frac{6S^3R_q^o}{S_{EQ}tq^2} \mathcal{A}(\kappa) \]

\[ \delta_{\gamma q}^{\text{slow}} \rightarrow \frac{6S^3(1 + 3R)R_q^o}{S_{EQ}t^2q^2} \mathcal{A}(\kappa) . \]

The rescaled dimensionless wave number is

\[ \kappa \equiv \frac{q \sqrt{2}}{S_{EQ}H_{EQ}} = \frac{(q/S_0) \sqrt{\Omega_R}}{H_0 \Omega_M} \]

where \( H_{EQ} = \sqrt{2}(H_0 \sqrt{\Omega_M)((S_0/S_{EQ})^3)^{3/2} \) and \( S_0/S_{EQ} = \Omega_M/\Omega_R \). Note, when perturbation just comes in at radiation-matter equality we have \( \sqrt{2}q/q_{EQ} \) and short wavelength considered in this section essentially means \( \kappa \gg 1 \).
The full solution up to the time of recombination is the sum of fast plus slow modes:

\[
\begin{align*}
\delta_{Dq} &= \frac{9SR^o_q}{SEQ} \mathcal{A}(\kappa) + 48\pi G \bar{\rho}_\gamma (2 + R)(1 + R)^{3/4} (S/q)^2 \hat{R}^o_q \cos \Theta \\
\psi_q &= -\frac{6SR^o_q}{SEQt} \mathcal{A}(\kappa) + 16 \sqrt{3} \pi G \bar{\rho}_\gamma (2 + R)(1 + R)^{1/4} (S/q) \hat{R}^o_q \sin \Theta \\
\delta_{u\gamma q} &= -\frac{6S^3R^o_q}{SEQtq^2} \mathcal{A}(\kappa) + \frac{S \sqrt{3}}{q (1 + R)^{3/4}} \hat{R}^o_q \sin \Theta \\
\delta_{\gamma q} &= \frac{6S^3(1 + 3R)R^o_q}{SEQt^2q^2} \mathcal{A}(\kappa) - \frac{3}{(1 + R)^{1/4}} \hat{R}^o_q \cos \Theta
\end{align*}
\]
A) Long wave length entering horizon well after matter-radiation equality
B) Short wave length entering horizon well before matter-radiation equality
C) Wave length entering the horizon abound matter-radiation equality
this dominated 1st acoustic peak around $\ell = 200$

Approximation: i) $n_B \sim 0$, ii) no damping can be solved exactly for all wave lengths
(neutrinos negligible): in matter dominated era
\[ \frac{d}{dt}\left(t^{4/3} \psi_q\right) = -\frac{2}{3} t^{-2/3} \delta_D q , \]

\[ \dot{\delta}_\gamma q - \left(q^2/S^2\right) \dot{\delta} u_{\gamma q} = -\psi_q , \]

\[ \dot{\delta}_D q = -\psi_q , \]

\[ \frac{d}{dt}\left(t^{-3/2} \delta u_{\gamma q}\right) = -\frac{1}{3} t^{-2/3} \delta_\gamma q , \]

with \( S \propto t^{2/3} \) with four independent solutions.

\[ \delta_D q = \frac{3}{2} \frac{q^2 t^2}{S^2} , \quad \psi_q = -\frac{q^2}{S^2} , \]

\[ \delta_\gamma q \equiv , \quad \delta u_{\gamma q} = -1 . \]
The fourth solution is more complicated and decays like $\delta_{Dq} \propto 1/t$ and $\psi_q \propto 1/t^2$. The solution which is satisfying the required boundary condition to the solution at earlier times is a linear combination of the first three solutions. Before matter-radiation equilibrium $S(t) \propto t^{1/2}$ different form $S(t) \propto t^{2/3}$ we just considered
for the matter dominated regime. It amounts to shift of zero time in solutions 2 and 3 by \( \Delta(\kappa) \) in the arguments of \( \sin \) and \( \cos \) below: the linear combination reads:

\[
\begin{align*}
\delta_{Dq} &= \frac{9 q^2 t^2 \mathcal{R}_q \mathcal{T}(\kappa)}{10 S^2}, \quad \psi_q = -\frac{9 q^2 t \mathcal{R}_q \mathcal{T}(\kappa)}{5 S^2}, \\
\delta_{\gamma q} &= \delta_{\nu q} = \frac{3 \mathcal{R}_q}{5} \left[ \mathcal{T}(\kappa) - \mathcal{S}(\kappa) \cos \left( q \int_0^t \frac{dt'}{\sqrt{3}S(t')} + \Delta(\kappa) \right) \right] \\
\delta_{u_{\gamma q}} &= \delta_{u_{\nu q}} = \frac{3 t \mathcal{R}_q}{5} \left[ -\mathcal{T}(\kappa) + \mathcal{S}(\kappa) \frac{S}{\sqrt{3}qt} \sin \left( q \int_0^t \frac{dt'}{\sqrt{3}S(t')} + \Delta(\kappa) \right) \right].
\end{align*}
\]
where \( S(\kappa), T(\kappa), \) and \( \Delta(\kappa) \) are dimensionless time dependent functions of \( \kappa \):

