CMB II: Primary fluctuations in the CMB

This lecture follows closely S. Weinberg's book Cosmology: Sects. 2.6, some essentials of Sects. 5, 6 and 7.

Part I: A first look

❶ Primary fluctuations and filtering of them

❷ The Sachs-Wolfe effect

❸ Evaluation of perturbations in general

❹ The role of horizons

❺ Beyond the Sachs-Wolfe effect

❶ Primary fluctuations and filtering of them

Primary fluctuations are the structures which we see in the CMB, which were imprinted at the times of last scattering, and which originate in the early universe. So to say a snapshot of the universe at the moment when it became transparent. Later in the history secondary effects got superimposed like the Sunyaev-Zel'dovich effect as well as the local motion of the earth the sun the galaxy etc. [see also Lect: 11,5]

Here we denote by $T(\hat{n})$ the primary component of the temperature field, observed in direction $\hat{n}$ (unit length vector). The fluctuations field observed as a projection on the sphere (of last scattering) conveniently is expanded in terms of spherical harmonics

$$\boxed{\Delta T(\hat{n}) = T(\hat{n}) - T_0 = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}) \;, \quad T_0 = \frac{1}{4\pi} \int \mathrm{d}^2\,\hat{n}\, T(\hat{n}) \;.}$$

T_0 is the present mean value of the CMB temperature ($\ell = 0, 1, 2, \cdots \;, \quad m = -\ell, \cdots \ell$). As T is real the $a_{\ell m}$ must satisfy the reality condition (note $Y^*_{\ell\,m}(\hat{n}) = Y_{\ell\,-m}(\hat{n})$)

$$\boxed{a^*_{\ell\,m} = a_{\ell\,-m}}$$

The quantities of cosmological interest are related to averages over all directions

$$\boxed{\langle\cdots\rangle = \frac{1}{4\pi}\int \mathrm{d}^2\hat{n}\cdots}$$

by definition

$$\langle \Delta T(\hat{n})\rangle = 0$$

The simplest non-trivial quantity is the two-point correlation

$$\boxed{\langle \delta T(\hat{n})\Delta T(\hat{n}')\rangle = \sum_{\ell m} C_\ell\, Y_{\ell\,m}(\hat{n}) Y_{\ell\,-m}(\hat{n}') = \sum_\ell c_\ell \left(\frac{2\ell+1}{4\pi}\right) P_\ell(\hat{n}\cdot\hat{n}')\ ,}$$

where P_ℓ are the Legendre polynomial. Using the orthogonality of the latter we can obtain

$$\boxed{C_\ell = \frac{1}{4\pi}\int \mathrm{d}^2\hat{n}\,\mathrm{d}^2\hat{n}'\, P_\ell(\hat{n}\cdot\hat{n}')\,\langle \Delta T(\hat{n})\Delta T(\hat{n}')\rangle\ .}$$

also given by $\langle a_{\ell m} a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\,\delta_{mm'}\,C_\ell$, which demonstrates that C_ℓ is real positive.

In practice we cannot average over different view positions of the CMB. What we observe is a quantity averaged over m but not position:

$$\boxed{C_\ell^{\mathrm{obs}} \equiv \tfrac{1}{2\ell+1} \sum_m a_{\ell m} a*_{\ell m} = \tfrac{1}{4\pi} \int \mathrm{d}^2\hat{n}\, \mathrm{d}^2\hat{n}'\, P_\ell(\hat{n}\cdot\hat{n}')\, \Delta T(\hat{n}) \Delta T(\hat{n}')}$$

In the Gaussian case, multi-point correlations are all determined by products of two-point correlations, as

$$\begin{aligned} \langle a_{\ell\, m} a_{\ell,-m} a_{\ell\, m'} a_{\ell\, -m'} \rangle &= \langle a_{\ell\, m} a_{\ell\, -m} \rangle \langle a_{\ell\, m'} a_{\ell\, -m'} \rangle \\ &+ \langle a_{\ell\, m} a_{\ell\, m'} \rangle \langle a_{\ell\, -m} a_{\ell\, -m'} \rangle \\ &+ \langle a_{\ell\, m} a_{\ell\, -m'} \rangle \langle a_{\ell\, -m} a_{\ell\, m'} \rangle \end{aligned}$$

and the fractional difference between the cosmological interesting C_ℓ and the observed C_ℓ^{obs}, the cosmic variance, given by

$$\left\langle \left(\frac{C_\ell - C_\ell^{\mathrm{obs}}}{C_\ell} \right)^2 \right\rangle = 1 - 2 + \frac{1}{(2\ell+1)^2\, C_\ell^2} \sum_{mm'} \langle a_{\ell\, m} a_{\ell,-m} a_{\ell\, m'} a_{\ell\, -m'} \rangle$$

may be calculated to be

$$\left\langle \left(\frac{C_\ell - C_\ell^{\rm obs}}{C_\ell} \right)^2 \right\rangle = \frac{2}{2\ell + 1}$$

while the off diagonal terms

$$\left\langle \left(\frac{C_\ell - C_\ell^{\rm obs}}{C_\ell} \right) \left(\frac{C_{\ell'} - C_{\ell'}^{\rm obs}}{C_{\ell'}} \right) \right\rangle = 0 \quad (\ell \neq \ell')$$

This proves that except for small ℓ actually $C_\ell^{\rm obs}$ is an accurate measure for C_ℓ, but limited by the cosmic variance. For different values of ℓ they are uncorrelated. This means that when $C_\ell^{\rm obs}$ is measured for all ℓ in some range $\Delta\ell$ in which C_ℓ varies little, the uncertainty due to the cosmic variance in C_ℓ obtained in this range is reduced by $1/\sqrt{\Delta\ell}$. Actually, measurements of C_ℓ for $\ell < 5$ likely tell us little about cosmology. On the other hand, foreground effects like the Sunayev-Zel'dovich effect are corrupting measurements for large $\ell > 2\,000$.

Fortunately, there is lots of structure in the range $5 < \ell < 2\,000$ that provides invaluable cosmological information.

Sources of primary anisotropies in CMB:

❖ 1. Intrinsic temperature fluctuations in the electron-nucleon-photon system at the time of last scattering at a redshift of about 1090.

❖ 2. The Doppler effect due to velocity fluctuations in the plasma at last scattering.

❖ 3. The gravitational redshift or blueshift due to fluctuations in the gravitational potential at last scattering: the Sachs-Wolfe effect.

