Renormalizing the Standard Model

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ABSTRACT

We review the renormalization of the Standard Model of electroweak interactions and go into details of calculating and renormalizing parameters and cross-sections. The main emphasis is on calculations for precision physics with Z bosons. Theoretical calculations are confronted with recent results from LEP.

I. THE STANDARD MODEL

1. Introduction

The known fundamental interactions of elementary particles (strong, weak and electromagnetic) derive from a *local gauge principle* (Weyl 1932, Yang-Mills 1954) with the gauge group [1, 2][1]

$$G_{loc} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \; . \tag{1}$$

The theory is essentially determined once the *matter fields* and their transformation laws under G_{loc} are specified. The real world is built from *massless* spin 1/2 particles, the *leptons and colored quarks*. Massless particles necessarily have fixed helicity

[1] $SU(2)_L \otimes U(1)_Y$ describes the electroweak interactions in an unified from[1], while $SU(3)_c$ describes Quantum Chromodynamics (QCD), the strong interactions of the hadrons in terms of the colored quarks and gluons [2]. Quarks and gluons are confined in hadrons, which correspond to color singlet states like the baryons (spin $\frac{1}{2}$ or $\frac{3}{2}$)

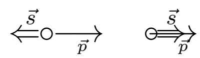
$$(q_1 q_2 q_3)_{\text{color singlet}} = \frac{1}{\sqrt{3!}} \varepsilon_{c_1 c_2 c_3} q_{1 c_1} q_{2 c_2} q_{3 c_3} \; ,$$

antibaryons the same in terms of antiquarks and mesons (spin 0 or 1)

$$(q_1 \bar{q}_2)_{\text{color singlet}} = \frac{1}{\sqrt{3}} \delta_{c_1 c_2} q_{1 c_1} \bar{q}_{2 c_2} \; .$$

Only hadrons show up as physical states.

(chirality). The relativistic massless Dirac field ψ decomposes into two independent Weyl fields a left-handed field $\psi_L = \frac{1-\gamma_5}{2}\psi$ and a right-handed field $\psi_R = \frac{1+\gamma_5}{2}\psi$:



ψ_L ψ_R

In relativistic quantum field theory locality and causality enforce particle-antiparticle pairing and the spin-statistics theorem to hold. For the chiral fields, this implies that a left-handed field ψ_L describes at the same time a left-handed particle and a right-handed antiparticle and a right-handed field ψ_R describes a right-handed particle and a left-handed antiparticle. If we count particles and antiparticles separately, using $\psi_R \simeq \psi_L^c$, we thus may consider all fields to be left-handed. If we use the labels r=red, g=green and b=blue for the quark colors, the list of particles in the first lepton - quark family reads[2]

$$\nu_{eL}, \bar{\nu}_{eL}, e_L^-, e_L^+, u_{Lr}, u_{Lg}, u_{Lb}, u_{Lr}^c, u_{Lg}^c, u_{Lb}^c, d_{Lr}, d_{Lg}, d_{Lb}, d_{Lr}^c, d_{Lg}^c, d_{Lb}^c$$

and there are two additional such families. These are 48 degrees of freedom described by the free matter Lagrangian

$$\mathcal{L}_{\text{ matter},0} = \sum_a \bar{\psi}_{La} i\gamma^\mu \partial_\mu \psi_{La} \ . \tag{2}$$

This Lagrangian has a global $U(48)$ symmetry. Nature has chosen the subgroup $G_{loc} \subset U(48)$ to be a *local* symmetry

$$\psi_L \to U(x)\ \psi_L, U \in G_{loc}.$$

This requires the existence of a set of gauge fields $V_{\mu i}$ which minimally couple to the fermions

$$\partial_\mu \psi_L \to D_\mu \psi_L = (\partial_\mu - i \sum_r g_r T_{ri} V_{\mu\, ri})\ \psi_L$$

By T_{ri} we denote the generators of the local group (r labeling the different group factors) and g_r are arbitrary coupling constants. Thus, the matter field interactions are determined to be

$$\mathcal{L}_{\text{matter,int}} = \sum_r g_r j_{ri}^\mu V_{\mu\, ri} \tag{3}$$

where

$$j_{ri}^\mu = \bar{\psi}_L \gamma^\mu T_{ri} \psi_L$$

[2]The recently observed neutrino oscillations require the neutrinos to have a tiny mass which must be different for the different flavors. This requires the existence of right-handed neutrinos $\nu_{\ell R} \equiv \bar{\nu}_{\ell L}$ in spite of the fact that they do not couple directly to gauge fields (i.e., they are singlets with respect to the SM gauge group).

GROUP	MULTIPLET	REPRESENTATION	
$SU(3)_c$:	LEPTONS	1	color singlet
	QUARKS	3	color triplet
	ANTIQUARKS	3^*	anticolor triplet
$SU(2)_L$:	$\left(\begin{array}{c}\nu_e\\ e^-\end{array}\right)_L, \left(\begin{array}{c}u\\ \tilde{d}\end{array}\right)_L$		
	$\left(\begin{array}{c}\nu_\mu\\ \mu^-\end{array}\right)_L, \left(\begin{array}{c}c\\ \tilde{s}\end{array}\right)_L$	$2=2^*$	weak isospin doublets
	$\left(\begin{array}{c}\nu_\tau\\ \tau^-\end{array}\right)_L, \left(\begin{array}{c}t\\ \tilde{b}\end{array}\right)_L$		
	$\nu_{eR}, e^-_R, u_R, d_R$		
	$\nu_{\mu R}, \mu^-_R, c_R, s_R$	1	weak isospin singlets
	$\nu_{\tau R}, \tau^-_R, t_R, b_R$		
$U(1)_Y$:	$Y = 2(Q - T_3)$	Abelian	weak hypercharge

Table 1: Matter field multiplets of the SM.

are the fermion currents. We observe that fermions talk to each other only via spin 1 gauge bosons.

In the unbroken phase, mass terms for fermions are forbidden, since $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$ is not $SU(2)_L \otimes U(1)_Y$ invariant.

The transformation properties of the fermions under G_{loc} are the simplest possible ones. Only the fundamental (the nontrivial representation of lowest dimension) and the trivial (singlet) representations show up. The multiplets and the associated weak quantum numbers are summarized in Tabs. 1 and 2.

	Doublets				Singlets			
	$(\nu_\ell)_L$	$(\ell^-)_L$	$(u,c,t)_L$	$(\tilde{d},\tilde{s},\tilde{b})_L$	$(\nu_\ell)_R$	$(\ell^-)_R$	$(u,c,t)_R$	$(d,s,b)_R$
Q	0	-1	$2/3$	$-1/3$	0	-1	$2/3$	$-1/3$
T_3	$1/2$	$-1/2$	$1/2$	$-1/2$	0	0	0	0
Y	-1	-1	$1/3$	$1/3$	0	-2	$4/3$	$-2/3$

Table 2: Matter fields and their $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ quantum numbers.

By $\tilde{q}$ we denoted the flavor eigenstate (often called weak eigenstate) fields, which correspond to the Cabibbo-Kobayashi-Maskawa (CKM) "rotated" mass eigenstate quark fields q: $\tilde{q}_d = U_{\rm CKM}\, q_d$ where $q_d = (d,s,b)$ is a horizontal vector in family space, i.e, one component in each of the 3 families [3]. Similarly, $\nu_e,\ \nu_\mu,\ \nu_\tau$ are the flavor eigenstate neutrino fields which are obtained by applying the Maki-Nakagawa-Sakata (MNS) neutrino mixing matrix (by convention in this case the lepton fields are kept fixed) $\nu_\ell = U_{\rm MNS}\, \nu_i$ to the mass eigenstate fields ν_i $(i = 1,2,3)$. Again, $\nu_\ell = (\nu_e, \nu_\mu, \nu_\tau)$ and ν_i are horizontal neutrino vectors [4]. For more on flavor mixing see below.

For the massless spin 1/2 gauge fields and the gauge couplings we will use the following notation,

group	fields		coupling
$SU(3)_c$:	$G_{\mu i}$	$i = 1, \cdots, 8$	g_s
$SU(2)_L$:	$W_{\mu a}$	$a = 1, 2, 3$	g
$U(1)_Y$:	B_μ		g' .

The pure gauge Yang-Mills Lagrangian is given by a sum of independent pieces from each group factor,

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} G_{\mu\nu i} G^{\mu\nu i} - \frac{1}{4} W_{\mu\nu a} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{4}$$

where

$$\begin{aligned} G_{\mu\nu i} &= \partial_\mu G_{\nu i} - \partial_\nu G_{\mu i} + i g_s f_{ijk} G_{\mu j} G_{\nu k} \\ W_{\mu\nu a} &= \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + i g \varepsilon_{abc} W_{\mu b} W_{\nu c} \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned}$$

are the non-abelian and abelian field strength tensors. The crucial consequence of non-abelian gauge invariance is that it requires the non-abelian fields to be self-interacting (they carry themselves non-abelian charge) and that the self-couplings are uniquely fixed once the couplings to the matter fields are determined. Thus one coupling constant determines three topologically different vertices (Fig. 1).

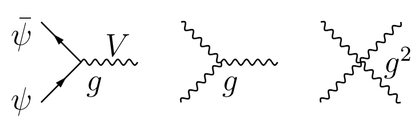


Figure 1: Interrelated interaction vertices of a gauge theory

In the following we will concentrate our considerations to the electroweak subgroup $SU(2)_L \otimes U(1)_Y$ which is broken in the real world to the electromagnetic abelian gauge group $U(1)_{\rm em}$known from QED.

The eigenstates of charge Q can be found easily. The W's have $Y = 0$ and hence $Q = T_3$, where T_3 denotes the 3rd component of weak isospin. The charge raising and lowering generators are obtained in the standard way. The shift operators

$$T_\pm = T_1 \mp i T_2 \; ; \; T_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; , \; T_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

satisfy the commutation relation

$$[T_3, T_\pm] = \pm T_\pm$$

and correspondingly the fields

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(W_{\mu 1} \mp i W_{\mu 2}) \tag{5}$$

carry charge ± 1. The fields $W_{\mu 3}$ and B_μ both have $Y = 0$ and $T_3 = 0$ and hence $Q = 0$ and thus can mix. The field which couples to the $Q = 0$ particle $\nu_{\ell L}$ we denote by Z_μ and the field orthogonal to it is the photon

$$\begin{aligned} Z_\mu &= \cos\Theta_W W_{\mu 3} - \sin\Theta_W B_\mu \\ A_\mu &= \cos\Theta_W B_\mu + \sin\Theta_W W_{\mu 3} \; . \end{aligned} \tag{6}$$

The weak mixing angle Θ_W is determined by $\tan\Theta_W = g'/g$ [3]. Note that due to this γ-Z mixing virtual γ's (light) and virtual Z's ("heavy light") are simultaneously produced in e^+e^- annihilation, for example. In terms of the physical fields, we may summarize the structure of the electroweak theory as follows:
The charged current (CC) has the form

$$J^+_\mu = J_{\mu 1} - i J_{\mu 2} = \bar{\nu}_\ell \gamma_\mu \left(1-\gamma_5\right) U_{\mathrm{MNS}}\, \ell + \bar{q}_u \gamma_\mu \left(1-\gamma_5\right) U_{\mathrm{CKM}}\, q_d \tag{7}$$

and exhibits quark flavor changing, through mixing by the unitary Cabibbo-Kobayashi-Maskawa matrix U_{CKM} as well as neutrino flavor mixing by the corresponding Maki-Nakagawa-Sakata matrix U_{MNS}. The $SU(2)_L$ currents have strict V-A (V=vector $[\gamma_\mu]$,A=axial-vector $[\gamma_\mu\gamma_5]$) form, which in particular implies that the CC is maximally parity (P) violating (Lee and Yang 1957). The mixing matrices exhibit a CP violating phase. The neutral current (NC) is strictly flavor conserving [5]

$$J^Z_\mu = J_{\mu 3} - 2\sin^2\Theta_W j^{em}_\mu = \sum_f \bar{\psi}_f \gamma_\mu (v_f - a_f \gamma_5) \psi_f \tag{8}$$

with

$$j^{em}_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f \tag{9}$$

the P conserving electromagnetic current. The sums extend over the individual fermion flavors f (and color). In our convention the NC vector and axial-vector neutral current coefficients are given by

$$v_f = T_{3f} - 2Q_f \sin^2\Theta_W \; , \quad a_f = T_{3f} \tag{10}$$

[3]Historically, the electroweak standard model gauge group has been introduced by Glashow in 1961. At that time only the charge changing weak currents J^+_μ and $J^-_\mu = (J^+_\mu)^\dagger$ were known. If one argues them to be the Noether currents which derive from a *symmetry*, where $SU(2)$ is the obvious candidate, the algebra of generators must be required to close

$$[T_+, T_-] = -2T_3 \; .$$

This implies that there must exist a neutral current associated with the 3rd generator T_3. Since the 3rd current cannot be identified with the electromagnetic current, $W_{\mu 3}$ cannot be identified with the photon and an extra abelian group factor was necessary in order to unify weak and electromagnetic interactions. In this way mixing and the weak mixing parameter $\sin^2\Theta_W$ was introduced.

where T_{3f} is the weak isospin ($\pm\frac{1}{2}$) of the fermion f[4]. The matter field Lagrangian thus takes the form

$$\mathcal{L}_{\text{matter}} = \sum_f \bar{\psi}_f i\gamma^\mu \partial_\mu \psi_f + \frac{g}{2\sqrt{2}}(J^+_\mu W^{\mu -} + h.c.) + \frac{g}{2\cos\theta_W} J^Z_\mu Z^\mu + e j^{em}_\mu A^\mu \qquad (11)$$

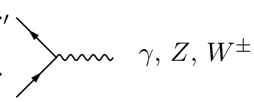
where $e = g \sin\Theta_W$ is the charge of the positron (unification condition). The discovery of the $W^\pm$ and Z bosons at the $p\bar{p}$ collider at CERN [6] directly confirmed these weak gauge boson couplings[5]. On the other hand the direct confirmation of the weak gauge boson self-interactions in the Yang-Mills part of the Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \frac{1}{4}(\partial_\mu W_{\nu i} - \partial_\nu W_{\mu i} + ig\varepsilon_{ikl} W_{\mu k} W_{\nu l})^2 \qquad (12)$$

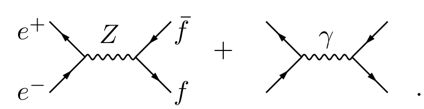
was possible only after 1996 at LEP-2 via the W-pair production process[6]. Phenomenologically we know that the $SU(2)_L \otimes U(1)_Y$ symmetry is broken by the mass

[4]Given the fermionic currents (7-9) belonging to the symmetry group $SU(2)_L \otimes U(1)_Y$ the SM is obtained as the *minimal renormalizable extension* of the low energy effective current-current type four-fermion weak interaction plus QED. That this simple "principle" would be so successful nobody really expected before the success story started after the proof of renormalizability by 't Hooft in 1972. Suddenly an electroweak theory was available with a plenitude of cosequences like neutral currents, non-Abelian gauge couplings, family structure (lepton - quark duality), Higgs sector, Yukawa pattern, mass generation by the Higgs mechanism, and last but not least the possibility to make precise predictions by including higher order quantum corrections.

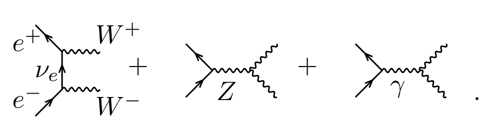
[5]Between 1934 when Fermi first introduced a chaged weak current and 1996 when LEP-2 started to investigate the Yang-Mills couplings, electroweak physics essentially was about determining the basic structure of the fermionic electroweak currents, which are characterized by the diagrams



and which have been establised by investigating processes like the ones displayed in Fig. 3 and by Z production and decay at LEP-1, dominated by the first of the two tree diagrams:



[6]W-pair production in e^+e^--annihilation above the W-pair threshold is dominated by the three tree diagrams:



The first diagram only exhibits the well established CC V-A interaction vertices, while the other two exhibit the new triple gauge couplings dictated by local gauge symmetry. A *direct* experimental test of the latter was possible only after the LEP upgrade 1996. The W's decay via the well-established pure V-A charged current interaction.

terms

$$\mathcal{L}_{\text{mass}} = -\sum_f m_f \bar{\psi}_f \psi_f + \frac{1}{2} M_Z^2 Z_\mu Z^\mu + \frac{1}{2} M_W^2 W_\mu^+ W^{-\mu} \tag{13}$$

of the physical particles. Since the mass terms are not $SU(2)_L \otimes U(1)_Y$ invariant this massive vector boson theory is not renormalizable.