\[
\kappa \equiv \frac{q \sqrt{2}}{S_{\text{EQ}} H_{\text{EQ}}} = \left( \frac{q/S_0}{H_0 \Omega_M} \right) \sqrt{\Omega_R} = \frac{19.3 (q/S_0) [\text{Mpc}^{-1}]}{\Omega_M h^2}
\]

is the dimensionless rescaled wave number. \( S_{\text{EQ}} H_{\text{EQ}} \) denote the previously calculated \( S \)-factor and Hubble constant at matter-radiation equality. Long and short wavelength distinction made earlier in terms of \( \kappa \) reads: in matter dominated era \( t = \frac{2}{3} H = (t/3H_{\text{EQ}})(S/S_{\text{EQ}})^{3/2} \) and \( \bar{\rho}_R/\bar{\rho}_M = S_{\text{EQ}}/S \) such that

\[
\frac{t^2 q^2}{S 62 \bar{\rho}_M} = \frac{4 \kappa^2}{9}.
\]

Hence

\[
t^2 q^2 / S^2 \gg \bar{\rho}_M/\bar{\rho}_R \leftrightarrow \kappa \gg 1
\]
For $\kappa \ll 1$ we can match the solution to the one given earlier for $t^2 q^2 / S^2 \ll \bar{\rho}_M / \bar{\rho}_R$ in the approximation $\bar{\rho}_B = 0$ and matching requires

$$T(\kappa) \to 1 , \quad S(\kappa) \to 1 , \quad \Delta(\kappa) \to 2\kappa / \sqrt{3} ; \quad \kappa \to 0$$

Similarly, for $\kappa \gg 1$ we can match the solution to the one given earlier for $t^2 q^2 / S^2 \gg \bar{\rho}_M / \bar{\rho}_R$ in the approximation $\bar{\rho}_B = 0$ and $\Gamma = 0$ and matching requires

$$T(\kappa) \to \frac{45}{2\kappa^2} \left[ -\frac{7}{2} + \gamma + \ln \left( \frac{4\kappa}{\sqrt{3}} \right) \right] , \quad S(\kappa) \to 5 , \quad \Delta(\kappa) \to 0 ; \quad \kappa \to \infty$$

We need solution of (⋆) (Subsect. 2) for general $y \equiv \bar{\rho}_M / \bar{\rho}_R$ and general $\kappa$, still in the approximation $\bar{\rho}_B = 0$. Rewrite Eqs. (⋆) in terms of $y$ and in dimensionless functions $f, g, d$ and $r$:

$$\delta_{Dq} = \kappa^2 \mathcal{R}^0_q d(y) / 4 , \quad \delta_{\gamma q} = \delta_{vq} = \kappa^2 \mathcal{R}^0_q r(y) / 4 , \quad \psi_q = (\kappa^2 H_{EQ} / 4 \sqrt{2}) \mathcal{R}^0_q f(y) , \quad \delta u_{\gamma q} = \delta u_{vq} = (\kappa^2 \sqrt{2} / 2 H_{EQ}) \mathcal{R}^0_q g(y)$$
obtaining:

\[ \sqrt{1 + y \frac{d}{dy} (y^2 f(y))} = - \frac{3}{2} \frac{d(y)}{y} - \frac{4}{3} \frac{r(y)}{y} , \]

\[ \sqrt{1 + y \frac{d}{dy} d(y)} = - y f(y) , \]

\[ \sqrt{1 + y \frac{d}{dy} r(y)} = - y f(y) + \frac{\kappa^2}{y} g(y) , \]

\[ \sqrt{1 + y \frac{d}{dy} \left( \frac{g(y)}{y} \right)} = - \frac{1}{3} r(y) . \]
with initial conditions as given by Eqs. (3) (Subsect. 2):

\[
d(y) \rightarrow r(y) \rightarrow y^2, \\
f(y) \rightarrow -2, \quad g(y) \rightarrow \frac{-y^4}{9}.
\]