❖ 4. Gravitational redshift or blueshift due to time-dependent fluctuations in the gravitational potential between the time of last scattering and the present: the so called integrated Sachs-Wolfe effect

A proper treatment of these effects require the theory of perturbations within general relativity. Here we take advantage of the fact that from the time when temperature dropped below about $10^4\ {}^\circ\mathrm{K}$ until the cosmological constant term became important at a redshift of order 1, the gravitational field was dominated by clod dark matter, which can be treated by Newtonian theory. This is possible for the Sachs-Wolfe and the integrated Sachs-Wolfe effects, which turn out to dominate the multipole coefficients C_ℓ for relatively small $\ell \lesssim 40$.

❷ The Sachs-Wolfe effect

In Newtonian perturbation theory the perturbation of the gravitational potential, when expresses as a function of the co-moving coordinate $\mathbf{x}$, is time-independent $\delta\phi(\mathbf{x})$. This perturbation has two effects:

❶ **gravitational redshift:** a photon emitted at the time of last scattering at point $\mathbf{x}$ will have its frequency (and hence its energy) shifted by a fractional amount $\delta\phi(\mathbf{x})$ (which is a direct energy shift). The temperature seen in direction $\hat{n}$ thus is shifted by

$$\boxed{\left(\frac{\Delta T(\hat{n})}{T_0}\right) = \delta\phi(\hat{n}\, r_L)}$$

where r_L is the radius of the sphere of last scattering.

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In Lect. 7 we obtained for the general time evolution of the density

$$\rho(t) = \rho_{0,\mathrm{crit}} \left\{ \Omega_\Lambda + \Omega_M \left(\frac{S_0}{S(t)} \right)^3 + \Omega_R \left(\frac{S_0}{S(t)} \right)^4 \right\}$$

with

$$\rho_{0,\mathrm{vac}} = \Omega_\Lambda \, \rho_{0,\mathrm{crit}} \;, \quad \rho_{0,\mathrm{mat}} = \Omega_M \, \rho_{0,\mathrm{crit}} \;, \quad \rho_{0,\mathrm{rad}} = \Omega_R \, \rho_{0,\mathrm{crit}} \;, \quad \rho_{0,\mathrm{crit}} = \frac{3\, H_0^2}{8\pi\, G} \;;$$

and the Friedmann equation

$$\dot{S}^2 = \frac{8\pi\, G}{3} \rho\, S^2 - k \quad \Leftrightarrow \quad H^2 \equiv \left(\frac{\dot{S}}{S} \right)^2 = \frac{8\pi\, G}{3} \rho - \frac{k}{S^2}$$

and division by H^2 yields at $t = t_0$

$$1 = \frac{8\pi G}{3H_0^2}\rho_0 - \frac{k}{H_0^2 S_0^2} \quad \Leftrightarrow \quad 1 = \Omega_\Lambda + \Omega_M + \Omega_R + \Omega_K \; ; \;\; \Omega_K \equiv -\frac{k}{H_0^2 S_0^2} \; .$$

Note that $w = p/\rho = -1, 0, 1/3$, respectively, for vacuum, matter and radiation. Therefore the present pressure is

$$p_0 = \frac{3H_0^2}{8\pi G}\left(-\Omega_\Lambda + \frac{1}{3}\Omega_R\right) \; ,$$

which may be used to represent the deceleration parameter

$$q_0 = \frac{8\pi G}{3H_0^2}\frac{\rho_0 + 3\,p_0}{2} = \frac{1}{2}\left(\Omega_M - 2\,\Omega_\Lambda + 2\,\Omega_R\right) \; .$$

Now, setting $x = S/S_0 = 1/(1+z)$, the Friedmann equation may be written in the form

$$
\begin{aligned}
\mathrm{d}t &= \frac{\mathrm{d}x}{H_0\, x\, \sqrt{\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_M\, x^{-3} + \Omega_R\, x^{-4}}} \\
&= \frac{-\mathrm{d}z}{H_0\, (1+z)\, \sqrt{\Omega_\Lambda + \Omega_K\, (1+z)^2 + \Omega_M\, (1+z)^3 + \Omega_R\, (1+z)^4}}\,.
\end{aligned}
$$

If we define the zero of time as the time of infinite redshift, then the time at which light was emitted to reach us with redshift z is

$$
t(z) = \frac{1}{H_0} \int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\, \sqrt{\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_M\, x^{-3} + \Omega_R\, x^{-4}}}\,.
$$

By setting $z = 0$ we obtain the present age of the universe $t_0 = t(z)|_{z=0}$. For distance measurements we in particular need the radial coordinate $r(z)$ of a source that is observed now with redshift z. Using the metric for radial light rays $\mathrm{d}s^2 = 0$, emitted at co-moving coordinates (t_1, r_1) and observed at the origin $r = 0$ at time t_0

$$c\,\mathrm{d}t = \pm S\,(t)\,\frac{\mathrm{d}r}{1 - kr^2} \quad \Rightarrow \quad \int_{t_1}^{t_0} \frac{c\,\mathrm{d}t}{S\,(t)} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1 - kr^2}}$$

we obtain [see Lect.: 5]

$$\begin{aligned} r(z) &= \mathscr{S}\left[\int_{t(z)}^{t_0} \frac{c\,\mathrm{d}t}{S\,(t)}\right] \\ &= \mathscr{S}\left[\frac{c}{S_0 H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\,\sqrt{\Omega_\Lambda + \Omega_K\,x^{-2} + \Omega_M\,x^{-3} + \Omega_R\,x^{-4}}}\right], \end{aligned}$$

where

$$\mathscr{S}[y] \equiv \begin{cases} \sin y & k = +1 \\ y & k = 0 \\ \sinh y & k = -1 \end{cases} .$$

Note that by definition $\Omega_K = 1 - \Omega_\Lambda - \Omega_M - \Omega_R$.

For a flat universe $k = 0$ we thus have for the radius of last scattering:

$$r_L = \frac{c}{S(t_0)H_0} \int_{1/(1+z_L)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_\Lambda\, x^4 + \Omega_K\, x^2 + \Omega_M\, x + \Omega_R}}$$

where $z_L \simeq 1090$ is the redshift of last scattering and t_0 the present time.