One can apply the same principle and couple the "to be massive" fields invariantly to a scalar field which develops a non-vanishing vacuum expectation value. Since we must break the $SU(2)_L$, we need a scalar field which transforms non-trivially under this group. The simplest choice is to take a complex doublet with weak hypercharge $Y = 1$

$$\Phi_b = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} = \frac{1}{\sqrt{2}}(H_s + i\tau_i\phi_i) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\rho_H}{\sqrt{2}} e^{i\frac{\tau_i}{2}\theta_i} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

and its Y-charge conjugate $\Phi_t = i\tau_2\Phi_b^*$

$$\Phi_t = \begin{pmatrix} \phi_0^* \\ -\ \phi^- \end{pmatrix} = \frac{1}{\sqrt{2}}(H_s + i\tau_i\phi_i) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\rho_H}{\sqrt{2}} e^{i\frac{\tau_i}{2}\theta_i} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{15}$$

the charge being determined by $Q = T_3 + Y/2$. In order to write down the gauge invariant Lagrangian for the scalars we need the covariant derivative, which is given by

$$D_\mu \Phi_b = (\partial_\mu - i\frac{g'}{2} B_\mu - i\frac{g}{2}\tau_a W_{\mu a})\Phi_b \tag{16}$$

and the Higgs Lagrangian takes the form (requiring renormalizability)

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\Phi_b)^+ (D^\mu\Phi_b) - \lambda \left(\Phi_b^+\Phi_b\right)^2 + \mu^2 \left(\Phi_b^+\Phi_b\right) . \tag{17}$$

Since the fermion doublets and the Higgs doublets have identical $SU(2)$ transformation properties, and taking into account the hypercharge assignments, we can write down the following invariant Yukawa type couplings (discarding family mixing for the moment)

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} = &-\left(G^\nu\, \bar{L}_\ell \Phi_t \nu_{\ell R} + h.c.\right) - \left(G^\ell\, \bar{L}_\ell \Phi_b \ell_R + h.c.\right) \\ &-\left(G^u\, \bar{L}_q \Phi_t u_R + h.c.\right) - \left(G^d\, \bar{L}_q \Phi_b d_R + h.c.\right) \end{aligned} \tag{18}$$

where $L_\ell = \binom{\nu_\ell}{\ell}$ and $L_q = \binom{u}{d}$ denote the left-handed lepton and quark doublets, respectively, and G^f Yukawa couplings. We choose $\mu^2 > 0$, such that the Higgs potential has a non-trivial minimum at $< H_s >= v > 0$ which represents the actual *ground state (vacuum) in the broken phase. Here, H_s is the neutral scalar component of the Higgs doublet*

$$H_s = (\phi_0 + \phi_0^*)/\sqrt{2}$$

and $H = H_s - v$ is the physical Higgs field with vanishing vacuum expectation value $< H >= 0$.

Exploiting the invariance of all terms in the Lagrangian we notice that we can gauge away the fields θ_i in the polar representation given in Eqs. (14,15), since the exponential is a $SU(2)$-matrix. This means that three (θ_i) of the four scalar fields (ρ_H, θ_i) are in fact unphysical. They are called Higgs ghosts or "would be Goldstone bosons". The gauge for which the ghosts are absent is called unitary *or* physical *gauge. While $\mathcal{L}_{\rm matter}$ and $\mathcal{L}_{\rm YM}$ remain unaffected by a gauge transformation, $\mathcal{L}_{\rm Higgs}$ and $\mathcal{L}_{\rm Yukawa}$ take a special simple form, because*

$$\Phi_b = \frac{H+v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in this gauge (identifying $\rho_H = H + v$). One gets

$$\begin{aligned}
\mathcal{L}_{\rm Higgs} &= \frac{1}{2}(\partial_\mu H \partial^\mu H) + \frac{(H+v)^2}{2v^2}(M_Z^2 Z_\mu Z^\mu + 2M_W^2 W_\mu^+ W^{-\mu}) \\
&\quad -\frac{\lambda}{4}H^4 - \lambda v H^3 - \frac{1}{2}m_H^2 H^2 \\
\mathcal{L}_{\rm Yukawa} &= -\sum_f m_f \bar{\psi}_f \psi_f \; (1 + \frac{H}{v})
\end{aligned} \tag{19}$$

and thus, $\mathcal{L}_{\rm Higgs} + \mathcal{L}_{\rm Yukawa} = \mathcal{L}_{\rm mass} + \mathcal{L}_H$ with

$$\begin{aligned}
\mathcal{L}_H &= \frac{1}{2}(\partial H)^2 - \frac{1}{2}m_H^2 H^2 \\
&\quad -\sum_f \frac{m_f}{v}\bar{\psi}_f \psi_f H + \frac{M_Z^2}{v} Z_\mu Z^\mu H + \frac{2M_W^2}{v} W_\mu^+ W^{-\mu} H + \cdots
\end{aligned} \tag{20}$$

as an extra piece, which renders the theory renormalizable.

The Higgs sector is completely unverified so far and its confirmation is a big challenge for experimental particle physics. The proof of renormalizability by G. 't Hooft [8] rejuvenated particle physics about 35 years ago and preceded the first phenomenological success of the SM which was the discovery of the neutral currents [9] in 1973.

A basic consequence of the Higgs mechanism is the validity of the following mass-coupling relations. The vector boson masses are given by

$$M_W = \frac{gv}{2}\,, \quad M_Z = \frac{gv}{2\cos\Theta_W} \quad . \tag{21}$$

The fermion masses and the Higgs mass are given by similar relations

$$m_f = \frac{G_f}{\sqrt{2}}\, v \ , \ \ m_H = \sqrt{2\lambda}\, v \ \ . \tag{22}$$

in terms of the Yukawa couplings G_f and of the Higgs coupling λ. In the standard model , the μ-decay constant G_μ is given by

$$G_\mu = \frac{g^2}{4\sqrt{2}M_W^2} = \frac{1}{\sqrt{2}v^2} = 1.16637(1) \times 10^{-5} \ \ (\ GeV)^{-2} \tag{23}$$

and thus the Higgs vacuum expectation value

$$v = (\sqrt{2}G_\mu)^{-1/2} = 246.2206(11) \ \ GeV$$

is a very precisely known quantity, frequently called the Fermi scale, which figures as a conversion factor between couplings and masses. One important consequence is that the existence of heavy particles requires strong couplings and for too heavy particles this leads to a breakdown of perturbation theory. With other words, particles with masses large as compared to the Fermi scale are unnatural in the minimal SM. The non-decoupling of heavy particles is a new feature characteristic of a spontaneously broken gauge theory. In contrast, in QED and QCD, where the couplings α and α_s and lepton and quark masses enter the Lagrangian as independent parameters, heavy particles decouple as asserted by the Appelquist-Carazzone theorem [10]. It infers that heavy states of mass M are essentially without a trace, more precisely effects are $O(E/M)$ or smaller, in reactions at energies $E \ll M$.
If we take for granted the SM, we can say that the existence of the Higgs condensate has been established. Like in superconductivity the Higgs could, in fact, be composite. It is certainly a very interesting question, whether there is an underlying "BCS-theory" for the standard model . In any case, phenomenologically one expects the SM to work as a low energy effective theory at scales below 1 TeV.
On a formal level the role played by the Higgs mechanism is the following: It

- *breaks $SU(2)_L \otimes U(1)_Y$ to $U(1)_{\rm em}$,*
- *generates the masses of the weak gauge bosons $W^\pm$, Z and the fermions,*
- *provides a "physical cut-off" to the massive vector boson gauge theory.*

The prize we have to pay is that

- *a neutral physical particle H must exist.*

*The mass of the Higgs is a free unknown parameter. At present the direct 95% CL lower limit for m_H from LEP experiments is $m_H \gtrsim$ 114 GeV. Precision measurements of the weak mixing parameter $\sin^2\Theta^\ell_{\rm eff}$ allows to constrain the possible values for the SM to ([**?**])*

$$m_H = 113^{+62}_{-42} \quad GeV \ \ (\text{LEP\&SLD}). \tag{24}$$

The one-sided 95% CL upper bound is $m_H < 237$ GeV. The discovery of the Higgs is expected at the future large hadron collider LHC, which presently is under construction at CERN.

3. Yukawa couplings

3.1 Quark masses and mixings

The most general form for $SU(2)_L \otimes U(1)_Y$ invariant couplings between fermions and scalars follows from the following transformation properties of the fields

$$\begin{array}{lcll} \Psi_{Lf} \doteq L_f & \rightarrow & U(x) L_f & \text{fermion doublet} \\ \Phi_{b,t} & \rightarrow & U(x) \Phi_{b,t} & \text{Higgs doublet} \\ f_R & \rightarrow & f_R & \text{fermion singlet .} \end{array}$$

Since we insist in renormalizability, the most general invariant Higgs fermion interaction is a complex linear combination of terms are of the form

$$(L_f \Phi_b f_R)_{ij} = \bar{u}^i_L d^j_R \phi^+ + \bar{d}^i_L d^j_R \phi_0 \ , \ \ (L_f \Phi_t f_R)_{ij} = \bar{u}^i_L u^j_R \phi^*_0 - \bar{d}^i_L u^j_R \phi^-$$

and their hermitian conjugates. Here, i, j = 1, 2, 3 are family indices and the quantum numbers of the right-handed singlets are fixed by weak hypercharge neutrality. Since each family is made up of fields with identical $SU(2)_L \otimes U(1)_Y$ transformation laws invariant Yukawa couplings are possible for combinations of fields from different families ($i \neq j$). The complete Yukawa Lagrangian for the quarks is then

$$\mathcal{L}^q_{\text{Yukawa}} \ = \ -\sum_{i,j=1}^{3} \left[G^u_{ij} \bar{L}_{qi} \Phi_t u_{jR} + G^d_{ij} \bar{L}_{qi} \Phi_b d_{jR} + \text{h.c.} \right] \qquad (25)$$

with G^u_{ij} and G^d_{ij} arbitrary complex 3 x 3 matrices.

With the fields having identical $SU(2)_L \otimes U(1)_Y$ quantum numbers one can form horizontal vectors. *For the quarks there are the 4 horizontal vectors $q_{uL}, q_{dL}, q_{uR}, q_{dR}$ where $q_u = (u, c, t)$ and $q_d = (d, s, b)$.*

In order to transform the fermion mass matrix (obtained by replacing $\phi^*_0 = \phi_0 = v/\sqrt{2}$, $\phi^+ = \phi^- = 0$) to diagonal form we must perform independent global unitary transformations of the 4 horizontal vectors. Whereas,

- *unitary transformations of $(q_u, q_d)_L$ as a doublet, q_{uR} and q_{dR}* do not change the *matter field Lagrangian,*
- *an independent transformation of q_{dL} leads to "mismatch"*

$$\tilde{q}_{dL} = U_{\text{CKM}}\, q_{dL}$$

of the quark fields in the charged current.

This leads us to the following form of the hadronic charged current

$$J_\mu^{CC} = (\bar{u}, \bar{c}, \bar{t})\, \gamma_\mu(1-\gamma_5)\, U_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{26}$$

given in Eq. (7) with the unitary 3×3 *matrix*

$$U_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{27}$$

which may be parametrized in terms of 3 rotation angles and a phase:

$$U_{\rm CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} .$$

where $c_{ij} = \cos\theta_{ij}$ *and* $s_{ij} = \sin\theta_{ij}$ *with* i *and* j *being the family labels. Without loss of generality one may assume all* c_{ij} *and* s_{ij} *to be positive and the phase* δ_{13} *to lie in the range* $0 \le \delta_{13} < 2\pi$[7].

This family mixing occurs if 4 independent unitary transformations are required to diagonalize the mass matrix, and this is the case if particles of the same charge all have different masses. This happens to be so for the quarks.

Due to unitarity, there is no mixing effect in the neutral current, since

$$\bar{\tilde{q}}_{dL}\tilde{q}_{dL} \equiv \bar{q}_{dL}q_{dL} \ .$$

This is called the GIM-mechanism explaining the absence of flavor-changing neutral currents (FCNC). In fact, in order to explain the absence of FCNC's, Glashow, Iliopoulos and Maiani had to propose, in 1970, the existence of a fourth quark, the charm quark c *as a doublet partner of the* s *quark. At that time only three quarks where known [5].*

The discovery of the J/ψ *in 1974 [11] revealed the completeness of the 2nd family with the charm quark* c*. The first 3rd family member showed up in 1975 with the discovery of the* τ *[12]. With the observation of the* Υ *[13] 1977 the existence of the bottom quark* b *could be established. After LEP experiments had established indirect constraints on the top quark mass (*$m_t = 166\ ^{+17}_{-19}\ ^{+19}_{-22}$ *GeV* (LEP 1993) *assuming* 60 *GeV* $< m_H <$ 1 *TeV) [14] direct evidence for the existence of the top quark*

[7]Present experimental results for the sines of the angles are $s_{12} = 0.2229 \pm 0.0022$ (sine of the Cabibbo angle), $s_{23} = 0.0412 \pm 0.0020$ and $s_{13} = 0.0036 \pm 0.0007$. The CKM phase corresponds to the angle $\gamma = \phi_3$ of the unitarity triangle and is restricted to $\delta_{13} = (1.02 \pm 0.22)$ radians $= 59^\circ \pm 13^\circ$. The pronounced hierarchy $s_{12} \gg s_{23} \gg s_{13}$ together with the fact that V_{ud} is close to unity allows us to write $V_{ud} \simeq c_{12},\ V_{us} \simeq s_{12},\ V_{ub} \simeq s_{13}\, e^{-i\delta_{13}},\ V_{cb} \simeq s_{23}$, and $V_{tb} \simeq c_{23}$ to a very good approximation.

has been found at the Tevatron in 1995 [15]. Thus all leptons and quarks for three complete families have been found. The mass of the top quark t turned out to be unexpectedly large [16]

$$m_t = 178.0 \pm 2.7 \pm 3.3 \quad GeV \;\; \text{(D0\&CDF)}. \tag{28}$$

First hints for a large top quark mass came from the discovery of the $B^0 - \bar{B}^0$ *oscillations in 1987 by ARGUS at DESY [17].*
We summarize the following **important consequences:**

- *i) all masses of quarks and leptons are independent*[8]
- *ii) the coupling of the Higgs boson to the fermions is universally proportional to each fermion mass, for bosons proportional to the square of each boson mass*
- *iii) there is quark flavor violation in charge exchange weak interactions*
- *iv) the phase in* U_{CKM} *is CP-violating and thus potentially capable of explaining the observed CP-violation in K-decays*[9] *(Cronin and Fitch 1964). At least 3 families are needed to "explain" CP-violation in this way. Recently CP violation in the B-system was found to be precisely as predicted by CKM mixing (BABAR at SLAC, Belle at KEK 2001):* $\sin(2\phi_1) = 0.78 \pm 0.08$ *where* $\phi_1 = \arg(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*})$.
- *v) flavor is conserved in neutral currents (GIM mechanism). This is strikingly supported by experiment, at least for the light flavors.*

Empirically the CKM matrix elements may be expanded in $\lambda = \sin\theta_{\mathrm{Cabibbo}} \simeq 0.22$ *with the following approximate sizes of the elements*

$$|V| \sim \left(\begin{array}{ccc} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{array} \right) .$$

[8] Leptons masses are $m_e = 0.510998902 \pm 0.000000021$ MeV, $m\mu = 105.6583568 \pm 0.0000052$ MeV, $m_\tau = 1776.99^{+0.29}_{-0.26}$ MeV. Current quark masses for the light quarks evaluated in the $\overline{\mathrm{MS}}$ scheme at the scale $\mu = 2$ GeV are: $m_u \approx 1.5 - 4.5$ MeV, $m_d \approx 5.0 - 8.5$ MeV, $m_s \approx 80 - 155$ MeV. The c-quark mass is estimated from charmonium and D masses. The "running" mass in the $\overline{\mathrm{MS}}$ scheme is $m_c(\mu = m_c) = 1.290^{+0.040}_{-0.045}$. The range $1.0 - 1.4$ GeV for the $\overline{\mathrm{MS}}$ mass corresponds to $1.47 - 1.83$ GeV for the pole mass [converted by two-loop perturbative QCD with $\alpha_s(\mu = m_c) = 0.39$.]. The b-quark mass is estimated from bottomonium and B masses. The "running" mass in the $\overline{\mathrm{MS}}$ scheme is $m_b(\mu = m_b) = 4.206 \pm 0.031$. The range $4.0 - 4.5$ GeV for the $\overline{\mathrm{MS}}$ mass corresponds to $4.6 - 5.1$ GeV for the pole mass [converted by two-loop perturbative QCD with $\alpha_s(\mu = m_b) = 0.22$.]. The top mass given above is the pole mass.

[9] The particle-antiparticle mixing of the neutral kaons $K^0 \leftrightarrow \bar{K}^0$ (Gell-Mann und Pais 1955) played a key role in revealing CP violation as an observable effect. In the B-meson system $B^0 \leftrightarrow \bar{B}^0$ mixing plays an analogous role.

The traditional parametrization reads

$$V_{\text{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3\,(\rho - \mathrm{i}\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3\,(1-\rho-\mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

with

$$s_{12} = \lambda = \frac{|V_{us}}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}\ , \quad s_{23} = A\lambda^2 = \lambda\left|\frac{V_{cb}}{V_{us}}\right|\ ,$$

$$s_{13}\mathrm{e}^{\mathrm{i}\delta} = V_{ub}^* = A\lambda^3\,(\rho + \mathrm{i}\eta) = \frac{A\lambda^3\,(\bar\rho + \mathrm{i}\bar\eta)\sqrt{1-A^2\lambda^4}}{\sqrt{1-\lambda^2}\,[1 - A^2\lambda^4\,(\bar\rho + \mathrm{i}\bar\eta)]}$$

with $\bar\rho + \mathrm{i}\bar\eta = -(V_{ud}V_{ub}^*)/(V_{cd}V_{cb}^*)$ *phase convention independent. Recent global CKM fits yield*

$$\lambda = 0.22537 \pm 0.00061\ , \quad A = 0.814^{+0.023}_{-0.024}\,,$$
$$\bar\rho\ = 0.117 \pm 0.021\ , \quad \bar\eta\ = 0.353 \pm 0.013\ .$$

The magnitudes of all nine CKM elements are

$$V_{\text{CKM}} = \begin{pmatrix} 0.97427 \pm 0.00014 & 0.22536 \pm 0.00061 & 0.00355 \pm 0.00015 \\ 0.22522 \pm 0.00061 & 0.97343 \pm 0.00015 & 0.0414 \pm 0.0012 \\ 0.00886^{+0.00033}_{-0.00032} & 0.0405^{+0.0011}_{-0.0012} & 0.99914 \pm 0.00005 \end{pmatrix}$$

The corresponding quark decay pattern is illustrated in the following diagram:

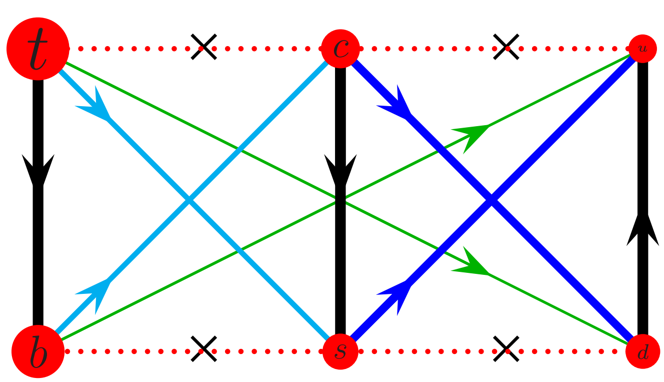


Figure 2: The CKM mixing hierarchy (29). FCNCs at tree level are forbidden [X].