Numerical integration of the DEQs yields the tabulated results:
<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$S$</th>
<th>$\mathcal{T}$</th>
<th>$\Delta$</th>
<th>$\kappa$</th>
<th>$S$</th>
<th>$\mathcal{T}$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0167</td>
<td>0.9948</td>
<td>0.1207</td>
<td>3</td>
<td>2.7839</td>
<td>0.4798</td>
<td>0.5334</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0551</td>
<td>0.9780</td>
<td>0.2240</td>
<td>3.5</td>
<td>2.9473</td>
<td>0.4311</td>
<td>0.5094</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1147</td>
<td>0.9569</td>
<td>0.3156</td>
<td>4</td>
<td>3.0970</td>
<td>0.3898</td>
<td>0.4854</td>
</tr>
<tr>
<td>0.4</td>
<td>1.1891</td>
<td>0.9339</td>
<td>0.3852</td>
<td>4.5</td>
<td>3.2346</td>
<td>0.3545</td>
<td>0.4659</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2680</td>
<td>0.9101</td>
<td>0.4423</td>
<td>5</td>
<td>3.3506</td>
<td>0.3241</td>
<td>0.4509</td>
</tr>
<tr>
<td>0.6</td>
<td>1.3529</td>
<td>0.8860</td>
<td>0.4800</td>
<td>5.5</td>
<td>3.4114</td>
<td>0.2976</td>
<td>0.4367</td>
</tr>
<tr>
<td>0.7</td>
<td>1.4388</td>
<td>0.8620</td>
<td>0.5148</td>
<td>6</td>
<td>3.5181</td>
<td>0.2726</td>
<td>0.4203</td>
</tr>
<tr>
<td>0.8</td>
<td>1.5195</td>
<td>0.8384</td>
<td>0.5336</td>
<td>6.5</td>
<td>3.5953</td>
<td>0.2531</td>
<td>0.4029</td>
</tr>
<tr>
<td>0.9</td>
<td>1.6081</td>
<td>0.8154</td>
<td>0.5531</td>
<td>7</td>
<td>3.6754</td>
<td>0.2361</td>
<td>0.3884</td>
</tr>
<tr>
<td>1.</td>
<td>1.6801</td>
<td>0.7930</td>
<td>0.5637</td>
<td>7.5</td>
<td>3.7473</td>
<td>0.2200</td>
<td>0.3782</td>
</tr>
<tr>
<td>1.2</td>
<td>1.8330</td>
<td>0.7502</td>
<td>0.5784</td>
<td>8</td>
<td>3.8015</td>
<td>0.2056</td>
<td>0.3695</td>
</tr>
<tr>
<td>1.4</td>
<td>1.9777</td>
<td>0.7104</td>
<td>0.5854</td>
<td>8.5</td>
<td>3.8432</td>
<td>0.1927</td>
<td>0.3590</td>
</tr>
<tr>
<td>1.6</td>
<td>2.1126</td>
<td>0.6734</td>
<td>0.5842</td>
<td>9</td>
<td>3.8865</td>
<td>0.1810</td>
<td>0.3465</td>
</tr>
<tr>
<td>1.8</td>
<td>2.2354</td>
<td>0.6391</td>
<td>0.5782</td>
<td>9.5</td>
<td>3.9380</td>
<td>0.1704</td>
<td>0.3350</td>
</tr>
<tr>
<td>2.</td>
<td>2.3451</td>
<td>0.6074</td>
<td>0.5700</td>
<td>10</td>
<td>3.9895</td>
<td>0.1608</td>
<td>0.3270</td>
</tr>
<tr>
<td>2.5</td>
<td>2.5895</td>
<td>0.5378</td>
<td>0.5537</td>
<td>11</td>
<td>4.0546</td>
<td>0.1440</td>
<td>0.3147</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$S$</td>
<td>$T$</td>
<td>$\Delta$</td>
<td>$\kappa$</td>
<td>$S$</td>
<td>$T$</td>
<td>$\Delta$</td>
</tr>
<tr>
<td>---------</td>
<td>--------</td>
<td>--------</td>
<td>----------</td>
<td>---------</td>
<td>--------</td>
<td>--------</td>
<td>----------</td>
</tr>
<tr>
<td>12</td>
<td>4.1172</td>
<td>0.1298</td>
<td>0.2962</td>
<td>55</td>
<td>4.7794</td>
<td>0.0153</td>
<td>0.1111</td>
</tr>
<tr>
<td>13</td>
<td>4.1841</td>
<td>0.1178</td>
<td>0.2850</td>
<td>60</td>
<td>4.7992</td>
<td>0.0134</td>
<td>0.1053</td>
</tr>
<tr>
<td>14</td>
<td>4.2175</td>
<td>0.1075</td>
<td>0.2747</td>
<td>65</td>
<td>4.8192</td>
<td>0.0118</td>
<td>0.0997</td>
</tr>
<tr>
<td>15</td>
<td>4.2676</td>
<td>0.0985</td>
<td>0.2604</td>
<td>70</td>
<td>4.8365</td>
<td>0.0105</td>
<td>0.0940</td>
</tr>
<tr>
<td>16</td>
<td>4.3135</td>
<td>0.0907</td>
<td>0.2541</td>
<td>75</td>
<td>4.8487</td>
<td>0.0094</td>
<td>0.0885</td>
</tr>
<tr>
<td>17</td>
<td>4.3336</td>
<td>0.0838</td>
<td>0.2438</td>
<td>80</td>
<td>4.8563</td>
<td>0.0084</td>
<td>0.0838</td>
</tr>
<tr>
<td>18</td>
<td>4.3796</td>
<td>0.0777</td>
<td>0.2339</td>
<td>85</td>
<td>4.8622</td>
<td>0.0077</td>
<td>0.0803</td>
</tr>
<tr>
<td>19</td>
<td>4.4043</td>
<td>0.0723</td>
<td>0.2296</td>
<td>90</td>
<td>4.8695</td>
<td>0.0070</td>
<td>0.0776</td>
</tr>
<tr>
<td>20</td>
<td>4.4233</td>
<td>0.0675</td>
<td>0.2195</td>
<td>95</td>
<td>4.8792</td>
<td>0.0064</td>
<td>0.0751</td>
</tr>
<tr>
<td>25</td>
<td>4.5271</td>
<td>0.0496</td>
<td>0.1920</td>
<td>100</td>
<td>4.8895</td>
<td>0.0059</td>
<td>0.0722</td>
</tr>
<tr>
<td>30</td>
<td>4.6051</td>
<td>0.0383</td>
<td>0.1713</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>4.6650</td>
<td>0.0305</td>
<td>0.1542</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>4.7087</td>
<td>0.0249</td>
<td>0.1396</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>4.7389</td>
<td>0.0209</td>
<td>0.1276</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>4.7605</td>
<td>0.0177</td>
<td>0.1182</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In fact in our approximation $\mathcal{T}$ is about 4% too low, because of the neglect of the neutrino anisotropic inertia. (Eisenstein & Hu Astrophys. J. 496 (1998) 605)
Two effects have yet to be taken into account:

- finite $\bar{\rho}_B$ i.e. finite $R$, $\delta_{Dq}$ and $\psi_q$ are not affected. The corrections in $\delta_{\gamma q}$ are to multiply by $1 + 3R$ the non-oscillatory term, by $(1 + R)^{-1/4}$ the cosine. For $\delta u_{\gamma q}$ the factor is $(1 + R)^{-3/4}$ for the sine. In the integrals $S(t) \sqrt{3} \rightarrow S(t) \sqrt{3(1 + R)}$,

- damping, i.e. finite $\Gamma$ which leads to $\exp(-\int_0^t \Gamma dt)$ factors of $\sin$ and $\cos$ in expressions for $\delta u_{\gamma q}$ and $\delta_{\gamma q}$

Explicit expressions follow below.
Weinberg’s semi analytic result

Using $\delta T / \bar{T} = \delta_{\gamma q} / 4 \bar{\rho}_\gamma = \delta_{\gamma q} / 3$, the form factors are given by

$$F(q) = \frac{1}{3} \delta_{\gamma q}(t_L) + \frac{S^2(t_L) \psi_q(t_L)}{3 q^2 t_L}$$
$$G(q) = -q \delta u_{\gamma q}(t_L) / S(t_L) + S(t_L) \psi_q / q$$

where
\[
\begin{align*}
\delta D_q(t_L) &= \frac{9q^2 t_L^2 R_q^o \mathcal{T}(\kappa)}{10S_L^2}, \\
\psi_q(t_L) &= \frac{-3q^2 t_L R_q^o \mathcal{T}(\kappa)}{5S_L^2}, \\
\delta \gamma_q(t_L) &= \frac{3 R_q^o}{5} \left[ \mathcal{T}(\kappa) (1 + 3 R_L) - (1 + R_L)^{-1/4} e^{-\int_0^{t_L} \Gamma dt} S(\kappa) \right. \\
&\quad \times \cos \left( \int_0^{t_L} \frac{q \, dt}{S(t) \sqrt{3 (1 + R(t))}} + \Delta(\kappa) \right), \\
\delta u_{\gamma q}(t_L) &= \frac{3 R_q^o}{5} \left[ -t_L \mathcal{T}(\kappa) + \frac{a_L}{\sqrt{3} q} (1 + R_L)^{-3/4} e^{-\int_0^{t_L} \Gamma dt} S(\kappa) \right. \\
&\quad \times \sin \left( \int_0^{t_L} \frac{q \, dt}{S(t) \sqrt{3 (1 + R(t))}} + \Delta(\kappa) \right). 
\end{align*}
\]
The analysis so far assumed a sharp time of last scattering. In reality we have a folding with the probability distribution for last scattering. The probability that last scattering has happened between \( t \) and \( t + dt \) is:

\[
F(q) = \frac{\mathcal{R}_q^0}{5} \left[ 3 \mathcal{T}(\kappa) R_L - (1 + R_L)^{-1/4} e^{-\int_0^{t_L} \Gamma dt} S(\kappa) \right. \\
\times \cos \left( \int_0^{t_L} \frac{q dt}{S(t) \sqrt{3} (1 + R(t))} + \Delta(\kappa) \right) \right] ,
\]

\[
G(q) = -\frac{\sqrt{3} \mathcal{R}_q^0}{5} (1 + R_L)^{-3/4} e^{-\int_0^{t_L} \Gamma dt} S(\kappa) \\
\times \sin \left( \int_0^{t_L} \frac{q dt}{S(t) \sqrt{3} (1 + R(t))} + \Delta(\kappa) \right) \right] .
\]
\[
P(t) \, dt = \frac{\exp\left[-(t-t_L)^2/2\sigma_t^2\right]}{\sigma_t \sqrt{2\pi}} \, dt
\]

\[
\cos[\sin]\left(\int_0^{t_L} \omega \, dt + \Delta\right) \rightarrow \int_{-\infty}^{+\infty} P(t) \, dt \, \cos[\sin]\left(\int_0^t \omega \, dt + \Delta\right)
\]

with \(\omega = q/S \sqrt{3(1+R)}\). The distribution \(P(t)\) is sharply peaked and one can expand the arguments of \(\cos[\sin]\) to first order in \(t - t_L\):

\[
\int_0^t \omega \, dt \simeq \int_0^{t_L} \omega \, dt + \omega_L(t - t_L)
\]

and the remaining integrals can be done easily

\[
\int_{-\infty}^{+\infty} P(t) \, dt \, \cos[\sin]\left(\int_0^t \omega \, dt + \Delta\right) \simeq \exp\left(-\omega_L^2 \sigma_t^2/2\right) \times \cos[\sin]\left(\int_0^t \omega \, dt + \Delta\right)
\]
Cosmology