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❷ gravitational change of expansion rate:

a perturbation of the gravitational potential also affects the expansion rate and hence the temperature. Since we are in the matter dominated era already $T \propto 1/S(t) \propto t^{-2/3}$ and $\delta\phi(\mathbf{x})$ leads to a shift in the redshift at the time of last scattering in direction $\hat{n}$ by a fractional amount

$$\boxed{\left(\frac{\delta z}{1+z}\right) = -\left(\frac{\delta S(t)}{S(t)}\right) \quad \Leftrightarrow \quad \left(\frac{\dot{S}(t)}{S(t)}\right) = H(t)}$$

to be evaluated at $T \simeq 3000\ {}^\circ\mathrm{K}$

$$\boxed{\left(\frac{\Delta T(\hat{n})}{T_0}\right) = -\tfrac{2}{3}\,\delta\phi(\hat{n}\,r_L)\ .}$$

The Sachs-Wolfe effect then is the net shift obtained by adding the two effects

$$\boxed{\left(\frac{\Delta T(\hat{n})}{T_0}\right)_{\mathrm{SW}} = \tfrac{1}{3}\,\delta\phi(\hat{n}\,r_L)\ .}$$

This is the correct result also obtained by the proper general relativistic calculation.

How does this look in Fourier space:

$$\delta\phi(\mathbf{x}) \quad = \quad \int \mathrm{d}^3q\,\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{x}}\,\delta\phi_{\mathbf{q}}$$

Using the Legendre expansion

$$\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{x}} = \sum_{\ell} (2\ell+1)\,\mathrm{i}^{\ell}\, P_{\ell}(\hat{q}\cdot\hat{n})\, j_{\ell}(qr)$$

in terms of spherical Bessel functions $j_\ell(z)$, which are given by $j_\ell(z) = (\pi/2z)^{1/2}\, J_{\ell+1/2}(z)$ in terms of normal Bessel functions. We then obtain:

$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)_{\mathrm{SW}} = \frac{1}{3}\sum_{\ell=0}^{\infty} \mathrm{i}^{\ell} \int \mathrm{d}^3q\, \delta\phi_{\mathbf{q}}\, j_\ell(qt_L)\, P_\ell(\hat{q}\cdot\hat{n})\;.$$

What we need to calculate is the two-point correlation of this object. In spite of the fact that $\delta\phi(\mathbf{x})$ depends on position, its distribution as seen by different observers in different parts of the universe in average is invariant under rotations and translations:

$$\langle\delta\phi(\mathbf{x})\delta\phi(\mathbf{y})\rangle = \mathscr{P}_\phi(|\mathbf{x}-\mathbf{y}|) \quad\Leftrightarrow\quad \langle\delta\phi_{\mathbf{q}}\delta\phi_{\mathbf{q}'}\rangle = \mathscr{P}_\phi(q)\,\delta^{(3)}(\mathbf{q}+\mathbf{q}')\;.$$

Since $\delta\phi(\mathbf{x})$ is real we have $\delta\phi^*_{\mathbf{q}} = \delta\phi_{-\mathbf{q}}$ which implies $\mathscr{P}_\phi(q)$ real and positive. Furthermore, we need the reflection $P_\ell(-z) = (-1)^\ell\, P_\ell(z)$ and orthogonality property of Legendre polynomials:

$$\int \mathrm{d}\Omega_{\hat{q}}\, P_\ell(\hat{q}\cdot\hat{n})\, P_{\ell'}(\hat{q}\cdot\hat{n}') = \left(\frac{4\pi}{2\ell+1}\right) \delta_{\ell\ell'}\, P_\ell(\hat{n}\cdot\hat{n}')$$

to calculate

$$\langle \Delta T(\hat{n}) \Delta T(\hat{n}')\rangle_{\mathrm{SW}} = \frac{4\pi^2 T_0^2}{9} \sum_\ell (2\ell+1)\, P_\ell(\hat{n}\cdot\hat{n}')\, \int_0^\infty \mathrm{d}q\, q^2\, \mathscr{P}_\phi(q)\, j_\ell^2(q r_L)\,,$$

which may be compared with the expansion in terms of the spherical harmonics and the C_ℓ:

$$\boxed{C_{\ell,\mathrm{SW}} = \frac{16\pi^2\, T_0^2}{9}\, \int_0^\infty \mathrm{d}q\, q^2\, \mathscr{P}_\phi(q)\, j_\ell(q r_L)}$$

If the fluctuations in the gravitational potential are caused by cold dark matter, the differential equation (DEQ) for $\delta\phi$ does not involve gradients and hence in Fourier space does not depend on the wave vector $\mathbf{q}$ and the $\mathbf{q}$ dependence of $\delta\phi_{\mathbf{q}}$ can only be the result of the initial condition. However, before the universe was dominated by cold dark matter when $q/S(t) > H(t)$ these arguments do not apply which actually means that our arguments are good for $\ell < 100$ in the following. For small enough q hence the arguments suggest a scaling ansatz conventionally written as (N_ϕ and n free parameters)

$$\boxed{\mathscr{P}_\phi(q) = N_\phi^2 q^{n-4}\;.}$$

where N_ϕ^2 is a positive constant. The integral may then be performed analytically using

$$\int_0^\infty \mathrm{d}s\; j_\ell^2(s)\; s^{n-2} = \frac{2^{n-2}\,\pi\,\Gamma(3-n)\,\Gamma(\ell+\frac{n-1}{2})}{\Gamma^2(\frac{4-n}{2})\,\Gamma(\ell+2-\frac{n-1}{2})}$$

and for $\ell < 100$ one obtains

$$\boxed{C_{\ell,\mathrm{SW}} = \frac{16\pi^3\,T_0^2\,N_\phi^2}{9}\,\frac{2^{n-2}\,\Gamma(3-n)\,\Gamma(\ell+\frac{n-1}{2})}{\Gamma^2(\frac{4-n}{2})\,\Gamma(\ell+2-\frac{n-1}{2})}\,r_L^{n-1}}$$

The parameters N and n are phenomenological parameters, which can be "measured" by fitting CMB data. However, even before this was possible after COBE, Harris (PRD 1970) and Zel'dovich (MNRAS 1972) estimated $\delta\phi$ from analyzing the large scale structure of matter, observed relatively close to the present. This is possible by applying the Poisson equation,

$$\nabla^2\delta\phi(\mathbf{r}) = \frac{1}{S^2(t)}\,\nabla^2\,\delta\phi(\mathbf{x}) = 4\pi\,G\,\delta\rho$$

where $\mathbf{r} = S\,(t)\,\mathbf{x}$ is the proper distance and $\mathbf{x}$ the co-moving coordinate. It allows us to relate matter density fluctuations to $\delta\rho$ to $\delta\phi$:

$$\langle\delta\rho(\mathbf{x},t)\delta\rho(\mathbf{x}',t')\rangle = \frac{1}{(4\pi\,G\,S\,(t)\,S\,(t'))^2} \times \int \mathrm{d}^3q\,q\,q^4\,\mathscr{P}_\phi(q)\,\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\;.$$

From this analysis of the observations Harris and Zel'dovich estimated

$$\boxed{n = 1 \quad \text{and} \quad N \approx 10^{-5}}$$

which is called the Harris-Zel'dovich form of $\mathscr{P}_\phi$.