Note: the u *quark is stable, the* s *and* b *quarks are metastable. Flavor changing neutral current transitions are allowed only as second (or higher) order transitions:*

e.g. $b \to s$ *is in fact* $b \to (t^*, c^*, u^*) \to s$, *where the asterix indicates "virtual transition".*

3.2 Neutrino masses and mixings

We notice that, according to Tab. 2, the right-handed neutrinos, if they exist, are sterile. *They do not carry any gauge charge and hence, in contrast to all other particles, do not interact via the spin 1 gauge bosons with the rest of the world. Hence, the right-handed neutrinos could be absent altogether. This would imply the leptonic CC to exhibit some very special properties: if* $\nu_{\ell R}$ *would not exist, then* $m_{\nu_\ell} = 0$ *and lepton numbers* L_ℓ *would be strictly conserved individually for each flavor* $\ell = e, \mu, \tau$. *For a long time this seemed to be supported by experiments. Today we know that this is true approximately only, although lepton-number violating processes such as* $\mu \to e\gamma$ *still are expected to have extremely small probability (see below). The observed neutrino mixing implies that neutrino masses must be non-vanishing and non-degenerate. Indeed, for small neutrino masses, the lepton-number violation is expected to be seen first in neutrino oscillations, which have been subject of extensive experimental searches (Davis since 1968, ... , Super-Kamiokande 1998-2001, SNO 2002, KamLAND 2003)*[10] *[**?**].*

It remains to be understood why neutrino masses are so small. One of the crucial questions which remains to be answered is whether the right-handed neutrinos are their own antiparticles (Majorana neutrinos). Note that only the sterile neutrinos may be Majorana particles and if so they would have their own bare mass terms. The latter typically would be expected to be large, since there is no symmetry constraining it in the unbroken phase of the SM.

Writing down all terms allowed by the SM gauge group. To a large extent things ecactly look like for the quarks, with the exception that now the right-handed neutrinos are gauge singlets:

$$\mathcal{L}^{(\nu\ell)}_{\mathrm{Yukawa}} = -\sum_{i,j=1}^{3} \left[G^\nu_{ij} \bar{L}_{\ell i} \Phi_t \nu_{jR} + G^\ell_{ij} \bar{L}_{\ell i} \Phi_b \ell_{jR} + h.c.\right] \tag{29}$$

with G^ν_{ij} *and* G^ℓ_{ij} *arbitrary complex 3 x 3 matrices. The mass-matrix obtained by*

[10]Present results may be summarized as follows: **a)** Solar neutrinos: $\Delta m^2_{12} \approx (7) \times 10^{-5}$ eV^2, $\tan^2\Theta_{12} \approx 0.4$ (large $\nu_e \leftrightarrow \nu_\mu$ mixing), $\sin^2 2\Theta_{13} < 0.067$. **b)** Atmospheric neutrinos: $\Delta m^2_{23} \approx (1.3 - 3.0) \times 10^{-3}$ eV^2, $\sin^2 2\Theta_{23} > 0.9$ (large angle $\nu_\mu \leftrightarrow \nu_\tau$ mixing). Main features are:
- smallness of ν masses: $m_\nu < 1 - 2$ eV, at least for one mass $m_\nu > \sqrt{\Delta m^2_{23}} > 0.04$ eV,
- hierarchy of Δm^2's : $|\Delta m^2_{12}/\Delta m^2_{23}| = 0.01 - 0.15$,
- no strong hierarchy of masses: $|m_2/m_3| > |\Delta m_{12}/\Delta m_{23}| = 0.18^{+0.22}_{-0.08}$,
- bi-large or maximal mixing between neighboring families (1-2) and (2-3),
- small mixing between remote families (1-3),

in any case $m_\nu \ll m_\ell, m_q$.

inserting the Higgs VEVs $\Phi_b = \frac{v}{\sqrt{2}}\binom{0}{1}$ *and* $\Phi_t = \frac{v}{\sqrt{2}}\binom{1}{0}$ *yields the Dirac mass terms*

$$\mathcal{L}^{(\nu\ell)\,\mathrm{D}}_{\mathrm{mass}} = -\sum_{i,j=1}^{3}\left[m^{\mathrm{D}}_{\nu ij}\bar{\nu}_{iL}\nu_{jR} + m^{\mathrm{D}}_{\ell ij}\bar{\ell}_{iL}\ell_{jR} + \mathrm{h.c.}\right] = \mathcal{L}^{\nu\,\mathrm{D}}_{\mathrm{mass}} + \mathcal{L}^{\ell\,\mathrm{D}}_{\mathrm{mass}}$$

with

$$m^{\mathrm{D}}_{\cdot ij} = \frac{v}{\sqrt{2}}G^{\cdot}_{ij} \quad .$$

These mass matrices may be diagonalized in the same manner as for the quarks. Consider first the lepton mass term

$$\mathcal{L}^{\ell\,\mathrm{D}}_{\mathrm{mass}} - \sum_{i,j=1}^{3}\left[m^{\mathrm{D}}_{\ell ij}\bar{\ell}_{iL}\ell_{jR} + \mathrm{h.c.}\right] = -\overline{\ell_L}\mathcal{M}_\ell\ell_R + \mathrm{h.c.} = -\overline{\ell^0}\mathcal{M}_\ell\ell^0 = -\bar{\ell}\mathcal{D}_\ell\ell$$

where $\ell = \ell_R + \ell_L$. *The mass matrix may be diagonalized by two distinct global unitary transformations*

$$\ell^0_{L,R} = A_{L,R}\ell_{L,R}$$

$$A_L^+\mathcal{M}_\ell A_R = \mathcal{D}_\ell = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} .$$

For the neutrinos the situation is more complex. Besides the Dirac mass term

$$\mathcal{L}^{\nu\,\mathrm{D}}_{\mathrm{mass}} = -\sum_{i,j=1}^{3}\left[m^{\mathrm{D}}_{\nu ij}\bar{\nu}_{iL}\nu_{jR} + \mathrm{h.c.}\right] = -\overline{\nu}_L\mathcal{M}^{\mathcal{D}}_\nu\nu_R + \mathrm{h.c.}$$

because the right-handed neutrinos ν_R *are gauge singlets (sterile) a Majorana mass term is admitted by the SM local gauge symmetry in the unbroken phase already*

$$\mathcal{L}^{\nu\,\mathrm{M}}_{\mathrm{mass}} = -\frac{1}{2}\sum_{i,j=1}^{3}\left[m^{\mathrm{M}}_{\nu ij}\overline{(\nu_{iR})^c}\nu_{jR} + \mathrm{h.c.}\right] = -\frac{1}{2}\overline{\nu^c_R}\mathcal{M}^{\mathcal{M}}_\nu\nu_R + \mathrm{h.c.}$$

The factor 1/2 accounts for the self-conjugacy $\nu^c_R = \eta\nu_R$ *(*η *a phase) of the Majorana neutrino as it is familiar from real scalar fields, for example*[11]. *Since no symmetry is protecting* m^M_ν *from being large it is natural to expect* $m^M_\nu \gg m^D_\nu,\, m^D_\ell$. *We may write the two 3 x 3 mass terms as one 6 x 6 one as follows:*

$$\mathcal{L}^{\nu}_{\mathrm{mass}} = -\frac{1}{2}\begin{pmatrix} \overline{(\nu^0_L)^c}, & \overline{\nu^0_R} \end{pmatrix}\begin{pmatrix} 0 & m^{D+} \\ m^D & m^M \end{pmatrix}\begin{pmatrix} \nu^0_L \\ (\nu^0_R)^c \end{pmatrix} + h.c.$$

[11]Under charge conjugation we have $\psi_{R(L)} \overset{C}{\to} \psi^c_{L(R)} = i\gamma^2\psi^*_{R(L)}$. A Majorana field satisfies $\psi^c_{L(R)} = i\gamma^2\psi^*_{R(L)} = \psi_{R(L)}$ up to a phase.

Upon diagonalizing the lepton and neutrino mass matrices the neutrino mixing matrix U_{MNS} is resulting in the leptonic charged current (7). It transforms the horizontal neutrino column vectors like $\nu_\ell = (U_{\text{MNS}})_{\ell j}\nu_j$ with $\ell = e, \mu, \tau$ labelling the weak eigenstates and $j = 1, 2, 3$ labelling the mass eigenstates. In general it exhibits 3 angles, 1 CP phase plus 2 Majorana phases[12] and may be written in the form $U_{\ell j}^{\text{Majorana}} = U_{\ell j}^{\text{Dirac}} \times \mathcal{A}$ where the diagonal matrix

$$\mathcal{A} = \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

accounts for the Majorana phases. The Dirac neutrino mixing matrix may be written as a product $U_{\text{MNS}}^{\text{D}} = R_{23}R_{13}R_{12}$ of three rotations R_{ij} in the planes (ij):

$$U_{\text{MNS}}^{\text{D}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{atmospheric}} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{+i\delta} & 0 & c_{13} \end{pmatrix} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{solar}}. \quad (30)$$

En bloc the MNS matrix for Majorana neutrinos thus reads

$$U_{\text{MNS}}^{M} = \begin{pmatrix} c_{12}c_{13}\, e^{i\frac{\alpha_1}{2}} & s_{12}c_{13}\, e^{i\frac{\alpha_2}{2}} & s_{13}\, e^{-i\delta_{13}} \\ (-s_{12}c_{23} - c_{12}s_{23}s_{13}\, e^{i\delta_{13}})\, e^{i\frac{\alpha_1}{2}} & (\; c_{12}c_{23} - s_{12}s_{23}s_{13}\, e^{i\delta_{13}})\, e^{i\frac{\alpha_2}{2}} & s_{23}c_{13} \\ (\; s_{12}c_{23} - c_{12}s_{23}s_{13}\, e^{i\delta_{13}})\, e^{i\frac{\alpha_1}{2}} & (-c_{12}s_{23} - s_{12}c_{23}s_{13}\, e^{i\delta_{13}})\, e^{i\frac{\alpha_2}{2}} & c_{23}c_{13} \end{pmatrix}.$$

xxxx

4. Experimental limits on the structure of weak currents

The properties of the weak currents have been established in a long history which started with Fermi in 1934. Here, we only mention some more recent of the fundamental experimental tests [20]:

- *V-A structure of the CC:*
  μ-decay provides the most sensitive clean direct tests for right-handed currents (e.g. $SU(2)_R \otimes SU(2)_L \otimes U(1)_{B-L}$ extension of the SM). The best limit for the transition amplitude is

$$\frac{A_{V+A}}{A_{V-A}} < 0.029 \quad (90\% CL)$$

[12]Considering N families, the unitary $N \times N$ mixing matrix has $2N^2 - N^2 = N^2$ free parameters. The N lepton and N neutrino fields multiplying the MNS matrix exhibit $2N$ free unobservable phases provided the neutrinos are of Dirac type. Only a change of the $2N-1$ relative phases affect the U_{MNS} matrix elements. Therefore only $N^2 - 2N + 1$ parameters are physical. The number of possible mixing angles is given by $N(N-1)/2$, the number of different pairs of fields. Thus the number of physical phases is: $N^2 - 2N + 1 - N(N-1)/2 = (N-1)(N-2)/2$. If the neutrinos are of Majorana type and hence self-conjugate fermions their phases are not arbitrary. In fact the Majorana phases are observable which means that there are only N free phases instead of $2N-1$ and thus for Majorana neutrinos we have $N-1$ extra phases.

- *absence of flavor-changing NC at tree level:*

$$\Gamma(K_L \to \mu^+\mu^-)/\Gamma(K_L \to all) = (7.15 \pm 0.16) \times 10^{-9}$$

puts limits on such interactions. The nonzero value for this rates is attributed to a combination of weak and electromagnetic interactions. Limits for charm-changing or bottom-changing NC's are much less stringent:

$$\begin{array}{lcl} \Gamma(D^0 \to \mu^+\mu^-)/\Gamma(D^0 \to all) & < & 4 \times 10^{-6} \\ \Gamma(B^0 \to e^+e^-)/\Gamma(B^0 \to all) & < & 6 \times 10^{-7} \end{array}$$

FCNC effects can be isolated in decays of hardons into lepton pairs only. In non leptonic decays there are equivalent CC transitions competing and FCNC cannot be isolated. Flavor-changing NC processes are allowed in higher orders (rare processes). The best test is expected from $K^+ \to \pi^+\nu\bar{\nu}$, which in the SM is a second order weak process with a branching fraction of (0.4 to 1.2) $\times 10^{-10}$. Present experimental result: $\Gamma(K^+ \to \pi^+\nu\bar{\nu})/\Gamma(K^+ \to all) = (1.6^{+1.8}_{-0.8}) \times 10^{-10}$

- *special properties of the lepton current:*
Present direct limits on the neutrino masses are:

$$\begin{array}{lcl} m_{\nu_e} < & 3 \text{ eV} & (\text{from } {}^3H \to {}^3He\, e^-\, \bar{\nu}_e) \\ m_{\nu_\mu} < & 170 \text{ keV} & (\text{from } \pi \to \mu\, \nu_\mu) \\ m_{\nu_\tau} < & 18 \text{ MeV} & (\text{from } \tau^- \to 3\pi\, \nu_\tau) \end{array}$$

The observed neutrino mixing (see below) requires nonzero and non degenerate small neutrino masses. L_ℓ conservation is tested by the branching fractions:

$$\begin{array}{lcl} R_{\mu\to e\gamma} < & 1.2 \times 10^{-11} & (\text{from } \mu \to e\gamma) \\ R_{\mu\to 3e} < & 1.0 \times 10^{-12} & (\text{from } \mu \to 3e) \end{array}$$

in purely leptonic processes. For semileptonic transitions the best limits come from conversion of muonic atoms $\mu^- + (Z, A) \to e^- + (Z, A)$ measured as

$$\Gamma(\mu^-\text{Ti} \to e^-\text{Ti})/\Gamma(\mu^-\text{Ti} \to \text{all}) < 4 \times 10^{-12}$$

For transitions of the type $\mu^- + (Z, A) \to e^+ + (Z - 2, A)$ the best limit is

$$\Gamma(\mu^-\text{Ti} \to e^+\text{Ca})/\Gamma(\mu^-\text{Ti} \to \text{all}) < 3.6 \times 10^{-11}$$

Majorana neutrinos, if they exist, would imply $\Delta L = 2$ transitions like the neutrinoless double beta decay $(Z, A) \to (Z+2, A) + e^- + e^-$. The best limit is

$t_{1/2} > 1.9 \times 10^{25}$ yr *(CL=90%) for ${}^{76}Ge$.*

Neutrino mixing (ν-oscillations $\nu_\ell \leftrightarrow \nu_{\ell'}$) has been established beyond doubts in the last few years (see below).

Open problems are the measurements of direct CP-violation (ε') in the K-meson system and CP-violation in the B-meson system [21]. We still do not know whether CP-violation is a phenomenon which has its "origin" in the CKM-phase solely, or if it's due to a new super-weak interaction outside the SM. Still unsolved is the solar neutrino problem [24]. The observed solar ν_e flux is too low. This could signal flavor mixing (causing conversion of ν_e into $\nu_{\mu,\tau}$ not visible to present detectors) of the neutrinos which is possible only if the neutrinos have different masses. Another possibility would be that the ν_e is unstable.
In summary: no deviations from the SM could be established until now.

5. Fixing the parameters of the SM

Besides the fermion masses, the CKM-mixing parameters and the Higgs mass the SM has 3 basic parameters g, g' and v. They are conventionally replaced by parameters which can be measured directly in a physical process. A specific choice of experimental data points as input parameters defines a renormalization scheme. *Like in QED a natural choice would be the fine structure constant and the physical particle masses (*on-shell scheme*):*

$$\alpha, M_W, M_Z, m_f, m_H.$$

Since M_W will not be known accurately at LEP1 we must use the precisely known μ-decay constant G_μ in place of M_W. Thus, we will use the parameter set

$$\alpha, G_\mu, M_Z, m_f, m_H$$

for accurate predictions of measurable quantities. In the pre-LEP era when M_Z was not known or known with rather limited accuracy from the $p\bar{p}$-collider, instead of M_Z the weak mixing parameter $\sin^2\Theta_W$ had to be used. For a study of low energy processes, this is still the adequate choice

$$\alpha, G_\mu, \sin_{\nu_\mu N(e)}, m_f, m_H \ .$$

The universal fine structure constant $\alpha = e^2/4\pi = 1/137.0359895(61)$ (determined in low momentum transfer Coulomb scattering), the Fermi constant G_μ (from the muon decay rate) and the weak mixing parameter $\sin^2\Theta_{\nu_\mu N(e)}$ (from low momentum transfer neutrino scattering).