The whole effects of the finite spread of the last scattering distribution is thus an additional damping of the fast oscillations contribution:

\[ \int_{0}^{t_L} \Gamma \, dt \rightarrow \int_{0}^{t_L} \Gamma \, dt + \omega_L^2 \sigma_i^2 / 2 = q^2 d_D^2 / S_L^2 , \]

where \( d_D \) is the damping length. The first term is the Silk damping discussed above, the second the Landau damping:

\[ d_D^2 = d_{\text{Silk}}^2 + d_{\text{Landau}}^2 , \]

\[ d_{\text{Silk}}^2 = S_L^2 \int_{0}^{t_L} \frac{t_\gamma}{6 S^2 (1 + R)} \left[ \frac{16}{15} + \frac{R^2}{(1 + R)} \right] \, dt , \]

\[ d_{\text{Landau}}^2 = \frac{\sigma_i^2}{6 (1 + R_L)} , \]

where \( t_\gamma = 1 / \omega_c \) the photon mean free time and \( R = 3 \bar{\rho}_B / 4 \bar{\rho} \omega_\gamma \).

Calculation of the Silk damping:
As \( R \propto S \)

\[
\tau_\gamma = \frac{1}{n_e \sigma_T c} = \frac{R^3}{n_B R_0^3 (1 - Y) X \sigma_T c}
\]

- \( R_0 \) present value of \( R = 3\Omega_B/4\Omega_\gamma \)
- \( Y \approx 0.24 \) fraction of nucleons in form of un-ionized helium around time of last scattering
- \( n_{B0} = 3H_0^2\Omega_B/8\pi G m_N \) present number density of baryons
- \( X = X(R) \) fractional ionization, calculated in Lect. 10 Subsect. [Recombination and last scattering].

Using

\[
dt = \frac{dR}{RH_0 \sqrt{\Omega_M(R_0/R)^3 + \Omega_R(R_0/R)^4}} = \frac{R \, dR}{H_0 \sqrt{\Omega_M} R_0^{3/2} \sqrt{R_{EQ} + R}}
\]

with
- \( R_{EQ} \equiv \Omega_R R_0/\Omega_M = 3\Omega_R \Omega_B/3\Omega_M \Omega_\gamma \) the value of \( R \) at matter-radiation equality.
The Silk damping length is then

\[
    d_{\text{Silk}}^2 = \frac{R_L^2}{6 \left(1 - Y\right) n_{B0} \sigma_T c H_0 \sqrt{\Omega_M} R_0^{9/2}} \times \int_0^{R_L} \frac{R^2 \, dR}{X(R) (1 + R) \sqrt{R_{\text{EQ}} + R}} \left[ \frac{16}{15} + \frac{R^2}{(1 + R)} \right].
\]

The standard deviation \( \sigma_t \) at time of last scattering is related to the standard deviation \( \sigma \) in the temperature at last scattering, calculated in Lect. 10, by \( \sigma_t = 3t_L \sigma / 2T_L \), such that

\[
    d_{\text{Landau}}^2 = \frac{3\sigma^2 t_L^2}{8 T_L^2 (1 + R_L)}. \]
\[ F(q) = \frac{\mathcal{R}_q}{5} \left[ 3 \mathcal{T} \left( qd_T / S_L \right) R_L - (1 + R_L)^{-1/4} S(qd_T / S_L) \times e^{-q^2d_T^2/S_L^2} \cos \left( qd_T / S_L + \Delta(qd_T / S_L) \right) \right] , \]

\[ G(q) = -\frac{\sqrt{3} \mathcal{R}_q}{5} (1 + R_L)^{-3/4} S(qd_T / S_L) \times e^{-q^2d_T^2/S_L^2} \sin \left( qd_T / S_L + \Delta(qd_T / S_L) \right) \right] . \]

Here \( d_T \) is defined by \( \kappa \equiv qd_T / S_L \) such that

\[
d_T = \frac{\sqrt{\Omega_R}}{(1 + z_L) H_0 \Omega_M} = \frac{0.0177}{\Omega_M h^2} \text{ Mpc} .
\]
The acoustic horizon distance at last scattering is

\[ d_H \equiv S_L \int_0^{t_L} \frac{dt}{S \sqrt{3 (1 + R)}} = \frac{2}{H_0 \sqrt{3 R_L \Omega_M} (1 + z_L)^{3/2}} \ln \left( \frac{\sqrt{1 + R_L} + \sqrt{R_{EQ} + R_L}}{1 + \sqrt{R_{EQ}}} \right) \]

with \( R_L = 3 \Omega_B / 4 \Omega_\gamma (1 + z_L) \).
What happened after last scattering: effects of re-ionization

As a last important correction we have to consider the reionization of neutral Hydrogen by ultraviolet light from the first generation of massive stars. Remember, after recombination and last scattering the relict baryonic matter finds itself in neutral atoms, about 75% in Hydrogen and about 25% in Helium. This normal neutral matter found itself as clouds of gas which started to collapse into stars as a result of gravitational instabilities and the existing fluctuations.