Adopting $n = 1$ yields the answer

$$\boxed{C_{\ell,\mathrm{SW}} \simeq \frac{8\pi^2 N_\phi^2 T_0^2}{9\,\ell\,(\ell+1)}}$$

which is scale-invariant in the sense that it is independent of r_L.

Often plots of the ℓ-dependence are presented as $\ell\,(\ell+1)\,C_\ell$ versus ℓ, which is suggested by the above result.

❸ Evaluation of perturbations in general

There are a number of other contributions to C_ℓ. In general perturbations depends on gradients and hence on $\mathbf{q}$. This has been discussed in the previous Lecture. The procedure to evaluate corresponding contributions to C_ℓ is straightforward, in principle:

❒ look at perturbations in Fourier space: $\delta f(\mathbf{x}) = \int \mathrm{d}^3 q \, \mathrm{e}^{\mathrm{i}\,\mathbf{x}\cdot\mathbf{q}} \, \delta f_{\mathbf{q}}$

❒ expand $\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot\hat{n}\,r_L}$ in the series of Legendre polynomials $P_\ell(\hat{q}\cdot\hat{n})$ with coefficients proportional to $j_\ell(q r_L)$

❒ our main interest is the region $\ell \gg 1$ where the cosmic variance can be neglected and which is the range of interest for cosmological information

❒ for $\ell \gg 1$ the integral over q is dominated by values of q of order ℓ/t_L.

The last statement results for corresponding properties of the spherical Bessel function $j_\ell(z)$: for $\ell \gg 1$ $j_\ell(z)$ is peaked at $z \simeq \ell$, for $\nu = \ell + 1/2 \to \infty \;, \quad z \to \infty \;;$

$\nu/z \neq 1$ fixed one obtains

$$j_\ell(z) \to \begin{cases} 0 & z < \nu \\ z^{-1/2}\,(z^2-\nu^2)^{-1/4}\,\cos\left(\sqrt{z^2-\nu^2} - \nu\arccos(\nu/z) - \frac{\pi}{4}\right) & z > \nu\,. \end{cases}$$

As a result C_ℓ for large enough ℓ represents the contributions of the Fourier components of the perturbations for $q \approx \ell/r_L$!

Interpretation:

- $\mathbf{k}_L \equiv \mathbf{q}/S(t_L)$ (note $S(t_L)\mathbf{x}$ is proper distance) is the physical wave number at time t_L
- C_ℓ for large ℓ reflects behavior of perturbations for $k_L \approx \ell/d_A$, where d_A is the angular diameter distance of the surface of last scattering

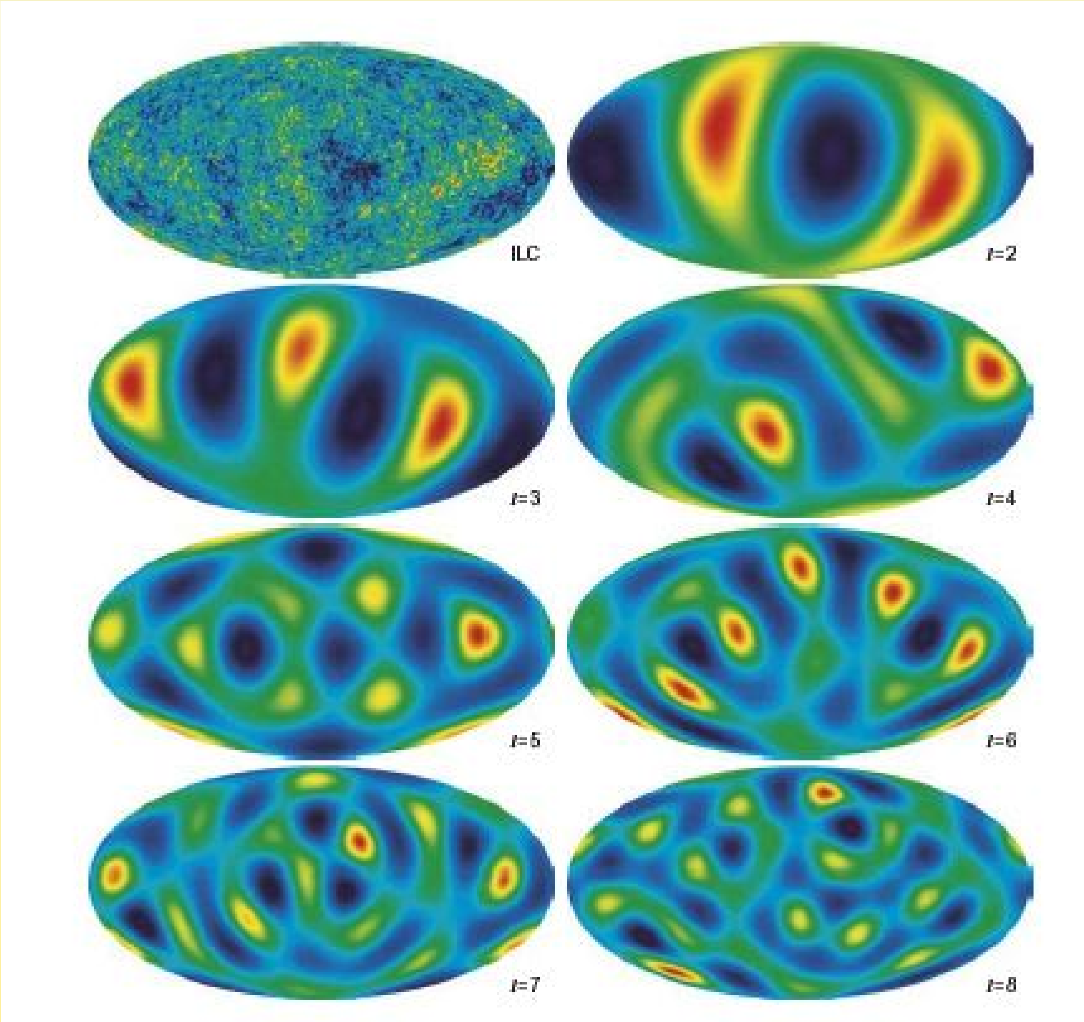
For a flat universe $k = 0$ we have:

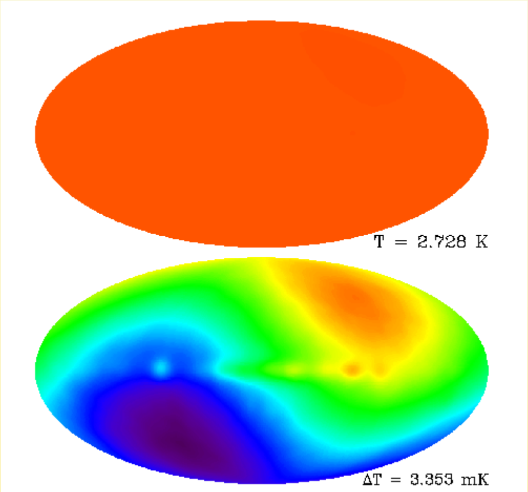
$$d_A \equiv r_L\, S(t_L) = \frac{c}{H_0\,(1+z_L)} \int_{1/(1+z_L)}^{1} \frac{\mathrm{d}x}{\sqrt{\Omega_\Lambda\, x^4 + \Omega_K\, x^2 + \Omega_M\, x + \Omega_R}}$$

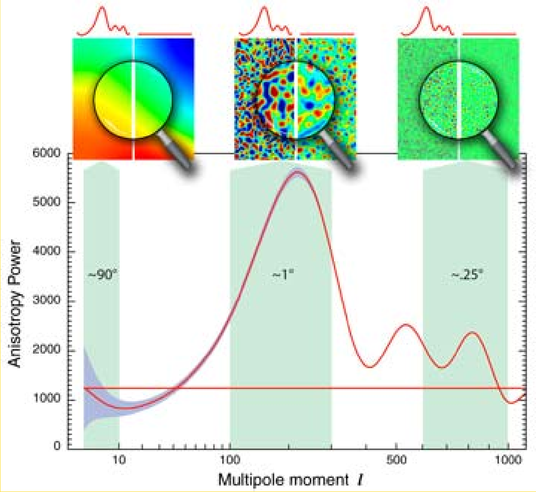
When $k \neq 0$ we have to take $\mathscr{S}[y]$ defined earlier in place of y on the r.h.s. Note $1 + z_L \equiv S(t_0)/S(t_L)$ and since $z_L \simeq 1090$ both $S(t_0)/S(t_L) > 1000$ and $r_L/d_A > 1000$ appear enhanced by a large factor. This factor is the growth factor of length scales due to the expansion of the universe since last scattering.

Angular distances map into angles of observation. For low $\ell = 2, 3$ correspond to angles greater than about 60°. In this range the cosmic variance [for $\ell = 2$ about 63%] and local effects make an interpretation difficult. Different physical mechanisms have different typical intrinsic length scales. A given mechanism thus produces patterns of typical length and this manifests itself as a peak in the C_ℓ vs. ℓ. Measurement by COBE and WMAP revealed pronounced and fairly pronounced peaks at about $\ell = 200,\ 500,\ 850$, which corresponds to angular sizes about $1^\circ,\ 0.4^\circ,\ 0.2^\circ$ [$\vartheta = \frac{180^\circ}{\ell}$] [2].

[2]Physicist usually associate spherical harmonics expansions with physics of angular momentum in the quantum mechanics of atoms (early encounters when leaning the hydrogen atom) as an expansion in S, P, D, . . . waves and, similarly, in the context of phase shift analysis of scattering processes. There typically only low values of ℓ are of interest and it might look "crazy" to as for angular momentum states with ℓ of order hundreds or thousand. In the present context ℓ values are directly related to angular resolution of geometrical patterns and the given angular sizes of primordial temperature patches (cosmological patchwork).







Resolving details requires large enough ℓ. Upper right panel: monopole (CMB mean value) [$\ell = 0$], dipole (earth's motion through the CMB) [$\ell = 1$].

Note, while $\ell = 0$ is the main cosmology effect, the relict radiation of the Big Bang, the next $\ell = 1$ already is a local effect (uninteresting to cosmology). It is fact or expected that low values of $\ell < 4$ are affected by the local constellation of the observer relative to the microwave background, which is attached to a freely falling frame. The high level of precision of CMB measurements requires a very precise understanding of "local physics" as well, in order to be able to distill the more interesting information, like the one relevant to cosmology.

➍ The role of horizons

Observational horizons play an important role in the interpretation of the CMB fluctuations data. Here first a remainder about cosmological horizons:

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In standard Robertson-Walker and Friedmann cosmologies the integral

$$\int_0^t \frac{c\,\mathrm{d}t'}{S(t')} = \int_0^{r_{\max}(t)} \frac{\mathrm{d}r}{\sqrt{1-kr^2}}$$

defines a maximum value of the RW-coordinate $r_{\max}(t)$ which is the maximum radial coordinate from which an observer can receive signals traveling at the speed of light, when t=0 is the moment of the Bog Bang. Actually, in standard cosmologies $S(t) \propto \sqrt{t}$ is universal for $t \to 0$, such that the integral on the l.h.s. converges such that $r_{\max}(t)$ is finite and defines the co-moving coordinate horizon. The proper distance horizon then is

$$\boxed{d_{\max}(t) = S(t) \int_0^{r_{\max}(t)} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = S(t) \int_0^t \frac{c\,\mathrm{d}t'}{S(t')}}$$

In a radiation dominated universe with $S(t) \propto t^{1/2}$ we would have $d_{\max}(t) = 2tc = c/H$, while for a mainly matter dominated universe $S(t) \propto t^{2/3}$, such that $d_{\max}(t) = 3tc = 2c/H$. At present we find

$$\boxed{d_{\max}(t_0) = \frac{c}{H_0} \int_0^1 \frac{\mathrm{d}x}{\sqrt{\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_M\, x^{-3} + \Omega_R\, x^{-4}}}}$$

which is the proper distance beyond which we cannot now see. $d_{\max}(t_0)$ is the particle horizon distance. As time goes on more and more objects can be seen appearing at the horizon by the observer. The objects (galaxies) are seen not as they are by now, but as they were at times $d_{\max}(t_0)/c$. Note that for the realistic case of a most times matter dominated universe the distance $d_{\max}(t) \simeq 3tc = 2c/H$ we see the galaxy in the state it was ~ 41.1 billion years ago when we take ~ 13.7 billion years for the age t of the universe. This seems paradoxical but is a simple consequence of the universal expansion of the universe [i.e. the factor $S(t)$], which at the horizon makes the galaxy look to move at speed of light while the galaxy is at rest relative to the matter in its vicinity (co-moving coordinates). In contrast light travels at speed of light also in the co-moving system, i.e. galaxies moves actually slower in terms of proper motion and hence requires more “time”.