We first discuss the relation between the different parameter sets. The low energy four-fermion processes are described by the effective Fermi-type Lagrangian

$$\mathcal{L}_{\rm eff} = -\frac{1}{\sqrt{2}}\left(G_\mu J^+_\mu J^{\mu -} + G_{NC} J^Z_\mu J^{\mu Z}\right) + e j^{em}_\mu A_\mu \tag{31}$$

which is the low energy effective form ($|q^2| \ll M_W^2, M_Z^2$) of

$$\mathcal{L}_{\rm int} = \frac{g}{2\sqrt{2}}\left(J^+_\mu W^{\mu -} + h.c.\right) + \frac{g}{2\cos\Theta_W} J^Z_\mu Z^\mu + e j^{em}_\mu A_\mu \ . \tag{32}$$

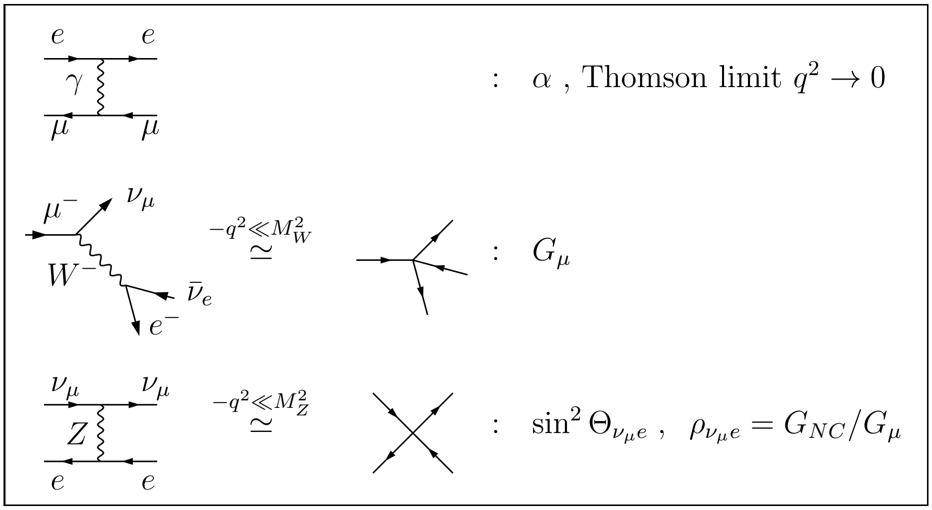


Figure 3: Parameters from low energy four-fermion processes

The electroweak unification condition and the parameter relations deriving from the processes shown in Fig. 3 read

$$\begin{array}{rrcccl} i) & \sqrt{4\pi\alpha} & = & e & = & g\sin\Theta_W \\ ii) & \sqrt{2}G_\mu & = & \frac{g^2}{4M_W^2} & = & \frac{1}{v^2} \\ & \sqrt{2}G_{NC} & = & \frac{g^2}{4M_Z^2\cos^2\Theta_W} & = & \rho_0\frac{1}{v^2} \\ iii) & \rho_0 & = & \frac{G_{NC}}{G_\mu} & = & \frac{M_W^2}{M_Z^2\cos^2\Theta_W} \equiv \rho_{tree} \end{array} \tag{33}$$

For the moment we have relaxed from the assumption $\rho_0 = 1$ valid in the minimal SM.

From the parameter relations we now obtain the tree level relation

$$\begin{aligned} \pi\alpha = \frac{e^2}{4} &\overset{i)}{=} \frac{g^2\sin^2\Theta_W}{4} \\ &\overset{ii)}{=} \sqrt{2}G_\mu M_W^2\sin^2\Theta_W \\ &\overset{iii)}{=} \sqrt{2}G_\mu M_W^2\left(1-\frac{M_W^2}{\rho_0 M_Z^2}\right). \end{aligned}$$

If radiative corrections are included this relation is modified into [31]

$$\sqrt{2}G_\mu M_W^2\left(1-\frac{M_W^2}{\rho_0 M_Z^2}\right) = \pi\frac{\alpha}{1-\Delta r}. \tag{34}$$

which is the defining equation for Δr (with ρ_0 kept fixed at its tree level value!). In the following we take $\rho_0 = 1$, as appropriate for doublet Higgses, such that by the last relation of Eq. (33)

$$\sin^2\Theta_W = 1 - \frac{M_W^2}{M_Z^2}. \tag{35}$$

The definition of Δr by Eq. (34) is conceptually very simple, all quantities involved have been measured and can be found in the particle data booklet.

Later, we will often use α and the physical particle masses as a convenient set of independent parameters. The Fermi constant is then a calculable quantity (μ-decay amplitude). Originally, the μ life-time τ_μ has been calculated within the framework of the effective four-point Fermi interaction. If one includes as usual the QED corrections (Fig. 4 displays the $O(\alpha)$ diagrams) to $O(\alpha^2)$ one obtains the result [18, 19]

$$\frac{1}{\tau_\mu} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} F(\frac{m_e^2}{m_\mu^2}) \left(1 + \frac{3}{5}\frac{m_\mu^2}{M_W^2}\right) \left[1 + \frac{\alpha(m_\mu)}{\pi}(\frac{25}{8} - \frac{\pi^2}{2}) + \frac{\alpha^2(m_\mu)}{\pi^2} C_2\right] . \qquad (36)$$

where

$$F(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x \, ,$$

$$C_2 = \frac{156815}{5184} - \frac{518}{81}\pi^2 - \frac{895}{36}\zeta(3) + \frac{67}{720}\pi^4 + \frac{53}{6}\pi^2 \ln 2 \, ,$$

and

$$\alpha(m_\mu)^{-1} = \alpha^{-1} \ - \frac{2}{3\pi} \ln \frac{m_\mu}{m_e} + \frac{1}{6\pi} \, .$$

This formula is used as the defining equation for G_μ in terms of the experimental μ life-time. Present data [20] yield the value given above. The Z-mass has been determined very accurately at LEP-1 [32]

$$M_Z = 91.1875 \pm 0.0021 \quad GeV \qquad (37)$$

*while the W mass we know from the $p\bar{p}$ colliders (experiments $UA2$ [33], CDF [34] and $D0$ [**?**]) and from LEP-2 [32]. Using their determination of the mass ration M_W/M_Z, for which common systematic errors drop out, together with the Z mass from LEP-1 one obtains*

$$M_W = 80.425 \ \ \pm 0.034 \qquad GeV \, . \qquad (38)$$

The various measurements of $\sin^2\Theta_W$ are collected in Tab. 1.

Table 1. $\sin^2\Theta_W$ measurements in NC processes [14,19,20,18]

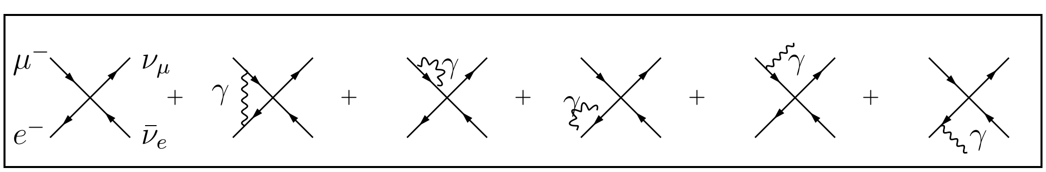


Figure 4: μ decay with $O(\alpha)$ QED corrections in the effective Fermi model

Measurement	$\sin^2\Theta_W$				
$\frac{M_W}{M_Z}$ $(p\bar{p})$	*0.2265*	$\pm$	*0.0062*		*(ave.)*
UA2	*0.2202*	$\pm$	*0.0084*	$\pm$	*0.0045*
CDF	*0.229*	$\pm$	*0.016*	$\pm$	*0.002*
$\left(\frac{\sigma_{NC}}{\sigma_{CC}}\right)_{\nu_\mu N}$	*0.232*	$\pm$	*0.006*		*(ave.)*
CDHS	*0.2275*	$\pm$	*0.005*	$\pm$	*0.005*
CHARM	*0.236*	$\pm$	*0.005*	$\pm$	*0.005*
P. V. in Cs	*0.215*	$\pm$	*0.007*	$\pm$	$[0.017]^{th}$
e^-D *(SLAC)*	*0.217*	$\pm$	*0.015*	$\pm$	$[0.013]^{th}$
$R_{\nu_\mu e} = \frac{\sigma_{\nu_\mu e}}{\sigma_{\bar{\nu}_\mu e}}$ *CHARM II*	*0.240*	$\pm$	*0.009*	$\pm$	*0.008*
assume $m_t = 140 \pm 40$ $\rightarrow$	*0.230*	$\pm$	*0.016*		
Γ_ℓ, A^ℓ_{FB} *LEP*	*0.2302*	$\pm$	*0.0025*		
assume $m_t = 140 \pm 40$ $\rightarrow$	*0.220*	$\pm$	*0.006*		

Assuming $\rho_{tree} = 1$, *as required by the minimal SM, recent global fits yield for the weak mixing angle and the top mass* ($68\%C.L.$)

$$\begin{array}{llll} \sin^2\Theta_W = 0.2273 \pm 0.0033 & , \quad m_t = 122^{+41}_{-32} & GeV & Ref.\ [35] \\ \sin^2\Theta_W = 0.2272 \pm 0.0040 & , \quad m_t = 139^{+33}_{-39} \pm 16 & GeV & Ref.\ [36] \\ \sin^2\Theta_e\ = 0.2325 \pm 0.0015 & , \quad m_t = 127 \pm 34 \pm 17 & GeV & Ref.\ [32] \end{array} \tag{39}$$

when $40\ \ GeV < m_H < 1\ \ TeV$.

A very important parameter in electroweak theory is the ρ*-parameter, defined by the neutral to charged current ratio at low energy. The* νN *scattering data yield the most sensitive determination of the* ρ*-parameter.*

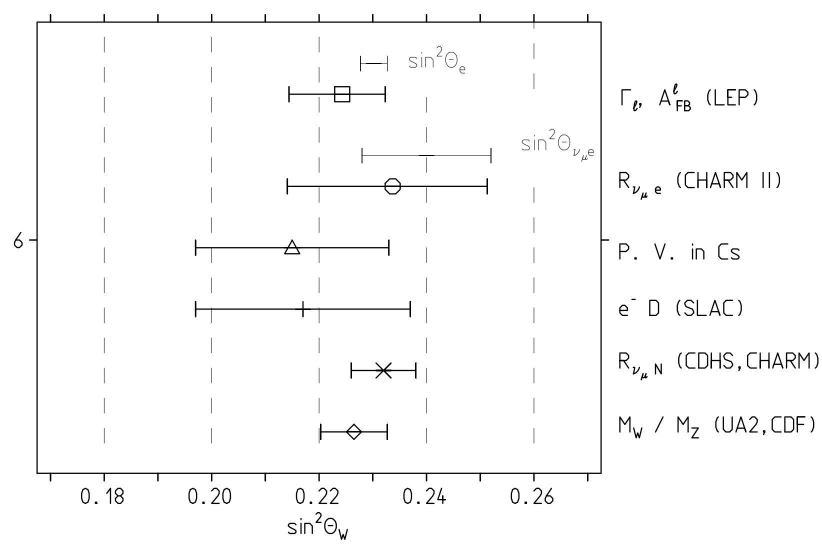


Figure 5: Comparison of various $\sin^2\Theta$ *measurements.*

Taking ρ and $\sin^2\Theta_W$ as independent parameters, a recent global fit to all NC-data [36] yields (the values indicated with an asterisk I have obtained by scaling with the theoretical predictions shown in Fig. 6)

m_t (GeV)	100	140	180	200	
$\sin^2\Theta_W$	0.2305	0.2260*	0.2207*	0.2215	± 0.0010
$\sin^2\Theta_W$ (SM)	0.23027	0.22580	0.22048	0.21741	
ρ_0	1.003	0.99996*	0.996*	0.994	± 0.003
ρ (SM)	1.00776	1.01082	1.01492	1.01737	

where the theoretical values (SM) are given for $m_H = 100$ GeV. $\rho_0 = \frac{\rho^{exp}}{\rho_{SM}}$ corresponds to ρ_{tree} if we ignore possible radiative corrections from non-standard physics. Thus ρ_0 is remarkably close to the minimal standard model value $\rho_{tree} = 1$.

These experimental results are extremely important constraints for possible deviations from the SM. For example, the measured value for $\sin^2\Theta_W$ is clearly in contradiction to the simplest grand unified model, namely, minimal SU(5), which predicts $\sin^2\Theta_W \simeq 0.211 - 0.218$. Independently, this theory has been ruled out by proton decay experiments. The bounds on the ρ-parameter allow to have additional scalar doublets or singlets which do not affect the minimal SM value $\rho_{tree} = 1$. However, possible Higgs triplet contaminations are limited because they implies $\rho_{tree} < 1$ and a pure triplet would give $\rho_{tree} = 1/2$.

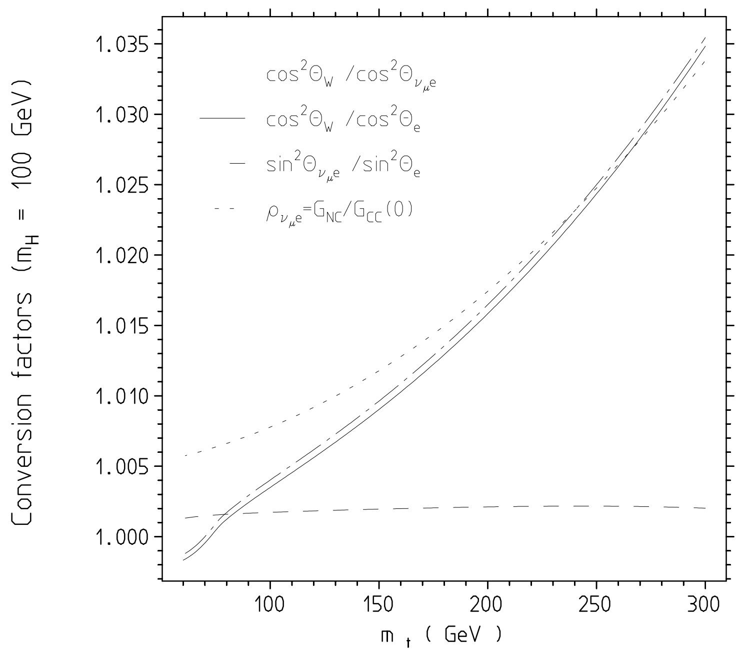


Figure 6: m_t-dependence of various $\sin^2\Theta$ conversions.

Since the discovery of the weak neutral current, almost two decades ago, the SM has been astonishingly successful and one has to wonder why. In the following we will discuss some important aspect of the SM in more detail with the hope to shed some more light on its unique structure.

Appendix A. Axial Vector Anomaly and Anomaly Cancellation.

Axial vector currents lead to the axial anomaly [37], which is associated with the triangle fermion loop diagram depicted in Fig. 5. More generally, anomalies show up in diagrams which exhibit an odd number of axial vector current vertices and which are UV divergent (and hence need regularization at intermediate steps). One can show that all anomalies are related to the triangle anomaly, which we briefly discuss now.

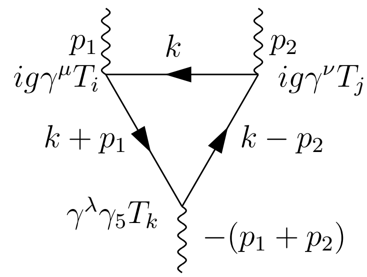


Figure 5: Triangle diagram exhibiting the axial anomaly

The amplitude for the triangle graph is given by the integral

$$\begin{aligned}\tilde{T}^{\mu\nu\lambda}_{ijk}(p_1,p_2) &= -iTr(T_jT_iT_k)\cdot \\ &\frac{g^2}{(2\pi)^4}\int d^4k Tr\left(\frac{1}{\not k-\not p_2+i\epsilon}\,\gamma^\nu\frac{1}{\not k+i\epsilon}\,\gamma^\mu\frac{1}{\not k+\not p_1+i\epsilon}\,\gamma^\lambda\gamma_5\right).\end{aligned}$$

Adding the diagram we obtain by interchanging the two vector vertices we get an amplitude which is bose symmetric

$$T^{\mu\nu\lambda}_{ijk}(p_1,p_2)=\tilde{T}^{\mu\nu\lambda}_{ijk}(p_1,p_2)+\tilde{T}^{\nu\mu\lambda}_{jik}(p_2,p_1)$$

and for which we impose *vector current conservation (condition on possible renormalization counter term)*

$$p_{1\mu}T^{\mu\nu\lambda}_{ijk}(p_1,p_2)=p_{2\nu}T^{\mu\nu\lambda}_{ijk}(p_1,p_2)=0\ .$$

It then turns out that the divergence of the axial vector current is non-vanishing and uniquely determined by the mass independent anomaly

$$-(p_1+p_2)_\lambda T^{\mu\nu\lambda}_{ijk}(p_1,p_2)=i\frac{g}{16\pi^2}D_{ijk}4\varepsilon^{\mu\nu\rho\sigma}p_{1\rho}p_{2\sigma}\neq 0 \tag{40}$$

(Adler, Bell and Jackiw 1969). We have introduced the abbreviation $D_{ijk}\equiv Tr(\{T_i,T_j\}T_k)$ *for the representation dependent coefficient of the anomaly. The result can be obtained as a matrix element of the anomalous divergence equation*

$$\partial_\lambda j^\lambda_{5k}(x)=\frac{g^2}{16\pi^2}D_{ijk}\tilde{G}^{\mu\nu}_i(x)G_{j\mu\nu}(x) \tag{41}$$

where $G_{i\mu\nu}$ is the (abelian or non-abelian) field strength tensor and $\tilde{G}_i^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{i\rho\sigma}$ its dual tensor. This is a very surprising result because the canonical Ward-Takahashi identities reading

$$\begin{aligned}
\partial_\mu\left(\bar{\psi}_1\gamma^\mu\psi_2\right)(x) &= i(m_1-m_2)\left(\bar{\psi}_1\psi_2\right)(x) \\
\partial_\mu\left(\bar{\psi}_1\gamma^\mu\gamma_5\psi_2\right)(x) &= i(m_1+m_2)\left(\bar{\psi}_1\gamma_5\psi_2\right)(x)
\end{aligned}$$

do not exhibit such a term and for massless fields both currents are conserved. The anomaly given above can be shown to be unaltered by higher order effects. Eq. (41) is thus the exact (non-perturbative) form of the axial anomaly (Adler and Bardeen 1969 , Gross and Jackiw 1972 and Korthals Altes and Perottet 1972). The crucial point about the anomaly is the fact that its presence spoils renormalizability af a theory! Only anomaly free theories are viable theories. The appearance of anomalies in a gauge field theory is strongly related to the fermion representations. Which representations are anomaly free?