The characteristic parameter is the optical depth $\tau_{\text{reion}}$ in the reionized plasma. Reionization happened at $z_{\text{reion}} \sim 10$. It affects the temperature anisotropy $\Delta T$ in two ways:

- Photons from the CMB have a small probability $1 - \exp(-\tau_{\text{reion}})$ to scatter with the free electrons which result from the re-ionization process. The scattering takes place at distances much smaller ($z_{\text{reion}} \sim 10$) than the radius of last scattering
Therefore it affects low $\ell$ only, where cosmic variance obscures the interpretation of observations in any case. We thus neglect this effect.

The probability of no scattering is $\exp(-\tau_{\text{reion}})$. For the two-point correlation it just means to multiply what we have calculated for $C_{TT,\ell}^S$ by the factor $\exp(-2 \tau_{\text{reion}})$.

The reionization effect causes a problem because it means that the over-all factor represented by the primordial fluctuations gets simply rescaled:

$$|R_q^o|^2 \rightarrow |R_q^o|^2 \exp(-2 \tau_{\text{reion}})$$

and the two effects of very different origin cannot be disentangled easily. One possibility is to fix the reionization factor form measurements of the polarization in the CMB (sketched in Lect. 11 Subsect. [Polarization]). First results suggest $\exp(-2 \tau_{\text{reion}}) \approx 0.8$, and we will adopt this.

The conventional parametrization, motivated by a phenomenological analysis of
matter density two-point correlations (see Lect. 12 Subsect. [2]), suggests

\[ |\mathcal{R}_q^o|^2 = N^2 q^{-3} \left( \frac{q/S_0}{k_R} \right)^{n_S-1} \]

where \( k_R \) is an arbitrary reference scale usually taken as \( k_R = 0.05 \text{MPC}^{-1} \). The exponent \( n_S \) is expected to depend on the wave number \( k \), it will turn out that actually \( n_S \) is not far from 1. The above parametrization lead to

\[ \left( \frac{\ell}{r_L} \right)^3 |\mathcal{R}^o_{\beta \ell/r_L}|^2 = \frac{N^2}{\beta^3} \left( \frac{\ell \beta}{k_R d_A (1 + z_L)} \right)^{n_S-1} \]

and the only “free” normalization of \( \mathcal{R}_q^o \) is \( N^2 k_R^{(1-n_S)} \).
Altogether

The elaborate analytic analysis of different regimes after gluing them together leads to the following result, valid for reasonably large $\ell$ (say $\ell > 20$, where we can ignore cosmic variance and the integrated Sachs-Wolfe effects, use the large $\ell$ approximation and treat the effect of reionization as a simple factor $\exp(-2\tau_{\text{reion}})$ we obtain the leading scalar contribution to the multipole coefficient as
\[
\frac{\ell (\ell + 1) C_{TT, \ell}^S}{2\pi} = \frac{4\pi T_0^2 N^2 e^{-2\tau_{\text{reion}}}}{25} \int_1^\infty d\beta \left( \frac{\beta \ell}{\ell_R} \right)^{n_S - 1} \times \left\{ \frac{1}{\beta^2 \sqrt{\beta^2 - 1}} \left[ 3T (\beta \ell / \ell_T) R_L \right. \right.
\]
\[
- (1 + R_L)^{-1/4} S(\beta \ell / \ell_T) e^{-\beta^2 \ell^2 / \ell_D^2} \cos (\beta \ell / \ell_H + \Delta(\beta \ell / \ell_T)) \left. \right] \right. 
\]
\[
+ \frac{3 \sqrt{\beta^2 - 1}}{\beta^4 (1 + R_L)^{3/2}} e^{-2\beta^2 \ell^2 / \ell_D^2} S^2(\beta \ell / \ell_T) \sin^2 (\beta \ell / \ell_H + \Delta(\beta \ell / \ell_T)) \right\}
\]

where

\[
\ell_D \equiv d_A / d_D , \quad \ell_T \equiv d_A / d_T , \quad \ell_H = d_A / d_H , \quad \ell_R = (1 + z_L) k_R d_A
\]
with $d_A$ the angular diameter of the surface of last scattering: for a flat universe $k = 0$

$$d_A \equiv r_L S(t_L) = \frac{c}{H_0 (1 + z_L)} \int_{1/(1+z_L)}^{1} \frac{dx}{\sqrt{\Omega_\Lambda x^4 + \Omega_K x^2 + \Omega_M x + \Omega_R}}$$

When $k \neq 0$ we have to take $\mathcal{S}[y]$ defined earlier in place of $y$ on the r.h.s. Here we neglect $\Omega_R$ and $\Omega_K = 1 - \Omega_\Lambda - \Omega_M$. If we assume that the integral over $\beta$ is dominated by values $\beta \approx 1$, neglect the Doppler term and for the moment neglect the term proportional to the transfer function $\mathcal{T}$ the result shows that $C_{TT,\ell}^S$ has

- peaks at $\chi(\ell) = \pi, 2\pi, 3\pi$, etc. where $\chi(\ell = \ell / \ell_H + \Delta(\ell / \ell_T))$ is the phase of the cosine for $\beta = 1$.