The photon we receive today (see ✲⋙, ✲⋙). More precise estimates yield that the edge of the observable universe is about ~ 46.5 billion light − years away. In other words, the co-moving coordinate "object" ct has no direct physical significance, in spite of the convention to call t age of the universe.

There is also an event horizon which is a horizon for event we will never be able to see (at no time in future). It is defined by

$$\lim_{T\to\infty} \int_t^T \frac{c\,\mathrm{d}t'}{S(t')} = \int_0^{r_{\mathrm{MAX}}(t)} \frac{\mathrm{d}r}{\sqrt{1-kr^2}}$$

If the integral $\int_t^\infty \frac{\mathrm{d}t'}{S(t')}$ is convergent $r_{\mathrm{MAX}}(t)$ is finite and any events at $r > r_{\mathrm{MAX}}(t)$ will never be observable by observer at $(r = 0, t)$. The corresponding proper distance of the event horizon is

$$\boxed{d_{\mathrm{MAX}}(t) = S(t) \int_0^{r_{\mathrm{MAX}}(t)} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = S(t) \int_t^\infty \frac{c\,\mathrm{d}t'}{S(t')}}$$

Exercise: Show that in a Friedmann cosmology with vanishing cosmological constant there is no event horizon, while with a non-vanishing cosmological constant an event horizon exists.

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For the discussion of CMB fluctuations the particle horizon is of interest because we have to distinguish fluctuation in the observable part of the universe and what happens behind the horizon. The relevant quantity is the Hubble variable $H(t) = \frac{\dot{S}}{S}(t)$, which measures the expansion rate. For the Fourier components of the fluctuations the relevant physical wave numbers are $q/S(t)$ and if these are small relative to the expansion rate H they are essentially independent of the wave number, in the sense that gradient terms are negligible in the DEQs describing the evolution of the perturbations. Thus all the q-dependence results from the initial condition (as illustrated in the example above). Such perturbation are said to be "outside the horizon" because the wavelength $\frac{2\pi S(t)}{q}$:

- in the radiation dominated era $q/S(t)$ decreased like $t^{-1/2}$

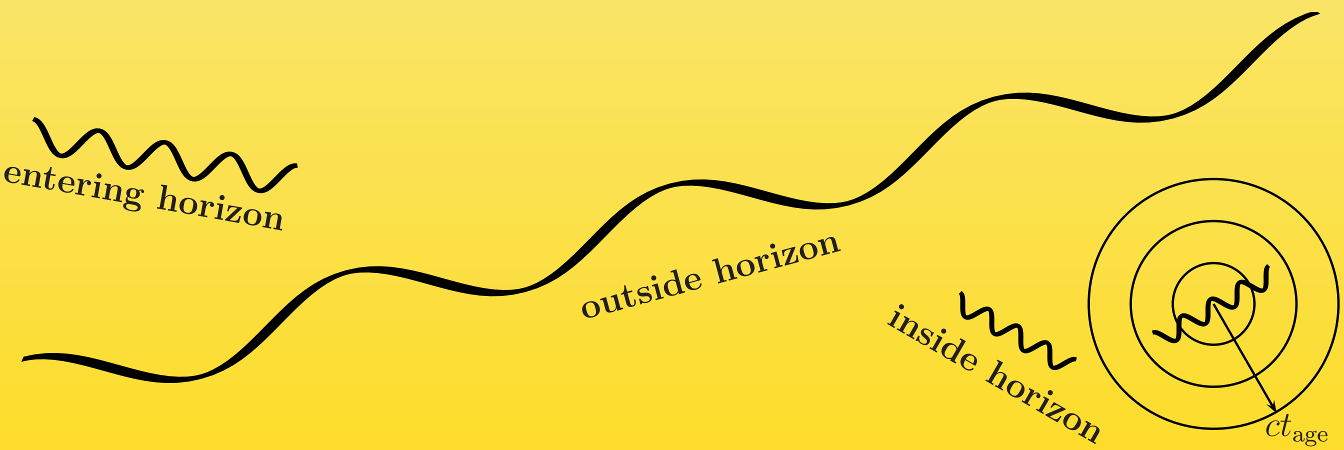
❒ in the matter dominated era $q/S(t)$ decreased like $t^{-2/3}$

❒ while $H(t) \propto 1/t$ in any case decreased faster.

The means

● the perturbations which were outside the horizon at earlier times subsequently moved into the observational part of the universe.

● perturbations with higher wave numbers enter the horizon earlier than those with lower wave numbers.



In fact the relevant horizon is not necessarily the usual one defined in terms of the propagation of light! In the photon-electron-baryon system which existed before recombination and decoupling plasma sound waves were propagating through the system and the relevant horizon in this case is the horizon defined by sound propagation and the speed of sound.

During the era of recombination, when radiation and matter were in thermal equilibrium, the speed of sound was

$$\boxed{v_s = \sqrt{\frac{\delta p}{\delta \rho}}\;,}$$

where δp and $\delta \rho$ are adiabatic fluctuations of pressure and density, respectively.

As we have shown earlier in this situation the specific entropy remains preserved:

$$\delta \sigma \propto \delta \left(\frac{\epsilon}{n_B}\right) + p\, \delta \left(\frac{1}{n_B}\right) = 0$$

ϵ is the thermal energy density, n_B the baryon number density and p the pressure.