- *Real representations ($R \sim R^*$) are anomaly free, since $D_{ijk} = 0$ for all real representations.*
  The groups which have only real representations are: $SO(2\ell+1)$ for ($\ell > 1$), $Sp(2\ell)$, G_2, F_4, E_7, E_8. In addition $D_{ijk} = 0$ also holds for $SO(2\ell)$ for ($\ell > 1$) with one exception: $SO(6) \simeq SU(4)$.

- *Since for any representation R one has $D_{ijk}(R) = D_{ijk}(R_0)\cdot K(R)$ where R_0 denotes the fundamental representation and $K(R)$ is a representation dependent invariant, all representations are anomaly free if $D_{ijk}(R_0) = 0$. In particular, this is the case for $SU(2)$, for which ($R_0 \sim R_0^*$), and for E_6.*

- *The groups $SU(n)$, ($n \geq 3$) have complex representations ($R \not\sim R^*$) and $D_{ijk}(R_0) \neq 0$. These groups are* not *anomaly save !*
  If we write

  $$j_i^\mu = \bar{\psi}_L\gamma^\mu T_{Li}\psi_L + \bar{\psi}_R\gamma^\mu T_{Ri}\psi_R \tag{42}$$

  at the γ_5-vertex and use

  $$\gamma_5\frac{1\pm\gamma_5}{2} = \pm\frac{1\pm\gamma_5}{2} \tag{43}$$

  we obtain

  $$D_{ijk} \equiv Tr(\{T_{Li},T_{Lj}\}T_{Lk}) - Tr(\{T_{Ri},T_{Rj}\}T_{Rk}) \tag{44}$$

  which tells us that left-handed and right-handed fields give independent contributions to the anomaly. Of particular interest for us is the color group $SU(3)_c$ and the quark representations. The quarks are in the fundamental representation 3, the antiquarks in 3^*. Under charge conjugation we have

  $$\psi_L \xrightarrow{C} \psi_L^c = i\gamma^2\psi_R^* \,.$$

Therefore it follows that ψ_L and ψ_R are in the same representation and hence $D_{ijk} \equiv 0$. Evidently, renormalizability of QCD requires parity conservation and thus the absence of axial current couplings.

- *Finally, anomalies are obtained from abelian axial current couplings. Here we have to worry about the $U(1)_Y$. Per doublet $\Psi = (\psi_1, \psi_2)$, using $Q = T_3 + Y/2$, $Q_1 - Q_2 = 1$ and $Q_{Ri} = Q_{Li}$, we get*

$$D = \sum_i (Y_{Li}^3 - Y_{Ri}^3) = -12Q_1 + 6 \tag{45}$$

which yields $D_{lepton} = 6$ and $D_{quark} = -6N_c\ (2Q_1^q - 1) = -6$.

As a consequence, we find that the $U(1)_Y$ subgroup of the standard model is renormalizable if and only if there is the lepton-quark family structure! This lepton-quark duality is one of the most surprising properties of the SM. Nature seems to take very seriously the mathematical consistency of the theory. Although a direct experimental "proof" for the existence of the top quark is still missing, there is strong indirect evidence for its existence.

Appendix B. How natural is the minimal SM?

We finally try to derive the SM by starting from some general assumptions [38]. Let us make the following assumptions:
1) local field theory
2) interactions follow from a local gauge principle
3) renormalizability
4) masses derive from the minimal Higgs system
5) ν_R is absent or if it exists it does not carry hypercharge.
We admit that the last assumption looks quit ad hoc, but nevertheless we make it. From the above assumptions the following picture develops:

- *For the gauge interactions, the simplest non-trivial possibility is that the fundamental massless matter fields group into doublets and triplets which are the fundamental representations of $SU(2)$ and $SU(3)$.*

- *Since fields are massless all fields can be chosen left-handed. Left-handed particles and left-handed antiparticles at this stage are uncorrelated.*

- *We must have* pairing *for particles that are going to be massive, since a mass term (we ignore the possibility to have Majorana fields here) has the form $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$. Notice that for massive particles, only, we know which left-handed antiparticle belongs to which left-handed particle to form a Dirac field.*

- *For $SU(3)_c$ triplets we* must *have pairing in order to avoid axial anomalies. $SU(3)$ is the simplest group having complex representations. This allows putting particles in 3 and antiparticles in the inequivalent 3^*. As a consequence a rich* *color singlet structure ($\equiv$ hadron spectrum) results. Furthermore, confinement requires $SU(3)_c$ to be unbroken!*

- *$SU(2)_L$ is anomaly free and hence there is no anomaly condition associated with this group. To generate mass we have to break $SU(2)_L$ by a Higgs mechanism. The simplest and natural possibility is to choose one Higgs field in the fundamental representation of $SU(2)_L$. There is no hypercharge for the moment. The Higgs field may be written in the form*

$$\Phi_b = \tilde{\Phi}\chi_b \; ; \; \chi_b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in terms of a 2×2 matrix field

$$\tilde{\Phi} = \frac{1}{\sqrt{2}}(H_s + i\tau_i\phi_i) \; .$$

The covariant derivative being given by

$$D_\mu\Phi_b = (\partial_\mu - i\frac{g}{2}\tau_a W_{\mu a})\Phi_b \; ,$$

the Higgs system Eq. (17) exhibits an extra global $SU(2)_R$-symmetry $\chi_b \to V^+\chi_b$. One easily checks that the transformation

$$\tilde{\Phi} \to U(x)\tilde{\Phi}V^+$$

with $U(x) \in SU(2)_{L,local}, V \in SU(2)_{R,global}$ leaves the Higgs Lagrangian invariant. This implies that the fields (W^+, W_3, W^-) form an isospin triplet with $M_Z = M_{W^\pm}$.
Now consider the fermions (still no hypercharge). Since L_f and Φ_b are doublets R_f must *be a singlet ! otherwise we would not be able to write down an invariant and renormalizable fermion-Higgs coupling. Therefore $SU(2)_L$ must be parity violating of V-A-type! The Yukawa term has the general form*

$$\mathcal{L}_{\text{Yukawa}} = -\bar{L}_f\tilde{\Phi}\begin{pmatrix} g_1 g_2 \\ g_3 g_4 \end{pmatrix} R_f + h.c.$$

with 4 complex couplings g_i and R_f a "doublet" having two right-handed singlets as entries. Although we have not used hypercharge to restrict these couplings, the existence of a global $SU(2)_R$-symmetry of the Higgs system allows transforming the Yukawa couplings

$$\tilde{\Phi}(\cdot)R_f \to \tilde{\Phi}V^+(\cdot)WR_f$$

to standard form, $V^+(\cdot)W$= real diagonal. Since $V \in SU(2)_R$ has 3 parameters and W is an arbitrary unitary matrix with 4 parameters we end up with one free parameter such that the system exhibits a global $U(1)$ invariance. This is not surprising since in the unitary gauge we always can end up only with $\mathcal{L}_{\text{Yukawa}}$ in the simple standard form Eq. (19).

- *The global $U(1)$ which is a consequence of the minimal Higgs mechanism may be* interpreted *as a global $U(1)_Y$. We are free to assign to Φ_b $Y = 1$, which means nothing else than that we measure Y in units of the Φ_b-hypercharge. Then*

$$\Phi_t = \tilde{\Phi}\chi_t \; ; \; \chi_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

has $Y = -1$, and we may write $\tilde{\Phi} = (\Phi_b, \Phi_t)$. Since we have the global $U(1)_Y$ for free, we may assume *this symmetry to be* local. *The covariant derivative for $\tilde{\Phi}$ now reads*

$$D_\mu \tilde{\Phi} = \partial_\mu \tilde{\Phi} + i\frac{g'}{2} B_\mu \tilde{\Phi} \tau_3 - i\frac{g}{2}\tau_a W_{\mu a}\tilde{\Phi}$$

and we find back the usual Higgs Lagrangian. The 3 real fields ϕ_a $a = 1, 2, 3$ can be gauged away and only 3 out of 4 gauge fields can acquire a mass. Hence there must exist one massless field, the photon! Evidently we obtain the relations $g' = g \tan\Theta_W$ and $\rho = M_W^2/(M_Z^2 \cos^2\Theta_W) = 1$! instead of $M_Z = M_{W^\pm}$ when $g' = 0$.
Now, what can we say about the hypercharge of the fermions?:
A left-handed doublet transforms like

$$L \to e^{i\frac{g'}{2}Y_L} L$$

where Y_L is arbitrary. By inspection of $\mathcal{L}_{\text{Yukawa}}$ we find for the hypercharges of the singlets: ψ_{1R} must have $Y_{1R} = Y_L + 1$ and ψ_{2R} must have $Y_{2R} = Y_L - 1$. One consequence is that $U(1)_Y$ must *violate parity. The astonishing thing is that the fermion current which couples to the photon preserves parity. By inspection we find*

$$\begin{aligned} D_\mu L_f &= (\partial_\mu - i\frac{g'}{2} Y_L B - \mu - i\frac{g}{2}\tau_3 W_{\mu 3} - \cdots) L_f \\ D_\mu R_f &= (\partial_\mu - i\frac{g'}{2} Y_L B - \mu - i\frac{g}{2}\tau_3 B_\mu - \cdots) R_f \end{aligned}$$

and the couplings of L_f and R_f to A_μ read

$$\begin{aligned} L_f &: \quad -i(g \sin\Theta_W \frac{\tau_3}{2} + g' \cos\Theta_W \frac{Y_L}{2}) A_\mu \\ R_f &: \quad -i(g' \cos\Theta_W \frac{\tau_3}{2} + g' \cos\Theta_W \frac{Y_L}{2}) A_\mu \, . \end{aligned}$$

Because we have $g' \cos\Theta_W = g \sin\Theta_W = e$ we find the Gell-Mann-Nishijima (GMN) relation

$$Q = T_3 + \frac{Y}{2}$$

as a consequence of a minimal Higgs structure! What we find is, that, whatever the hypercharge of L_f is L_f and R_f must couple identically to photons. Thus QED must be parity conserving! Furthermore the charges of the upper (1) and lower (2) components of the doublets satisfy

$$Q_{Li} = Q_{Ri}\ ,\ Q_1 - Q_2 = 1 \text{ and } Q_1 + Q_2 = Y_L\ .$$

So far we have no charge quantization. Here we need the last assumption.

- *If ν_R does not exist we have to set $Y_{\nu R} = 0$ and consequently we must have $Y_{\nu L} = -1 = Y_{\ell L} = 0$ and $Q_\nu = 0$, $Q_\ell = -1$. For the $U(1)_Y$ anomaly cancellation we need lepton-quark duality and the charges of the quarks must have their known values if they appear in three colors. One thus must have the usual charge quantization.*

We finally summarize the consequences of the assumptions stated above:

- *Breaking $SU(2)_L$ by a minimal Higgs automatically leads to a global $U(1)_Y$, which can be gauged,*
- *parity violation of $SU(2)_L$,*
- *custodial symmetry relation $\rho = M_W^2/(M_Z^2 \cos^2\Theta_W) = 1$,*
- *existence of the photon,*
- *parity conservation of QED,*
- *the validity of the Gell-Mann-Nishijima relation,*
- *family structured fermions, and*
- *charge quantization.*

The SM local symmetries permit quark-lepton family replica and we know that three families exist as it is required to admit CP violation in the simplest possible way. All members of the three families have been discovered by now.
Note that the SM local symmetry structure does not exclude right-handed neutrinos to exist and to exhibit masses. For some time one has been assuming lepton flavor conservation by imposing ad hoc an extra global $U(1)_e \otimes U(1)_\mu \otimes U(1)_\tau$ symmetry, which implied neutrino masses to be vanishing and flavor symmetry to be unbroken. This has been motivated by the non-observation of flavor transitions like $\mu \to e\gamma$ and many other "missing" flavor transitions. The existence of neutrino oscillations tells us that neutrinos must have masses which requires that right-handed neutrinos do exist as gauge singlets. The ν_R's hence do not couple to the gauge boson sector as a consequence of the SM's gauge symmetry structure. The existence of neutrino masses (although so tiny that up to date they escape any direct observation) requires right-handed neutrinos to have Yukawa couplings to the SM Higgs field. It also implies

flavor violations, which however are too small (vanishing in the limit of vanishing neutrino masses) to be experimentally accessible up to now.
Epilogue: unlike Maxwell's electromagnetism, which unified electrical and magnetical laws and predicted electromagnetic waves, the electroweak theory is not a true unification, it rather regulates the mixing of electromagnetic and weak interaction phenomena. At the heart is $\gamma - Z$ *mixing and* Z *resonance (heavy-light) physics which manifests itself most convincingly in electron-positron annihilation into* Z *bosons.*
Addendum on the neutrino masses and neutrino mixing:
As already mentioned, in the SM of electroweak interactions, neutrinos originally were assumed to be massless. Progress in neutrino oscillation experiments (Davis et al. since 1968, ... , Super-Kamiokande 1998-2001, SNO 2002, KamLAND 2003) by measuring solar [22] and atmospheric [23] neutrinos fluxes have established that the known neutrinos are massive and mix with one another.
After the experimental results reported by Super-Kamiokande [23], SNO [25], KamLAND [26] and experiments with neutrinos from nuclear reactors [27], we now have very good knowledge of 5 parameters associated with neutrino mixing:

$$\Delta m^2_{atm} \simeq 2.5 \times 10^{-3} \text{ eV}^2, \tag{46}$$
$$\Delta m^2_{sol} \simeq 6.9 \times 10^{-5} \text{ eV}^2, \tag{47}$$
$$\sin^2 2\theta_{atm} \simeq 1, \tag{48}$$
$$\tan^2 \theta_{sol} \simeq 0.46, \tag{49}$$
$$|U_{e3}| < 0.16. \tag{50}$$

The last 3 numbers tell us that the neutrino mixing matrix is rather well-known, and to a very good first approximation, it is given by

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c & -s & 0 \\ s/\sqrt{2} & c/\sqrt{2} & -1/\sqrt{2} \\ s/\sqrt{2} & c/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{51}$$

where $\sin^2 2\theta_{atm} = 1$ *and* $U_{e3} = 0$ *have been assumed, with* $s \equiv \sin\theta_{sol}$, $c \equiv \cos\theta_{sol}$.
To allow neutrinos to be massive theoretically, a starting point is the observation by Weinberg [28] that standard model fields provide a unique dimension-five operator, namely

$$\mathcal{L}_\Lambda = \frac{f_{ij}}{2\Lambda}(\nu_i\phi^0 - e_i\phi^+)(\nu_j\phi^0 - e_j\phi^+) + H.c.\,, \tag{52}$$

which generates a Majorana neutrino mass matrix given by

$$(\mathcal{M}_\nu)_{ij} = \frac{f_{ij}v^2}{\Lambda}, \tag{53}$$

where v *is the vacuum expectation value of* ϕ^0. *This also shows that whatever the underlying mechanism for the Majorana neutrino mass, it has to be "seesaw" in character, i.e.* v^2 *divided by a large mass [29].*

The seesaw mechanism works as follows: a light neutrino for each flavor has a very heavy not yet discovered heavy partner of the same flavor. Thus

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\left(\overline{\nu}\,\overline{N}\right)\tilde{\mathcal{M}}\left(\begin{array}{c}\nu\\ N\end{array}\right)\text{h.c.}\,,\quad \tilde{\mathcal{M}} = \left(\begin{array}{cc} m_\nu & 0\\ 0 & M\end{array}\right) \tag{54}$$

in the mass eigen basis.
The doublet partner of the light left-handed neutrino, the lepton ℓ has a Dirac mass M. The heavy right-handed partner neutrino has mass $B \gg M$.
The 2×2 light-heavy neutrino mixing-matrix arises in a natural manner within the SM by considering the most general mass matrix allowed by gauge invariance of the action, and the corresponding charges of the lepton and neutrino fields.
When neutrino masses have been assumed to be strictly zero neutrino mass couplings to the Higgs had to be assumed to be vanishing. As a consequence extra flavor symmetries $U(1)_\ell$ ($\ell = e, \mu, \tau$) were assumed to be exact, compatible with corresponding flavor transition experiments like $\mu \to e\gamma$, which one still is searching for. Neutrino oscillation experiments finally established the fact that neurinos must be massive as well, like all other fermions. Still, the tininess of neutrino masses is a striking feature which demands an explanation.
Neutrino mass terms in the SM can be introduced like other mass terms provided one introduces SM singlet neutrino fields which do not couple to any gauge fields. They thus only enter in linear form in the neutrino Yukawa interaction term, as done in the first term of (18).
In the symmetric phase left and right handed fermi fields are actually Weyl spinors χ, ψ, ... A left-handed SM neutrino-lepton weak isospin doublet field is represented by $L = \binom{\chi}{\psi}$, as present in the minimal SM without neutrino masses. Let η denote the right-handed singlet neutrino Weyl spinor. There are three forms one can write a neutrino mass term

$$\frac{1}{2}B'\,\chi^\alpha\chi_\alpha\,,\ \ \frac{1}{2}B\eta^\alpha\eta_\alpha \quad \text{and } M\eta^\alpha\chi_\alpha$$

and theire complex conjugates, which collectivelly may be written as a quadratic form

$$\frac{1}{2}\left(\begin{array}{cc}\chi & \eta\end{array}\right)\left(\begin{array}{cc}B' & M\\ M & B\end{array}\right)\left(\begin{array}{c}\chi\\ \eta\end{array}\right)$$