- the presence of the positive term $3\mathcal{T} R_L$ arising from the inertia of the baryonic plasma enhances and slightly shifts the peaks for $\chi = \pi, 3\pi$, etc. where the cosine is negative and reduces the peaks for $\chi = 2\pi, 4\pi$, etc. where the cosine is positive.
for large $\ell$ the Silk and Landau damping factor $\exp(-\beta^2 \ell^2 / d_D^2)$ damps out the peaks

This pattern obtained with the formula above yields a fairly good approximation the the exact result from numerical simulations.
7 Reading off the cosmological parameters

Input: $T_0$ which fixes $\Omega_\gamma h^2$ and $\Omega_R h^2 = \Omega_\gamma h^2 + \Omega_\nu h^2$, and also $t_L$, $z_L$ and $\sigma$ which depend weakly on cosmological parameters, we find:

- $R_L \propto \Omega_B h^2$
- $d_{\text{Silk}}^2 \propto (\Omega_B h^2)^{-7/2}(\Omega_M h^2)^{-1/2}$ and it is a complicated but not very sensitive function of $\Omega_B h^2$ and $\Omega_M h^2$.
- $d_{\text{Landau}}^2$ depends on $\Omega_B h^2$ through a factor $(1 + R_L)^{-1}$
- $d_T \propto (\Omega_M h^2)^{-1}$
- $d_H \propto (\Omega_B h^2)^{-1/2}(\Omega_M h^2)^{-1/2}$ modulo a slowly varying log
- $d_A$ only depends on $H_0$, $\Omega_\Lambda$, $\Omega_M$, $\Omega_B$ and $\Omega_K$, apart from a dependence on $\Omega_B h^2$ and $\Omega_M h^2$

Remark on this last statement: the effect of observationally allowed values of $\Omega_\Lambda$ or $\Omega_K$ would be quite negligible before recombination. So their influence on $C_{TT,\ell}^S$
is limited to their influence on the propagation of light after recombination, that is, on $d_A$

**Numbers:**

\[
\begin{align*}
\Omega_M h^2 &= 0.13299, \quad \Omega_B h^2 = 0.02238, \quad h = 0.71992, \quad \Omega_\Lambda = 1 - \Omega_M \\
n_s &= 0.95820, \quad k_R = 0.05 \text{ Mpc}^{-1}, \quad N^2 = 1.736 \times 10^{-10}, \quad e^{-2\tau_{\text{reion}}} = 0.80209
\end{align*}
\]

With $T_0 = 2.725 \, ^\circ\text{K}$ we get $\Omega_\gamma h^2 = 2.47 \times 10^{-5}$ and with 3 flavor of massless neutrinos $\Omega_R h^2 = 1.681 \, \Omega_\gamma h^2 = 4.15 \times 10^{-5}$

Recombination parameters: for $\Omega_B h^2 = 0.02$ and $\Omega_M h^2 = 0.15$ we have

\[
1 + z_L = 1090, \quad \sigma = 262 \, ^\circ\text{K}, \quad t_L = 370,000 \, \text{yrs}.
\]
Furthermore, for the given values of $\Omega_M h^2$ and $\Omega_B h^2$ we have

\begin{align*}
R_0 &= 679.6, \quad R_L = 0.6234, \quad R_{EQ} = 0.2121 \\
\ell_T &= 97.60, \quad \ell_H = 96.15, \quad \ell_D = 1598, \quad \ell_R = 708.
\end{align*}
The scalar multipole coefficient \( \ell (\ell + 1) C_\ell^S / 2\pi \) in square microKelvin, vs. \( \ell \), for the cosmological parameters given in above. Weinberg’s formula, the hydrodynamic approximation, is indicated by the dashed curve, while for comparison the solid curve gives the more accurate large scale computer calculation by CAMB.
Comparison of theory and WMAP observations for the multipole coefficient $\ell (\ell + 1) C_\ell^S / 2\pi$ in square microKelvin vs. $\ell$, from Komatsu, E., et.al., 2011, ApJS, 192, 18. Shown are the seven-year WMAP data together with data from CBI and ACBAR. Fits indicate the sensitivity to the primordial $^4\text{He}/\text{H}$ ratio $Y_p$. 
Cosmology

WMAP “Wilkinson Microwave Anisotropy Probe” link

Next generation experiment started: “The Planck Mission”
Update: the results from ESA’s Planck satellite

reproduced from: †, see also: ‡
The Planck CMB fluctuation pattern: imprinted on the sky when the universe was just 380 000 years (after B.B.) old.
The Planck power spectrum: (the acoustic peaks)
The Planck cosmic parameters update:

The Planck data set a new value for the rate at which the Universe is expanding today, known as the Hubble constant:
$H_0 = 67.15 \pm 3.6 \text{ (km/s)/Mpc}$
This is significantly less than the current standard value $H_0 = 74.2 \pm 3.6 \text{ (km/s)/Mpc}$ in astronomy.

The data imply that the age of the Universe is 13.82 billion years, in place of 13.7 billion years.
Planck anomaly?

“When compared to the best fit of observations to the standard model of cosmology, Planck’s high-precision capabilities reveal that the fluctuations in the cosmic microwave background at large scales are not as strong as expected. The graphic shows a map derived from the difference between the two, which is representative of what the anomalies could look like.”
8 Full calculations with CMBFAST or CAMB

A reliable calculation of perturbations about a RW-metric cosmology requires a treatment within general relativity and Boltzmann transport theory for a proper account of the era of recombination. For State of the Art calculations the following computer codes are available:

- **CMBfast** by U. Seljak & M. Zaldarriaga. *

  CMBFAST Web Interface Form: *

- **CAMB** Code for Anisotropies in the Microwave Background by A. Lewis & A. Challinor *

  [This code (fortran 90) is based on CMBFAST (fortran 77)]

**Overall Aims**

1) To remove the sources of distortion ➞ hear pure sound
2) To listen to times before (and after) the CMB.
To achieve this we need to calculate the early universe

**Early Simplicity**

Surprisingly, the early universe was much simpler than it is today, for several reasons:

1) Everything was spread out uniformly.
2) Atoms & light were in thermodynamic equilibrium, which is a particularly simple physical state.
3) Changes due to expansion are also simple.