By the fact that there are so much more photons than relict baryons both energy and pressure are dominated by the radiation:

$$\epsilon = a\,T^4\,,\quad p = \frac{1}{3}\,a\,T^4$$

and therefore for adiabatic fluctuations

$$3\,\frac{\delta T}{T} = \frac{\delta n_B}{n_B} = \frac{\delta \rho_B}{\rho_B}$$

where ρ_B is the baryon mass density. This gives the speed of sound

$$v_s = \left(\frac{4\,a\,T^3\,\delta T/3}{\delta\rho_B + 4\,a\,T^3\,\delta T}\right)^{1/2} = \frac{1}{\sqrt{3\,(1+R)}}$$

where

$$\boxed{R \equiv \frac{3\rho_B}{4\,a\,T^4}\;.}$$

Since $\rho_B \propto S(t)^{-3}$ and $T \propto S(t)^{-1}$, we have $R \propto S(t)$ and hence $\mathrm{d}t = \mathrm{d}R/HR$ or more precisely

$$\mathrm{d}t = \frac{\mathrm{d}R}{RH_0\sqrt{\Omega_M\,(R_0/R)^3 + \Omega_R\,(R_0/R)^4}} = \frac{R\mathrm{d}R}{H_0\,\sqrt{\Omega_M}R_0^{3/2}\,\sqrt{R_{\mathrm{EQ}} + R}}$$

where $R_{\mathrm{EQ}} = \Omega_R R_0/\Omega_M = 3\Omega_R\Omega_B/4\Omega_M\Omega_\gamma$ is the R-value at matter-radiation equality. The acoustic horizon distance then is given by

$$\boxed{d_H \equiv S_L\,\int_0^{t_L} \frac{c\,\mathrm{d}t}{S(t)\,\sqrt{3\,(1{+}R)}} = R_L\,\int_0^{t_L} \frac{c\,\mathrm{d}t}{R\,\sqrt{3\,(1{+}R)}}}$$

which can be integrated analytically:

$$\boxed{d_H = \frac{2c}{H_0\,\sqrt{3\,R_L\,\Omega_M}\,(1{+}z_L)^{3/2}}\;\ln\left(\frac{\sqrt{1{+}R_L}+\sqrt{R_{\mathrm{EQ}}{+}R_L}}{1{+}\sqrt{R_{\mathrm{EQ}}}}\right)\;,}$$

where $R_L = 3\Omega_B/4\Omega_\gamma\,(1 + z_L)$ is the value of R at last scattering.

What do we learn:

- gradients become important when $k_L \approx 1/d_H$
- since C_ℓ are dominated by ℓ-values ℓ/d_A, gradients become important when $\ell \simeq \ell_H$ where $\ell_H = d_A/d_H$
- ℓ_H is independent of H_0 because both d_A and d_H are proportional to H_0.

Estimating ℓ_H: note that $d_A \propto (1+z_L)^{-1}$ while $d_H \propto (1+z_L)^{-3/2}$ such that $\ell_H \propto (1+z_L)^{1/2} \simeq 33$. For a more precise estimate we use cosmological parameters from supernova observations and from cosmological nucleosynthesis: $\Omega_M = 0.26$, $\Omega_\Lambda = 0.74$ and $\Omega_B = 0.043$. This yields:

✦ $R_L = 0.62$, $R_{\mathrm{EQ}} = 0.21$

✦ $d_A = 3.38\, H_0^{-1}\,(1+z_L)^{-1}$, $d_H = 1.16\, H_0^{-1}\,(1+z_L)^{-32}$

✦ $\Longrightarrow$ $\boxed{\ell_H = 2.9\ \sqrt{1+z_L} \simeq 96}$

❺ Beyond the Sachs-Wolfe effect

In Lect. 12 we considered Newtonian theory of perturbations in density $\delta\rho$, velocity $\delta\mathbf{v}$ and gravitational potential $\delta\phi$ in curved RW-metric space: continuity

$$\frac{\mathrm{d}\delta\rho_{\mathbf{q}}}{\mathrm{d}t} + 3\,H\,\delta\rho_{\mathbf{q}} + \mathrm{i}\,\frac{1}{S\,(t)}\,\bar{\rho}\,\mathbf{q}\cdot\delta\mathbf{v}_{\mathbf{q}} = 0\,,$$

$$\frac{\mathrm{d}\delta\mathbf{v}_{\mathbf{q}}}{\mathrm{d}t} + H\,\delta\mathbf{v}_{\mathbf{q}} = -\mathrm{i}\,\frac{1}{S\,(t)}\,\mathbf{q}\,\delta\phi_{\mathbf{q}}\,,$$

$$\mathbf{q}^2\,\delta\phi_{\mathbf{q}} = -4\pi\,G\,S^2(t)\,\delta\rho_{\mathbf{q}}$$

with solutions which are classified conveniently according to their transformation properties under 3-dimensional rotations:

Vector modes: if no scalar modes $\delta\rho_{\mathbf{q}}$, $\delta\phi_{\mathbf{q}}$ and $\mathbf{q}\cdot\delta\mathbf{v}_{\mathbf{q}}$ vanish and the pure vector satisfies

$$\frac{\mathrm{d}\delta \mathbf{v}_{\mathbf{q}}}{\mathrm{d}t} + H\,\delta \mathbf{v}_{\mathbf{q}} = 0$$

with solution

$$\boxed{\delta \mathbf{v}_{\mathbf{q}} \propto 1/S\,(t)}$$

which means that vector modes simply decay and usually they are ignored.

Scalar modes: in these modes velocity perturbations $\delta \mathbf{v}(\mathbf{r}, t)$ can be expressed as a gradient with respect to $\mathbf{x} = \mathbf{r}/S\,(t)$ of a scalar perturbation $\delta u(\mathbf{r}, t)$ such that

$$\frac{\mathrm{d}\delta\rho_{\mathbf{q}}}{\mathrm{d}t} + 3\,H\,\delta\rho_{\mathbf{q}} - \frac{1}{S\,(t)}\bar{\rho}\,\mathbf{q}^2\,\delta u_{\mathbf{q}} = 0\,,$$

$$\frac{\mathrm{d}\delta u_{\mathbf{q}}}{\mathrm{d}t} + H\,\delta u_{\mathbf{q}} = -\frac{1}{S\,(t)}\delta\phi_{\mathbf{q}} = \frac{4\pi\,G\,S\,(t)}{\mathbf{q}^2}\delta\rho_{\mathbf{q}}\,.$$

For $k = 0$ we have $4\pi\,G\,\bar{\rho} = 3H^2/2$, and with $\bar{\rho} \propto S^{-3}$ and the definition of H one may eliminate $\delta u_{\mathbf{q}}$ obtaining

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(S^2\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\rho_{\mathbf{q}}}{\bar{\rho}}\right)\right) - \frac{2}{3}H^2S^2\left(\frac{\delta\rho_{\mathbf{q}}}{\bar{\rho}}\right) = 0}$$