Since the singlet spinor η is uncharged under all SM gauge couplings the parameter B can take arbitrary values. The parameter M is forbidden by electroweak gauge symmetry and can only appear after electroweak symmetry breaking through the Higgs mechanism. This implies that M is naturally $O(v)$. Since χ has weak isospin 1/2 like the Higgs field H and η has weak isospin zero b' must be zero.

$$\mathcal{L}_{\text{Yukawa}} = \frac{y}{\sqrt{2}}\,\eta L\epsilon H^* + \cdots \tag{55}$$

The light-heavy neutrino mixing matrix thus must take the form

$$\begin{pmatrix} 0 & M \\ M & B \end{pmatrix} \tag{56}$$

which has eigenvalues

$$\lambda_{\pm} = \frac{1}{2}\,(B \pm \sqrt{B^2 + 4\,M^2})\ . \tag{57}$$

While the large eigenvalue is $\lambda_+ \approx B$ the small one is

$$\lambda_- \approx -M^2/B\ . \tag{58}$$

Note that $\lambda_+\lambda_- = -M^2$. If one of the eigenvalues goes up, the other goes down, and vice versa. Therefore the name "seesaw" mechanism. Since the Majorana mass B is not protected by a symmetry (i.e. there exists no broken symmetry which requires $B = 0$ in the symmetry limit) such that B can be as large as the Planck scale $M_{\rm Planck} \simeq 10^{19}$ GeV. This serves to explain why the observed light neutrinos are very light, $O(1\ \mathrm{eV})$, for appropriately large B.
Weyl fields:

ν_L	destroys	a LH chiral neutrino	and	creates	a RH antineutrino	,
$\overline{\nu}_L$	creates	"	and	destroys	"	,
ν^c_L	creates	"	and	destroys	"	,
$\overline{\nu}^c_L$	destroys	"	and	creates	"	,

Neutrino mass term in weak eigen basis

$$\mathcal{L}_{\rm mass} = -\frac{1}{2}\,(\overline{\nu}_L\,\overline{\nu}^c_R)\;\mathcal{M}\begin{pmatrix} \nu^c_L \\ \nu_R \end{pmatrix} + \mathrm{h.c.}\ ,\ \ \mathcal{M} = \begin{pmatrix} m^L_M & m_D \\ m_D & m^R_M \end{pmatrix} \tag{59}$$

with conjugate fields $\nu_L \overset{\rm h.c.}{\longleftrightarrow} \overline{\nu}_L$, $\nu_R \overset{\rm h.c.}{\longleftrightarrow} \overline{\nu}_R$, $\nu^c_L \overset{\rm h.c.}{\longleftrightarrow} \overline{\nu}^c_L$, $\nu^c_R \overset{\rm h.c.}{\longleftrightarrow} \overline{\nu}^c_R$. Thus

$$\mathcal{L}_{\rm mass} = -\frac{1}{2}\,(\overline{\nu}_L\,\overline{\nu}^c_R)\;\mathcal{M}\begin{pmatrix} \nu^c_L \\ \nu_R \end{pmatrix} - \frac{1}{2}\,(\overline{\nu}^c_L\,\overline{\nu}_R)\;\mathcal{M}\begin{pmatrix} \nu_L \\ \nu^c_R \end{pmatrix}\ . \tag{60}$$

which yields the Dirac mass term if we identify the charge conjugate L[R] fields with the corresponding R[L] fields i.e. $\nu^c_L \equiv \mu_R$ etc.

$$-m_D\;(\overline{\nu}_L\nu_R + \overline{\nu}_R\nu_L) \tag{61}$$

and a Majorana mass term

$$-\frac{1}{2}\,m^L_M\;(\overline{\nu}_L\nu^c_L + \overline{\nu}^c_L\nu_L) - \frac{1}{2}\,m^R_M\;(\overline{\nu}_R\nu^c_R + \overline{\nu}^c_R\nu_R) \tag{62}$$

Geometrical mean reads $m^R_M m^L_M = m^2_D$ which implies the mass hierarchy $M \approx m^R_M \gg m_D > m^L_M \approx 0$. For given m_D a higher value of m^R_M implies a lower value of m^L_M and vice versa.
Eigenvalue equation $(m^L_M - \lambda)(m^R_M - \lambda) - m^2_D = 0$ with solutions

$$\lambda_{\pm} = \frac{1}{2}\,\left(m^R_M + m^L_M\right) \pm \frac{1}{2}\sqrt{(m^R_M + m^L_M)^2 - 4\,(m^R_M m^L_M - m^2_D)} \tag{63}$$

II. QUANTIZATION AND REGULARIZATION

1. Gauge fixing

The quantum field theory associated with the classical gauge invariant Lagrangian

$$\begin{aligned}\mathcal{L}_{inv} &= \mathcal{L}_{matter} + \mathcal{L}_{YM} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} \\ &= \mathcal{L}_{inv}^{bilinear} + \mathcal{L}_{inv}^{int}\end{aligned}$$

in the broken phase $\Phi_b = \Phi + \frac{v}{\sqrt{2}}\binom{0}{1}$

$$SU(2)_L \otimes U(1)_Y \to U(1)_{e.m.}$$

may be defined by writing down the path-integral representation

$$Z\{J, \bar{\chi}, \chi, \cdots\} = \int \mathcal{D}V_{\mu i}\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int(\mathcal{L}_{eff}+JV+\bar{\chi}\psi+\bar{\psi}\chi+\cdots)} \tag{64}$$

for the generating functional of the time-ordered Green functions. By J, $\bar{\chi}$, χ, $\cdots$ *we denote the classical source functions. If we would try to choose* $\mathcal{L}_{eff} = \mathcal{L}_{inv}$ *the functional integral would not exist. The problem is known from QED. Because of the gauge invariance of the action* $S_{inv} = i \int d^4x\mathcal{L}_{inv}$ *the equations of motion do not determine the gauge fields uniquely. In order to get non-degenerate equations of motion we have to fix a gauge. A convenient choice is the* linear covariant *'t Hooft gauge* ($R_\xi - gauge$)*. Each gauge field has associated a gauge function*

$$\begin{array}{lcll} W_\mu^\pm & : & C^\pm = -\partial_\mu W^{\mu\pm} \pm i\xi_W M_W \phi^\pm & (=0) \\ Z_\mu & : & C_Z = -\partial_\mu Z^\mu - \xi_Z M_Z \phi & (=0) \\ A_\mu & : & C_A = -\partial_\mu A^\mu & (=0) \end{array} \tag{65}$$

and one adds to the invariant Lagrangian the bilinear Lorentz-invariant gauge fixing *part*

$$\mathcal{L}_{GF} = -\frac{1}{\xi_W}C^+C^- - \frac{1}{2\xi_Z}C_Z^2 - \frac{1}{2\xi_A}C_A^2 \ . \tag{66}$$

The ξ_i*'s are independent gauge parameters. For notational convenience we will take them equal,* $\xi_W = \xi_Z = \xi_A = \xi$*. Of course physics must be independent of* ξ *! The extra terms in the gauge functions containing the Higgs ghosts have been chosen such that the non-diagonal (mixed) terms*

- $\mathcal{L}_{inv}^{(W,\phi)} = i\ M_W\partial_\mu W^{\mu+}\phi^- + h.c. + M_Z\partial_\mu Z^\mu\phi$

drop out in the sum $\mathcal{L}_{inv}^{bilinear} + \mathcal{L}_{GF}$*. In this way we achieve a diagonalization of the terms bilinear in* $\partial_\mu W_a^\mu$ *and* ϕ_a *with the consequence that the Higgs ghosts get a gauge dependent mass. The mass term obtained is*

- $\mathcal{L}^{(\phi)}_{mass} = -\xi_W M_W^2 \phi^+\phi^- - \frac{1}{2}\xi_Z M_Z^2 \phi^2$.

The gauge dependent masses are another direct indication that the Higgs ghosts ("would be Goldstone bosons") cannot be physical. We now obtain the well-defined gauge boson propagators

$$D_V^{\mu\nu}(p,\xi) \equiv -i\left(g^{\mu\nu} - (1-\xi_V)\frac{p^\mu p^\nu}{p^2 - \xi_V M_V^2 + i\epsilon}\right)\frac{1}{p^2 - M_V^2 + i\epsilon} \tag{67}$$

for $V = W_\mu^\pm, Z_\mu, A_\mu$ and with $M_A = m_\gamma = 0$. For $\xi = 1$ we have the 't Hooft-Feynman gauge where the propagators take the particularly simple form

$$\frac{-ig^{\mu\nu}}{p^2 - M_V^2 + i\epsilon} . \tag{68}$$

The renormalizable R_ξ-gauge (R-gauge) provides a one parameter interpolating family of gauges with the unitary gauge as a limiting case. For $\xi \to \infty$ we indeed get the physical U-gauge propagator

$$-i\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{M_V^2}\right)\frac{1}{p^2 - M_V^2 + i\epsilon}\ ;\ V = W, Z$$

which is purely transverse on the mass-shell $p^2 = M_V^2$. If we write the R-gauge propagator in the form

$$D_V^{\mu\nu}(p,\xi) = -i\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{M_V^2}\right)\frac{1}{p^2 - M_V^2 + i\epsilon} - i\frac{p^\mu p^\nu}{M_V^2}\frac{1}{p^2 - \xi_V M_V^2 + i\epsilon}$$

we observe that the first term is the unitary piece while the second term is a kind of Pauli-Villars cut-off term. The ghost propagators are given by

$$D_V^\phi(p) = \frac{i}{p^2 - \xi_V M_V^2 + i\epsilon} \tag{69}$$

and freeze out $D_V^\phi(p) \to 0$ as $\xi \to \infty$ (unitary gauge). It is rather amusing to see how the "gauging away" of the Higgs ghosts works at the level of the Feynman diagrams.

2. Faddeev-Popov ghosts

Unlike in QED adding $\mathcal{L}_{GF}$ to the invariant Lagrangian spoils gauge invariance, unitarity and renormalizability of S-matrix! If we compare the classical abelian with the non-abelian gauge transformations

$$\begin{array}{lcl} U(1) & : & A_\mu \to A_\mu + \partial_\mu\omega \\ & & C_A \to C_A - \Box\omega \\ SU(2) & : & W_{\mu a} \to W_{\mu a} + \partial_\mu\omega_a - g\epsilon_{abc}W_{\mu c}\omega_b \\ & & C_a \to C_a - \Box\omega_a + g\epsilon_{abc}\partial^\mu(W_{\mu c}\omega_b) \end{array}$$

we observe that the harmless extra term in the abelian gauge function is replaced by a non-trivial and non-harmless extra term in the non-abelian case. Faddeev and Popov [39] have found the way out of the dilemma. The restauration of the gauge symmetry can be achieved by taking into account the functional determinant obtained in the functional integral under a gauge transformation of the fields (integration variables). If we define the functional integral as follows, with a Faddeev-Popov determinant,

$$\int \mathcal{D}W_{\mu a} Det\left(\frac{\delta C_a}{\delta \omega_b}\right) e^{i\int\left(\mathcal{L}_{inv}-\frac{1}{2\xi}C_a^2\right)d^4x}$$

one easily checks that now the functional integral is independent on the specific choice of the gauge function C_a. By introducing anticommuting scalars, the FP ghost fields $\bar{\eta}_a$ and η_a, we may represent the FP-determinant as a Berezin integral over Grassmann variables (algebra of anticommuting c-numbers) [40]

$$Det\left(\frac{\delta C_a}{\delta \omega_b}\right) = \int \mathcal{D}\eta\mathcal{D}\bar{\eta} e^{i\int \mathcal{L}_{FP} d^4x}$$

with

$$\mathcal{L}_{FP} = \bar{\eta}_a M_{ab} \eta_b \; ; \quad M_{ab} \doteq \frac{\delta C_a}{\delta \omega_b} \quad . \tag{70}$$

As a result we find the proper functional integral quantization

$$\int \mathcal{D}W_{\mu a}\mathcal{D}\eta\mathcal{D}\bar{\eta} e^{i\int \mathcal{L}_{eff} d^4x}$$

with the "quasi invariant" effective Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L}_{inv} + \mathcal{L}_{GF} + \mathcal{L}_{FP} \; . \tag{71}$$

In the following we will use a somewhat more compact notation. We treat $SU(2)_L \otimes U(1)_Y = G$ as a single gauge group G with generators T_A and structure constants f_{ABC}. The gauge fields and the FP-ghosts are denoted by

$$\begin{array}{lll} G_{A\mu} & : & W_\mu^\pm, Z_\mu, A_\mu \\ \eta_A & : & \eta^\pm, \zeta, \aleph \end{array} \qquad A : \pm, Z, \gamma \; .$$

Using this notation the FP-Lagrangian reads

$$\mathcal{L}_{FP} \;\; = \;\; \bar{\eta}^\mp M_{\pm B}\eta_B + \bar{\zeta} M_{ZB}\eta_B + \bar{\aleph} M_{\gamma B}\eta_B$$

with

$$M_{AB}\eta_B = \frac{\partial C_A}{\partial G_{C\mu}}\frac{\delta G_{C\mu}}{\delta \omega_B}\eta_B + \frac{\partial C_A}{\partial \Phi_C}\frac{\delta \Phi_C}{\delta \omega_B}\eta_B \; .$$

Since the quantities associated with a gauge transformation of fields, which appear in the last equation, will be used for a discussion of the Slavnov-Taylor identities later, we list them in detail here:

- *The gauge variations are given by:*

$$\frac{\delta G_{A\mu}}{\delta\omega_B}\eta_B \doteq D_{\mu AB}\eta_B$$

$$\begin{aligned}
D_{\mu\pm B}\eta_B &= \partial_\mu\eta^\pm \pm i\left[e(W_\mu^\pm\aleph - A_\mu\eta^\pm) - g\cos\Theta_W(W_\mu^\pm\zeta - Z_\mu\eta^\pm)\right] \\
D_{\mu ZB}\eta_B &= \partial_\mu\zeta - ig\cos\Theta_W(W_\mu^-\eta^+ - W_\mu^+\eta^-) \\
D_{\mu\gamma B}\eta_B &= \partial_\mu\aleph + ie(W_\mu^-\eta^+ - W_\mu^+\eta^-)
\end{aligned} \tag{72}$$

$$\frac{\delta\Phi_A}{\delta\omega_B}\eta_B \doteq (D\Phi_A)_B\eta_B$$

$$\begin{aligned}
(D\phi^\pm)_B\eta_B &= \pm\left[e\phi^\pm\aleph + \frac{g\,(\sin^2\Theta_W - \cos^2\Theta_W)}{2\cos\Theta_W}\phi^\pm\zeta + \frac{g}{2}(H + i\phi)\eta^\pm\right] \\
(D\phi)_B\eta_B &= -i\left[\frac{g}{2\cos\Theta_W}H\zeta + \frac{g}{2}(\phi^+\eta^- - \phi^-\eta^+)\right] \\
(DH)_B\eta_B &= i\frac{g}{2\cos\Theta_W}\phi\zeta + \frac{g}{2}(\phi^+\eta^- - \phi^-\eta^+)
\end{aligned} \tag{73}$$

$$\frac{\delta\Psi_i}{\delta\omega_B}\eta_B \doteq (D\Psi_i)_B\eta_B$$

$$\begin{aligned}
(D\psi_{\nu_\ell})_B\eta_B &= i\left[-\frac{g}{2\cos\Theta_W}\;\zeta\;\psi_{L\nu_\ell} + \frac{g}{\sqrt{2}}\eta^+\;\psi_{L\ell}\right] \\
(D\psi_\ell)_B\eta_B &= i\left[\frac{g}{\sqrt{2}}\;\eta^-\psi_{L\nu_\ell} + \frac{g}{2\cos\Theta_W}\;\zeta\;\psi_{L\ell} - e\aleph\psi_\ell - e\tan\Theta_W\zeta\;\psi_\ell\right]
\end{aligned} \tag{74}$$

- *For $M_{AB}\eta_B$ we obtain:*

$$\begin{aligned}
M_{\pm B}\eta_B &= -\Box\eta^\pm - \xi M_W^2\eta^\pm - \xi\tfrac{M_W^2}{v}\;(H \pm i\phi)\;\eta^\pm \\
&\quad +\tfrac{1}{2}\xi M_W g\cos\Theta_W\;\;((1-\tan^2\Theta_W)\;\zeta - 2\tan\Theta_W\aleph)\;\phi^\pm \\
&\quad \pm ig\cos\Theta_W\partial^\mu\left(W_\mu^\pm\;(\zeta - \tan\Theta_W\aleph)\;-(Z_\mu - \tan\Theta_W A_\mu)\;\eta^\pm\right) \\
M_{ZB}\eta_B &= -\Box\zeta - \xi M_Z^2\zeta - \xi\tfrac{M_Z^2}{v}H\zeta \\
&\quad -\xi\tfrac{M_Z M_W}{v}\;(\phi^+\eta^- + \phi^-\eta^+) \\
&\quad +ig\cos\Theta_W\partial^\mu(W_\mu^-\eta^+ - W_\mu^+\eta^-) \\
M_{\gamma B}\eta_B &= -\Box\aleph - ie\;\partial^\mu\;(W_\mu^-\eta^+ - W_\mu^+\eta^-) \quad .
\end{aligned} \tag{75}$$