For this reason, it is possible to calculate quite accurately most of the properties of the young Universe.

Doing this results in a “Computer Simulation”
Huge effort to simulate the young Universe over past 20yrs

Most well-known simulation is called “CMBFAST”, written mainly by Uros Seljak & Mattias Zaldarriaga

The program follows the evolution of four coupled fluids:
dark matter & neutrinos (collisionless components)
protons/electrons & photons (collisional components)

Input some cosmological parameters, including an initial sound spectrum & follow its evolution, to give:

\( P(k, t) \) the time evolving sound/pressure spectrum
\( k \) is a 3-D spatial frequency, not projected angular freq: \( \ell \)

The various distortions can then be “added” \( C_\ell \)
which is the “observed” CMB patches sound spectrum
Evolution of \( \rho_D \) and \( \rho_B \) in \( \Lambda \)CDM B.B. cosmology. Between entering the horizon and decoupling, the dark matter perturbation (dashed line) grows logarithmically, before the growth accelerates in matter domination. On the other hand, between entering the horizon and decoupling, the perturbation in the baryon-photon fluid (solid line) oscillates rapidly. After decoupling, it grows rapidly to match the dominant matter perturbation, the dark matter mode.
Further reading: S. Weinberg, Cosmology
Epilogue: the role particle physics in cosmology versus laboratory/accelerator particle physics

- In hot Big Bang cosmology looking back in time, in the early universe, all atoms where ionized, all nuclei disintegrated and even hadrons appear ’dissolved” into a quark gluon plasma: this was the era of elementary particle physics, with quarks, leptons, neutrinos and the gauge bosons: gluons, weak gauge bosons and the photons (plus all their antiparticles).

- One can check that conditions were such that the expansion rate, governed by gravitational effects, was always slow in comparison to elementary particle reaction rates, such that all particle species were in thermal equilibrium. Furthermore densities, and pressure were such that chemical potentials were negligible. This implies that elementary particle reactions at given temperatures were as seen in accelerator experiments at the equivalent energy.

- This implies that we can follow back in time the evolution of the universe as far...
as we understand the physics of elementary processes.

- However, this does not mean that we could proof from cosmological observations what kind of Lagrangian governs elementary particle interactions.

- On step further: we will not learn from any whatever detailed investigation of cosmological observations what is the theory beyond the Standard Model, extending the SM to higher energies.

- On the other hand cosmology very well can tell us physics beyond the SM, like dark matter, B violation etc. But it does not tell us what specific extensions of the SM we are confronted with.

- In other words, what kind of dark matter we “see” mediated by gravity very likely only laboratory experiments at the high energy frontier can tell us.

- In some cases, like in Big Bang nucleosynthesis, where the baryon matter density can be determined from the observed light element abundances as well as
from the fluctuation pattern in the cosmic microwave background it is possible to draw pretty detailed conclusions on the number of light neutrinos and the neutron life time which affect the $^4\text{He}/\text{H}$ ratio predictions sensitively.

- In spite of the deep interrelation of cosmology and particle physics ("cosmic bridge") the two are fields have a different primary focus and are based on very different methods and provide answers to different questions.
COSMOLOGY

A Schematic Outline of the Cosmic History

- The Big Bang
  - The Universe filled with ionized gas
- The Universe becomes neutral and opaque
  - The Dark Ages start
- Galaxies and Quasars begin to form
  - The Reionization starts
- The Cosmic Renaissance
  - The Dark Ages end
- Reionization complete, the Universe becomes transparent again
- Galaxies evolve
- The Solar System forms
- Today: Astronomers figure it all out!

Time since the Big Bang (years)
- ~ 300 thousand
- ~ 500 million
- ~ 1 billion
- ~ 9 billion
- ~ 13 billion

S.G. Djorgovski et al. & Digital Media Center, Caltech
Dark Ages after last scattering (epoch of recombination):

between 150 million to 1 billion years reionization of atoms takes place. The first stars and quasars form from gravitational collapse. The intense radiation they emit reionizes the intergalactic gas (hydrogen) of the surrounding universe. From this point on, most of the universe is composed of plasma. Stars cannot be seen until reionization becomes negligible due to ongoing expansion and decreasing matter density (dilution).
The first billion years

What is the Reionization Era?
A Schematic Outline of the Cosmic History

Recombination: hydrogen becomes neutral
Thermal decoupling: hydrogen cools
  - Simple physics
  - Great for cosmology!
Structures form: hydrogen heated
  - First stars/galaxies
  - “Astrophysics” begins
Reionization: hydrogen ionized
  - Universe becomes transparent to UV light
Only neutral hydrogen in dense clumps

As far as we can see.
Previous, this was the last lecture.