Since for $k = 0$ $S\,(t) \propto t^{2/3}$ and $H = 2/3t$ we may replace $S^2 = t^{4/3}$ and $H^2S^2 = t^{-2/3}$ and the general solution is a linear combination of powers $t^{2/3}$ and t^{-1}. It is reasonable to assume that at the time of last scattering the t^{-1} mode was negligible such that $\delta\rho_{\mathbf{q}}/\bar{\rho} \propto t^{2/3}$ and since $\bar{\rho} \propto S^{-3}(t)$ we have

$$\boxed{\delta\rho_{\mathbf{q}} \propto S^{-2}(t)\ ,}$$

and the Poisson equation then shows that

$$\boxed{\delta\phi_{\mathbf{q}} \quad \text{is time independent.}}$$

A reliable calculation of perturbations about a RW-metric cosmology requires a treatment within general relativity which is rather elaborate and is beyond the scope of these lectures. State of the Art calculations are referredt to at the end of the lecture.

Here as a first step we only perform some order of magnitude considerations based on Newtonian perturbation theory in curved RW-metric space:

Doppler effect

due to velocity fluctuations in the plasma at last scattering:

as discussed above the dominant contributions come from compression modes, for which the velocity is a gradient of a scalar. One can show that the Doppler effect contribution to C_ℓ does not interfere with the SW effect because it is proportional to the wave vector $\mathbf{k}_L$. In the integral over q in this case is suppressed by a factor $\mathbf{k}_L\, d_H$. In fact for multipoles $\ell \ll \ell_H$ the contribution from the Doppler effect will be suppressed by a factor $((\ell/d_A)\, d_H)^2 = \ell^2/\ell_H^2$, relative to the WS-effect.

Intrinsic temperature fluctuations

in the electron-nucleon-photon system at the time of last scattering at a redshift of about $z_L = 1090$:

above we have seen that intrinsic temperature fluctuation is 1/3 of the plasma density, which we can identify with the baryon mass density

$$\boxed{\frac{\delta T}{T} = \frac{1}{3}\,\frac{\delta \rho_B}{\rho_B}\ .}$$

In the appropriate general relativistic approach one can show that outside the horizon the dominant perturbations are adiabatic and the fractional plasma density may be identified with the total matter density fluctuation ratio

$$\boxed{\frac{\delta\rho_B}{\rho_B} \sim \frac{\delta\rho}{\bar{\rho}} \;.}$$

The latter again is related to the perturbation in the gravitational potential via Poisson's equation. Evaluated at the time of last scattering, we have in Fourier space

$$\delta\rho_{\mathbf{q}}(t_L) = -\frac{q^2}{4\pi\, G\, S^2(t_L)}\, \delta\phi_{\mathbf{q}} = -\frac{k_L^2}{4\pi\, G}\, \delta\phi(t_L)$$

where again $k = q/S(t)$. The total mass density $\bar{\rho}(t_L)$ at last scattering is related to the horizon size d_H by

$$\bar{\rho}(t_L) \simeq \frac{3H^2(t_L)}{8\pi\, G} \approx \frac{1}{2\pi\, G\, d_H^2} \;,$$

and hence the order of magnitude of the intrinsic temperature fluctuations is related to the perturbation in the gravitational potential by

$$\boxed{\frac{\delta T(t_L)}{\bar{T}(t_L)} = \frac{1}{3}\,\frac{\delta\rho(t_L)}{\bar{\rho}(t_L)} \approx k_L^2\, d_H^2\, \delta\phi_{\mathbf{q}}\ .}$$

Hence, again, we expect the contributions in the q integral to C_ℓ to be suppressed at low wave numbers like $\mathbf{k}_L^2 d_H^2$. The interference of this contribution with the SW-effect for $\ell \ll \ell_H$, like the Doppler effect, is smaller by a factor $((\ell/d_A)\, d_H)^2 = \ell^2/\ell_H^2$, relative to the WS-effect.

Integrated Sachs-Wolfe effect

which is the gravitational shift in the spectrum due to time-dependent fluctuations in the gravitational potential between the time of last scattering and the present:

If $\delta\phi$ is truly time independent there is no “integrated Sachs-Wolfe effect”. In a time independent potential (time independent ripples) redshift and blueshift cancel each other (relative to a constant background potential). There are small time variations because radiation still makes a contribution the the gravitational field some time after last scattering and at late times the gravitational constant comes into play and causes some time dependence. The late-time integrated SW-effect mainly affects C_ℓ for $\ell \lesssim 10$.

Conclusion: for $10 < \ell, 50$ the C_ℓ's are dominated by the Sachs-Wolfe effect.

Apart from $\ell = 0$ the $T_0 = 2.725\ ^\circ\mathrm{K}$ CMB and the $\ell = 1$ anisotropy due to the earth's motion through the CMB, the first detection of the anisotropy was achieved by COBE in 1992, measured the Planck distribution for $2.7\ ^\circ\mathrm{K}$ and a temperature fluctuation $30 \pm 5\ \mu\mathrm{K}$ and an angular distribution consistent with Harrison-Zel'dovich spectrum. After four years in 1996 values of C_ℓ had been measured from $\ell = 2$ to $\ell = 40$. A fit of the $C_{\ell,\mathrm{SW}}$ as a function of n for $\ell \geq 4$ gave the result

$$\boxed{n = 1.13 \pm^{0.3}_{0.4}}$$

consistent with $n = 1$. The result for C_ℓ is often written as

$$\boxed{C_\ell = \frac{24\pi Q^2}{5\,\ell\,(\ell+1)}\ ,}$$

in terms of a quadrupole moment Q. Fitting C_ℓ for $10 \leq \ell \leq 40$ gave

$$\boxed{Q = 18 \pm 1.4\ \mu\mathrm{K}}$$

and a comparison with $C_{\ell,\mathrm{SW}}$ given earlier yields

$$\boxed{N_\phi = \sqrt{\frac{27}{5\pi}}\frac{Q}{T_0} = (8.7 \pm 0.7) \times 10^{-6}\ .}$$

Surprisingly, the multipole coefficients for $\ell = 2$ and $\ell = 3$ were found to be much lower than expected from the $C_{\ell,\mathrm{SW}}$ fit for $\ell \geq 4$ extrapolated to the lower ℓ. What it means is that correlations in the temperature fluctuations at angles greater than about 60° seem to be absent. Problems may be due to foreground effects and/or cosmic variance.

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