Now, the FP-Lagrangian $\mathcal{L}_{FP}$ is explicitly given by the sum of the last three terms multiplied from the left by $\bar{\eta}^{\mp}$, $\bar{\zeta}$ and $\bar{\aleph}$, respectively. A warning should be made here, $\mathcal{L}_{FP}$ is not hermitian ! Thus in contrast to ordinary fermion loops, FP ghosts contribute differently for ghosts running clock wise or counter-clock wise in a diagram. By the above expressions the FP-ghost propagators read

$$\Delta_V^{\eta}(p,\xi) = \frac{i}{p^2 - \xi_V M_V^2 + i\epsilon} \tag{76}$$

and thus look the same as the Higgs ghost propagators. However, they obey Fermi statistics such that there is a factor (-1) per FP-ghost loop! in the Feynman rules. Similarly to the Higgs ghosts, the FP-ghosts freeze out ($\Delta_V^{\eta} \to 0$) in the unitary gauge limit $\xi \to \infty$ with one exception. The FP-ghost partner of the photon $\Delta^{\aleph}(p) = \frac{i}{p^2+i\epsilon}$ remains in the game. In addition, there are two interaction terms $-\xi \frac{M_W^2}{v} \bar{\eta}^{\mp}\eta^{\pm}\,(H \pm i\phi)$ and $-\xi \left[\frac{M_Z^2}{v}\bar{\zeta}\zeta H + \frac{M_Z M_W}{v}\bar{\zeta}\,(\phi^+\eta^- + \phi^-\eta^+)\right]$ which have a coupling proportional to ξ which in the U-gauge limit give rise to the so called Lee-Yang terms [41]. Since we are not going to consider calculations in the unitary gauge we need not care further about these terms.

$$\boxed{\textit{Quantization complete !}}$$

3. Becchi-Rouet-Stora (BRS) symmetry

The local gauge invariance of the functional integral

$$\int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}e^{i\int \mathcal{L}_{eff}d^4x} \tag{77}$$

yields relations between Green functions, the Slavnov-Taylor (ST) [42] identities. They generalize the Ward-Takahashi (WT) [43] identities which derive from global symmetries.
The ST-identities provide the tool needed for proofs of

i) gauge invariance

ii) unitarity

iii) renormalizability

of the S-matrix. ST-identities may be obtained from the BRS-symmetry [44] of $\mathcal{L}_{eff}$. The idea behind BRS-symmetry is to dispose of the as yet undefined transformation properties of the FP-ghost fields $\bar{\eta}$ and η such that

$$\delta^{BRS}\mathcal{L}_{eff} = 0 \ . \tag{78}$$

In order to achieve this it is natural to demand the following relations to hold:

$$\begin{array}{lrcl} i) & \delta\mathcal{L}_{inv} & = & 0 \\ ii) & \delta\mathcal{L}_{GF} & = & -\frac{1}{\xi}C_A M_{AB}\omega_B \\ iii) & \delta\mathcal{L}_{FP} & = & \delta\bar{\eta}M_{AB}\eta_B + \bar{\eta}_A\delta(M_{AB}\eta_B) \\ & & = & -\delta\mathcal{L}_{GF} \\ iv) & \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta} & & \text{invariant} \ . \end{array}$$

A solution for this set of conditions may be obtained as follows: 1) Introduce anticommuting global c−number variables $\delta\lambda$, $\delta\lambda$ anticommuting with η and $\bar{\eta}$, *and identify* $\omega_B = \eta_B\delta\lambda$. *Thus*

- $\delta G_A = D_{AB}\eta_B\delta\lambda$

where G_A *can be a gauge field, a scalar or a fermi field with* $D_{AB}\eta_B$ *given in the previous subsection. 2) Assume* η *to transform according to the regular representation, thus*

- $\delta\eta_A = -\frac{1}{2}gf_{ABC}\eta_B\eta_C\delta\lambda$

where a permutation symmetry factor 1/2 (antisymmetry of f_{ABC} *and anticommutativity of the* η*'s) has been taken into account. 1) and 2) imply* $\delta(M_{AB}\eta_B) = 0$. *We thus take the freedom to choose: 3) The field* $\bar{\eta}$ *transforms as*

- $\delta\bar{\eta}_A = -\frac{1}{\xi}C_A\delta\lambda$

such that conditions i) to iii) are satisfied. One can show iv) to be true for the above set of transformations which define the BRS-transformation.

4. ST-identities

The BRS invariance of $\mathcal{L}_{eff}$ *allows a simple derivation of the ST-identities. Performing a change of integration variables in the functional integral does not change the value of the integral. If we choose an (infinitesimal) BRS-transformation we get*

$$\begin{aligned} Z\{J,\bar{\beta},\beta\} &= \int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}e^{i\int(\mathcal{L}_{eff}+JG+\bar{\eta}\beta+\bar{\beta}\eta)d^4x} \\ &= \int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}e^{i\int(\cdots)}e^{i\int(J\delta G+\delta\bar{\eta}\beta+\bar{\beta}\delta\eta)d^4z} \\ &= \int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}e^{i\int(\cdots)}[1+i\int dz(J\delta G+\delta\bar{\eta}\beta+\bar{\beta}\delta\eta)(z)] \ . \end{aligned} \tag{79}$$

In the second step we have used $\delta\lambda^2 = 0$ *which implies that terms higher than linear vanish if we expand the exponential. Using* $\delta\lambda\beta = -\beta\delta\lambda$ *etc. we can write*

$$\int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta} \ i\int dz\left\{J_A D_{AB}\eta_B + \frac{1}{\xi}C_A\beta_A - \frac{1}{2}\bar{\beta}_A g f_{ABC}\eta_B\eta_C\right\}(z) \times e^{i\int d^4y(\mathcal{L}_{eff}+J_AG_A+\bar{\eta}_A\beta_A+\bar{\beta}_A\eta_A)} \equiv 0 \ .$$

Taking the functional derivative

$$i\frac{\delta}{\delta\beta_C(x)}\cdots|_{\bar\beta=\beta=0}$$

we obtain the ST-identities

$$\int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar\eta\ \ i\int dz\left\{\bar\eta_C(x)J_A D_{AB}\eta_B + i\frac{1}{\xi}C_A\delta_{CA}\delta(x-z)\right\}(z) \times e^{i\int d^4y(\mathcal{L}_{eff}+J_AG_A)} \equiv 0\ . \tag{80}$$

For the time-ordered Green functions, by applying

$$(-i)^N\frac{\delta^{(N)}}{\delta J_{A_1}(x_1)\cdots\delta J_{A_N}(x_N)}\cdots|_{J=0}$$

to the functional ST-identity, this implies

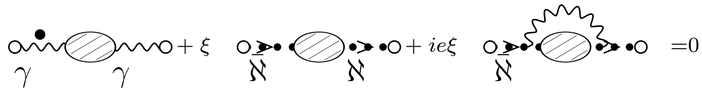
$$\frac{1}{\xi}<0|TC_C(x)G_{A_1}(x_1)\cdots G_{A_N}(x_N)|0> = \sum_i <0|TG_{A_1}(x_1)\cdots\underbrace{\bar\eta_C(x)(D_{A_iB}\eta_B)(x_i)}_{\text{replacing }\ G_{A_i}(x_i)}\cdots G_{A_N}(x_N)|0>\ . \tag{81}$$

As an example we obtain for the gauge boson propagators

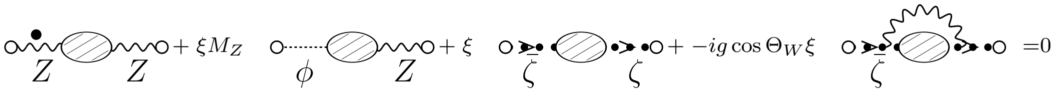
$$<0|TC_C(x)G_A(y)|0>\ \ =\ \ \xi<0|T\bar\eta_C(x)(D_{AB}\eta_B)(y)|0>\,, \tag{82}$$

or, for the individual fields, (by a $\bullet$ *we indicates a derivative of a field)*

$$-<T\partial_\mu A^\mu(x)A_\nu(y)>=\xi<T\bar\aleph(x)\partial_\nu\aleph(y)>+ie\xi<T\bar\aleph(x)(W_\mu^-\eta^+-W_\mu^+\eta^-)(y)>$$



$$-<T\partial_\mu Z^\mu(x)Z_\nu(y)>-\xi M_Z<T\phi(x)Z_\nu(y)> = \xi<T\bar\zeta(x)\partial_\nu\zeta(y)>-ig\ \cos\Theta_W\ \xi<T\bar\zeta(x)(W_\mu^-\eta^+-W_\mu^+\eta^-)(y)>$$



For the mixed cases $<T\partial_\mu A^\mu Z_\nu>$ *and* $<T\partial_\nu Z^\mu A_\nu>$ *we get similar relations. The ST-identities tell us how the gauge terms like* $\partial_\mu Z^\mu$ *cancel against Higgs and*

FP-ghosts! Before we derive another set of relations for gauge field propagators, we consider the <u>*FP-ghost propagator*</u>*. We have*

$$\begin{aligned} & \int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}\ \bar{\eta}_C(x)\frac{\delta}{\delta\bar{\eta}_A(z)}e^{\int(\mathcal{L}_{eff}+JG)dx} \\ = \ & i\int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}\ \bar{\eta}_C(M_{AB}\eta_B)(z)e^{i\int(\cdots)} \\ = \ & -\int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}\delta_{CA}\delta(x-z)e^{i\int(\cdots)} \end{aligned} \tag{83}$$

where the last step is a partial integration. Taking a functional derivative for vanishing sources, we find

$$< 0|T\bar{\eta}_C(x)(M_{AB}\eta_B)(z)G_{A_1}(x_1)\cdots|0 >= i\delta_{CA}\delta(x-z) < 0|TG_{A_1}(x_1)\cdots|0 > , \tag{84}$$

which is the FP-ghost propagator in the standard form. We may use this result in order to get time-ordered Green functions with multiple insertions of gauge functions C_A. To this end, in the derivation of the ST-identities, we add a source term for C_A by replacing $J_AG_A \to J_AG_A + L_AC_A$. This implies a substitution $J_AD_{AB}\eta_B \to J_AD_{AB}\eta_B + L_AM_{AB}\eta_B$ in the above derivation and taking the functional derivative

$$-i\frac{\delta}{\delta L_A(y)} \quad \cdots \quad |_{J=L=0}$$

of the modified functional ST-identity, we find

$$\begin{aligned} \tfrac{1}{\xi} < 0|TC_C(x)C_A(y)|0 > \ &= \ < 0|T\bar{\eta}_C(x)(M_{AB}\eta_B)(y)|0 > \\ &= \ i\delta_{CA}\delta(x-y) \ . \end{aligned} \tag{85}$$

Inserting the specific forms of the gauge functions we arrive at the equations

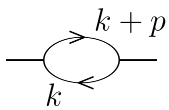
$$\begin{aligned} & < T\partial_\mu A^\mu(x)\partial_\nu A^\nu(y) >= -i\xi\delta(x-y) \\ & < T\partial_\mu A^\mu(x)\partial_\nu Z^\nu(y) > + \xi M_Z < T\partial_\mu A^\mu(x)\phi(y) >= 0 \\ & < T\partial_\mu Z^\mu(x)\partial_\nu Z^\nu(y) > + \xi M_Z < T\partial_\mu Z^\mu(x)\phi(y) > \\ & + \xi M_Z < T\phi(x)\partial_\nu Z^\nu(y) > + \xi^2 M_Z^2 < T\phi(x)\phi(y) >= -i\xi\delta(x-y) \end{aligned}$$

for the longitudinal parts of the gauge field propagators. One can use these ST-identities to prove that longitudinal amplitudes in propagators drop out in physical amplitudes. Of course similar relations are valid for vertex functions and higher Green functions.

5. Dimensional Regularization

So far we have ignored that quantities like the path integral and Green functions etc. are mathematically <u>*illdefined*</u>*. We assume the theory to be defined by its formal power series expansion in $\mathcal{L}int$. The perturbative definition is acceptable if the expansion is well defined order by order in the perturbative expansion and if*

this expansion is <u>*renormalizable*</u> *i.e. it can be made finite by a redefinition of the parameters (parameter renormalization) and the fields (multiplicative wave function renormalization).*
Starting with the Feynman rules of the classical quantized Lagrangian, called bare Lagrangian, the formal perturbation expansion is given in terms of ultraviolet (UV) divergent Feynman integrals. As an example consider the scalar self-energy diagram



$$= \frac{1}{(2\pi)^4}\int d^4k \frac{1}{k^2-m_1^2+i\varepsilon}\frac{1}{(k+p)^2-m_2^2+i\varepsilon} \overset{|k|\gg m_1,m_2}{\sim} \int \frac{d^4k}{k^4}$$

which is divergent because the integral does not fall-off sufficiently fast at large k. In order to get a well-defined perturbation expansion the theory must be <u>*regularized*</u>[13]*. The regularization should respect as much a possible the symmetries (ST-identities) of the "classical theory". Two regularizations are known to respect local gauge symmetries (up to possible violation of chiral properties):*

1. *Lattice regularization, which makes possible the application of methods known from statistical mechanics. In particular it makes possible a non-perturbative approach (Monte Carlo simulation of lattice gauge theories) [45].*

2. *Dimensional regularization (DR), which is suitable for the perturbative approach [46].*

Since we are interested in perturbative calculations we need to discuss dimensional regularization only. The basic observations behind DR are the following:

i) Feynman rules formally look the same in different space-time dimensions $d = n(integer)$

ii) Feynman integrals converge the better the lower d is.

5. 1. Dyson power counting

The action

$$S = i\int d^dx \mathcal{L}_{eff}$$

[13]Often one simply chooses a cut-off (upper integration limit in momentum space) to make the integrals converge by "brute force". A cut-off may be considered to parametrize our ignorance about physics at very high momentum or energy. If the cut-off Λ is large with respect to the energy scale E of a phenomenon considered, $E \ll \Lambda$, the cut-off dependence may be removed by considering only relations between low-energy quantities (renormalization). Alternatively, a cut-off may be interpreted as the scale where one expects new physics to enter and it may serve to investigate how a quantity (or the theory) behaves under changes of the cut-off (renormalization group). In most cases simple cut-off regularization violates symmetries badly and it becomes a difficult task to make sure that one obtains the right theory when the cut-off is removed by taking the limit $\Lambda \to \infty$ after renormalization.

measured in units of $\hbar = 1$ *is dimensionless and therefore* $dim\ \mathcal{L}_{eff} = d$ *in mass units. The inspection of the individual terms yields the following dimensions for the fields:*

$$
\begin{array}{lcl}
\bar{\psi}\gamma^\mu\partial_\mu\psi & : & dim\ \psi = \frac{d-1}{2} \\
(\partial_\mu G_{i\nu} - \cdots)^2 & : & dim\ G_{i\mu} = \frac{d-2}{2} \\
\partial_\mu\Phi^+\partial^\mu\Phi & : & dim\ \Phi = \frac{d-2}{2} \\
\bar{g}_0\bar{\psi}\gamma^\mu T_i\psi G_{i\mu} & : & dim\ \bar{g}_0 = \frac{4-d}{2} \Rightarrow \bar{g}_0 = g_0\mu^{\epsilon/2} \\
\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu G_i^\mu + \cdots)^2 & : & dim\ \xi = 0 \\
\mathcal{L}_{FP} = -\bar{\eta}\Box\eta + \cdots & : & dim\ \eta = \frac{d-2}{2}
\end{array}
$$

where $\epsilon = 4 - d$, g_0 *denotes the dimensionless bare coupling constant (*$dim\ g_0 = 0$*) and* μ *is an arbitrary mass scale. The dimension of time ordered Green functions in momentum space is then given by (the Fourier transformation* $\int d^dqe^{-iqx}\cdots$ *gives* $-d$ *for each field):*

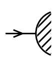
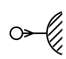
$$dimG^{(n_B,2n_F)} = n_B\frac{d-2}{2} + 2n_F\frac{d-1}{2} - (n_B + 2n_F)d \tag{86}$$

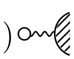
where

$$
\begin{array}{lcl}
n_B & : & \#\text{of boson fields} : G_{i\mu}, \Phi, \eta \\
2n_F & : & \#\text{of Dirac fields (in pairs)} : \psi\cdots\bar{\psi}\ .
\end{array}
$$

It is convenient to split off trivial factors which correspond to external propagators and four-momentum conservation:

- amputation of external legs : $G^{(n_B,2n_F)} \to G^{(n_B,2n_F)amp}$

   $= -i(\not{p} - m)$  : $dimG \to dimG + 1$

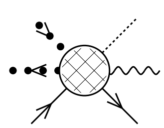
   $= -i(p^2 - m^2)$  : $dimG \to dimG + 2$

- $d-$momentum conservation : : $G^{(n_B,2n_F)} = (2\pi)^d\delta^{(d)}(\sum p_{ext})\hat{G}^{(n_B,2n_F)}$

yields for the proper amputated vertex functions

$$dim\hat{G}^{amp} = d - n_B\frac{d-2}{2} - 2n_F\frac{d-1}{2}\ . \tag{87}$$

A generic Feynman diagram represents a Feynman integral

 $\Longleftrightarrow I_\Gamma(p) = \int \frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dk_m}{(2\pi)^d}J_\Gamma(p,k)\ .$

The convergence of the integral can be inspected by looking at the behavior of the integrand for large momenta: For $k_i = \lambda\hat{k}_i$ *and* $\lambda \to \infty$ *we find*

$$\Pi_i d^dk_iJ_\Gamma(p,k) \to \lambda^{d(\Gamma)}$$

where

$$d(\Gamma) = d - n_B \frac{d-2}{2} - 2n_F \frac{d-1}{2} + \sum_{i=1}^{n} (d_i - d) \tag{88}$$

is called the superficial divergence *of the 1pi diagram* Γ*. The sum extends over all (n) vertices of the diagram and* d_i *denotes the dimension of the vertex i. The* $-d$ *at each vertex accounts for d-momentum conservation. For a vertex exhibiting* $n_{i,b}$ *boson fields,* $n_{i,f}$ *fermi fields and* l_i *derivatives of fields we have*

$$d_i = n_{i,b} \, \frac{d-2}{2} + n_{i,f} \, \frac{d-1}{2} + l_i$$

such that

$$d_i - d = \begin{cases} d-4 & \textit{for quartic vertices} \\ \frac{d-4}{2} & \textit{for triple vertices} \end{cases}$$

and

$$\sum_{i=1}^{n} (d_i - d) = (2n_4 + n_3) \, \frac{d-4}{2}$$

for a diagram with n_4 *quadrilinear and* n_3 *trilinear vertices. Note that in a renormalizable theory at most four field can meet at a given vertex. Also note that the type of fields or derivative of fields joining at the vertices does not matter.*

By Euler's loop theorem the number of loops can be calculated by

$$L = N_{\text{int}} - V + 1$$

in terms of the total number of internal lines N_{int} *and the total number of vertices* V *of a graph*[14]*. In our case the number of all lines of a Feynman-graph is* $4n_4 + 3n_3 = 2N_{\text{int}} + N_{\text{ext}}$ *and we obtain (*n_B *= number of external boson lines,* n_F *= number of external fermion pairs)*

$$2n_4 + n_3 = 2L - 2 + N_{\text{ext}} = 2L - 2 + n_B + 2n_F \quad .$$

Utilizing this relation may write (88) in the alternative form

$$d(\Gamma) = 4 - n_B - 2n_F \frac{3}{2} + L\,(d-4) \quad . \tag{89}$$

[14] A particle physicist's derivation of the Euler relation is the following: for a connected graph of vertices joined by lines, we assign to each internal line an d-momentum. At each vertex we have d-momentum conservation and thus $N_{\text{int}} - V$ independent momenta. However, one of the momentum conservation relations just represents the conservation of the total external momentum. Thus there are $N_{\text{int}} - V + 1$ independent momenta. The latter are the momenta to be integrated over in a corresponding Feynman amplitude, and the number of independent momenta corresponds to the number of loops L.

The result can be easily understood: the loop expansion of an amplitude has the form

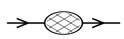
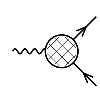
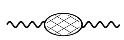
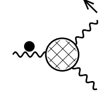
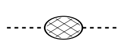
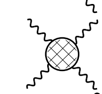
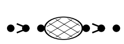
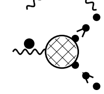
$$A^{(L)} = A^{(0)}\,[1 + a_1\,\alpha_g + a_2\,\alpha_g^2 + \cdots + a_L\,\alpha_g^L + \cdots]$$

where $\alpha_g = \frac{g^2}{4\pi}$ is the conventional expansion parameter. $A^{(0)}$ is the tree level amplitude which coincides with the result in $d = 4$. An additional loop requires at least two additional triple vertices $O(g)$ or a quartic vertex $O(g^2)$ (as is typical for gauge couplings and vertices in Yang-Mills theories), such that an additional loop requires two powers of g. Since the dimension of the bare $\bar{g}_0 = g_0\mu^{\frac{4-d}{2}}$ is $[g] = \frac{4-d}{2}$ each power of α_g indeed means an additional contribution $-2[g] = d - 4$ to the dimension of the integral (explicit factors $\mu^{(4-d)}$ per α_g in the coefficients a_L).

We now are ready to formulate the convergence criterion which reads:

$$\begin{array}{ll} I_\Gamma & \text{convergent} \;\bowtie\; d(\gamma) < 0 \;\;\forall\;\; 1pi \text{ sub} - \text{diagrams } \gamma \subset \Gamma \\ I_\Gamma & \text{divergent} \;\bowtie\; \exists\;\; \gamma \subset \Gamma \text{ with } d(\gamma) \geq 0 \;\;. \end{array}$$

In $d \leq 4$ dimensions, a renormalizable theory has the following types of primitively divergent diagrams (i.e., diagrams with $d(\Gamma) \geq 0$ which may have divergent subintegrals)[15]*:*

	$d(\Gamma)$ d	$(d=4)$		$d(\Gamma)$ d	$(d=4)$
	$d-3$	(1)		$d-4$	(0)
	$d-2$	(2)		$d-3$	(1)
	$d-2$	(2)		$d-4$	(0)
	$d-2$	(2)		$d-3$	(1)

$+(L_\Gamma - 1)(d-4)$ *for a diagram with $L_\Gamma (\geq 1)$ loops. The table shows the non-trivial leading one-loop $d(\Gamma)$ to which per additional loop a contribution $(d-4)$ has to be added (see (89)). Thus the dimensional analysis tells us that convergence improves for $d < 4$. For a renormalizable theory we have*

[15] For the time being we resrict ourselves to the consideration of the non-trivial Green functions, the two- and more-point functions, which are needed for the construction of the scattering-matrix S. Vacuum diagrams, characterized by $n_B = 0, N_F = 0$ (no external legs, $d(\Gamma) \leq d$), usually need not be considered, since the omission of all vacuum diagrams is equivalent to the proper normalization of the non-trivial Green functions and of the S-matrix (i.e., $< 0|S|0 >= 1$). In field theories with spontaneous symmetry breaking, or theories which did undergo a Higgs mechanism scalar one-point functions (tadpoles) may exhibit a non-vanishing vacuum expectation value (e.g., $< 0|H|0 >= v \neq 0$). Corresponding diagrams are characterized by $n_B = 1, n_F = 0$ (one external scalar leg, $d(\Gamma) \leq d/2 + 1$) and must be taken into account as well.

- *$d(\Gamma) \leq 2$ for $d = 4$.*

In lower dimensions

- *$d(\Gamma) < 2$ for $d < 4$*

a renormalizable theory becomes super-renormalizable, while in higher dimensions

- *$d(\Gamma)$ unbounded! $d > 4$*

and the theory is called non-renormalizable[16]*.*

5. 2. Dimensional regularization

Dimensional regularization of theories with spin is defined in three steps.
1. *Start with Feynman rules formally derived in $d = 4$.*
2. *Generalize to $d = 2n > 4$! This intermediate step is necessary in order to treat the vector and spinor indices appropriately.*

1) For fermions we need the $d = 2n$ dimensional Dirac algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1} \; ; \;\; \{\gamma^\mu, \gamma_5\} = 0 \tag{90}$$

where γ_5 must satisfy $\gamma_5^2 = \mathbf{1}$ and $\gamma_5^+ = \gamma_5$ such that $\frac{1}{2}(\mathbf{1} \pm \gamma_5)$ are the chiral projection matrices. The metric has dimension d

$$g^{\mu\nu} g_{\mu\nu} = g_\mu^\mu = d \;\; ; \;\; g_{\mu\nu} = \begin{pmatrix} 1 & 0 & \cdots & \\ 0 & -1 & & \\ \vdots & & \ddots & \\ & & & -1 \end{pmatrix} \; . \tag{91}$$

By $\mathbf{1}$ we denote the unit matrix in spinor space. In order to have the usual relation for the adjoint spinors we furthermore require

$$\gamma^{\mu +} = \gamma^0 \gamma^\mu \gamma^0 \;\; .$$

Simple consequences of this d-dimensional algebra are:

$$\begin{array}{lcl} \gamma_\alpha \gamma^\alpha & = & d\,\mathbf{1} \\ \gamma_\alpha \gamma^\mu \gamma^\alpha & = & (2-d)\,\gamma^\mu \\ \gamma_\alpha \gamma^\mu \gamma^\nu \gamma^\alpha & = & 4g^{\mu\nu}\,\mathbf{1} + (d-4)\,\gamma^\mu\gamma^\nu \\ \gamma_\alpha \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\alpha & = & -2\gamma^\rho\gamma^\nu\gamma^\mu + \; (4-d)\gamma^\mu\gamma^\nu\gamma^\rho \text{ etc.} \end{array} \tag{92}$$

[16]Such theories (they often show up as effective theories in low energy expansions, like chiral perturbation theory, for example, which describes the low energy behavior of QCD) can be renormalized (i.e., made finite, in spite of the naming), but only at the expense of a proliferation of new types of counter terms and corresponding new adjustable parameters, showing up in each order of the perturbation (low energy) expansion.

Traces of strings of γ-matrices are very similar to the ones in 4-dimensions. In $d = 2n$ dimensions one can easily write down $2^{d/2}$ dimensional representations of the Dirac algebra [47]. Then

$$\begin{array}{lcl} Tr\mathbf{1} & = & f(d) = 2^{d/2} \\ Tr\prod_{i=1}^{2n-1}\gamma^{\mu_i}(\gamma^5) & = & 0 \\ Tr\gamma^\mu\gamma^\nu & = & f(d)\; g^{\mu\nu} \\ Tr\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma & = & f(d)\;(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\;\; \text{etc.} \end{array} \tag{93}$$

One can show that for renormalized quantities *the only relevant property of $f(d)$ is $f(d) \to 4$ for $d \to 4$. Very often the convention $f(d) = 4$ (for any d) is adopted. Bare quantities and the related MS or $\overline{MS}$ quantities depend upon this convention (by terms proportional to* $\ln 2$*).*

In anomaly free theories we can assume γ_5 to be fully anticommuting! But then

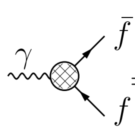
$$Tr\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 = 0 \;\; \textit{for all} \;\; d \neq 4!$$

The 4-dimensional object

$$4i\varepsilon^{\mu\nu\rho\sigma} = Tr\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 \;\; \text{for} \;\; d = 4 \tag{94}$$

cannot be obtained by dimensional continuation if we use an anticommuting γ_5 [48].

Since fermions do not have self interactions they only appear as closed fermion loops, which yield a trace of γ-matrices, or as a fermion strings connecting an external $\psi \cdots \bar{\psi}$ pair of fermion fields. In a transition amplitude $|T|^2 = Tr(\cdots)$ we again get a trace. Consequently, in principle, we have eliminated all γ's! Commonly one writes a covariant tensor decomposition into invariant amplitudes, like, for example,



$$= -ie\left\{\gamma^\mu A_1 + \gamma^\mu\gamma_5 A_2 + i\sigma^{\mu\nu}\frac{q_\nu}{2m}A_3 + \cdots\right\}$$

where μ is an external index.

2) External momenta (and external indices) must be taken $d = 4$ dimensional because four functions cannot be analytic continuation of three etc. The following rules apply:

External momenta :	$p^\mu = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$	4 dimensional
Loop momenta :	$k^\mu = (k^0, \cdots k^{d-1})$	d dimensional
	$k^2 = (k^0)^2 - (k^1)^2 - \cdots - (k^{d-1})^2$	
	$pk = p^0k^0 - \vec{p}\cdot\vec{k}$	4 dimensional etc.

3. *Interpolation in d to complex values and extrapolation to $d < 4$.*

Loop integrals now read

$$\mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \cdots$$

with μ an arbitrary scale parameter. The crucial properties valid in DR independent of d are: (F.P. = finite part)

a) $\int d^d k k_\mu f(k^2) = 0$
b) $\int d^d k f(k+p) = \int d^d k f(k)$
which is <u>not</u> true with UV cut − off's
c) If $f(k) = f(|\, k\,|)$:
$\int d^d k f(k) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty dr r^{d-1} f(r)$

d) *For divergent integrals, by analytic subtraction, :*
F.P. $\int_0^\infty dr r^{d-1+\alpha} \equiv 0$ *for arbitrary* α
so-called minimal subtraction (MS). Consequently

$$F.P. \int d^d k f(k) = F.P. \int d^d k f(k+p) = F.P. \int d^d(\lambda k) f(\lambda k) \quad .$$

This implies that

dimensionally regularized integrals behave like convergent integrals *and formal manipulations are justified. Starting with d sufficiently small, by partial integration, one can always find a representation for the integral which converges for $d = 4 - \varepsilon$, $\varepsilon > 0$ small.*

In the following we discuss DR for one-loop integrals in some detail.

<u>*One-loop integrals:*</u>
An integral of the general form

$$I_\Gamma^{\mu_1 \cdots \mu_m}(p_1, \cdots, p_n) = \int d^d k \frac{\prod_{j=1}^m k^{\mu_j}}{\prod_{i=1}^n ((k+p_i)^2 - m_i^2 + i\varepsilon)} \tag{95}$$

has superficial degree of divergence

$$d(\Gamma) = d + m - 2n \leq d - 2$$

where the bound holds (see Eq. (89) and Footnote [15] on page 45) for two- or more-point functions in <u>renormalizable</u> theories and $d \leq 4$. I_Γ is convergent for $d(\Gamma) < 0$ in $d = 4$. If $d(\Gamma) \geq 0$ in $d = 4$, consider $d = n$ integer ≥ 4 because of vectors k^{μ_i}. We split the objects into:
Vectors in physical subspace ($\mu = 0, 1, 2, 3$): $\hat{p}, \hat{k}, \cdots$
Vectors in the $d - 4$ dimensional complement ($\mu = 4, \cdots,\ d - 4$): $\bar{p}, \bar{k}, \cdots$.

Correspondingly, the notation is as follows:

$$\begin{array}{lll} \text{External momenta} & : & p_i = \hat{p}_i;\ \bar{p}_i = 0 \\ \text{Loop momenta} & : & k = \hat{k} + \bar{k};\ \hat{k}\cdot\bar{k} = 0 \\ & & p_i k = \hat{p}_i \hat{k} \\ & & k^2 = \hat{k}^2 + \bar{k}^2 = \hat{k}^2 - \omega^2;\ \omega = \mid \bar{k} \mid \\ \text{Metric tensor} & : & g^{\mu\nu} = \hat{g}^{\mu\nu} + \bar{g}^{\mu\nu};\ \hat{g}\cdot\bar{g} = 0 \\ & & g^\mu_\mu = d,\ \hat{g}^\mu_\mu = 4,\ \bar{g}^\mu_\mu = d-4 \ \ . \end{array}$$

In physical amplitudes the indices of the integral $I_\Gamma^{\mu_1\cdots\mu_m}$ *are either external (if we resort to a covariant decomposition e.g.) or contracted. The possibilities are:*
1. contraction with an external momentum $p_i^{\mu_j} : k^{\mu_j} \to \hat{k}^{\mu_j}$ $(pk = \hat{p}\hat{k})$
2. the index is external (e.g. γ *matrix)* $\hat{g}^{\mu_j\alpha} : \hat{g}^{\mu_j}_\alpha k^\alpha = \hat{k}^{\mu_j}$
3. an index pair is contracted (with $g_{\mu\nu}$*)* $: g_{\mu\nu}k^\mu k^\nu = k^2$.
In the first two cases the k^{μ_i}*'s can be taken four dimensional, in the last case we obtain an integral of the form*

$$\int d^dk \frac{k^2}{(k+p)^2 - m^2 + i\varepsilon} \frac{k^{\mu_1}\cdots k^{\mu_{m-2}}}{\prod_{i=1}^{n-1}((k+p_i')^2 - m_i^2 + i\varepsilon)} \ \ .$$

We write

$$k^2 = (k+p)^2 - m^2 - (2\hat{p}\hat{k} + \hat{p}^2 - m^2)$$

such that

$$\frac{k^2}{(k+p)^2 - m^2 + i\varepsilon} = 1 - \frac{(2\hat{p}\hat{k} + \hat{p}^2 - m^2)}{(k+p)^2 - m^2 + i\varepsilon} \ \ .$$

In this way all the one-loop integrals reduce to a sum of integrals of the form

$$I_\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_m}(\hat{p}_1, \cdots, \hat{p}_n) = \int d^dk \frac{\prod_{j=1}^m \hat{k}^{\mu_j}}{\prod_{i=1}^n((k+\hat{p}_i)^2 - m_i^2 + i\varepsilon)}$$

with

$$d^dk = d^4\hat{k}d^{d-4}\bar{k} = d^4\hat{k}\omega^{d-5}d\omega d\Omega_{d-4} \ \ .$$

In the $d-4$ *dimensional complement the integrand depends on* ω *only! The angular integration over* $d\Omega_{d-4}$ *yields*

$$\int d\Omega_{d-4} = S_{d-4} = \frac{2\pi^{\varepsilon/2}}{\Gamma(\varepsilon/2)}\ ; \ \ \varepsilon = d-4\ ,$$

which is the surface of the $d-4$ *dimensional sphere. Using this result we get (discarding 4-dimensional indices)*

$$I_\Gamma(p_i) = \int d^4\hat{k} J_\Gamma(d, \hat{p}, \hat{k})$$

where

$$J_\Gamma(d,\hat p,\hat k) = S_{d-4}\int_0^\infty d\omega\omega^{d-5} f(\hat p,\hat k,\omega)\ .$$

Now this integral can be analytically continued to <u>*complex*</u> *values of d. For the ω-integration we have*

$$d^\omega(\Gamma) = d-4-2n$$

i.e. the ω-integral converges if

$$d < 4+2n \ \ (*)\ .$$

On the other hand the condition for convergence of I_Γ is

$$d(\Gamma) = d+m-2n < 0$$

i.e. $d < 2n-m$ but then $(*)$ is also true. As a result we find that
• *for a* <u>*renormalizable*</u> *theory $d(\Gamma)\le 2$ in $d\le 4$ and hence all integrals converge for* <u>*Re $d<2$*</u>*.* <u>*However:*</u>

$$\int_0^\infty d\omega\omega^{d-5} f(\hat p,\hat k,\omega)$$

is <u>*infrared divergent*</u> *for Re $d < 4$*[17]*. The integral has*

$$\text{domain of convergence}:\ \ 4 < d < 4+2n$$

and is

$$\underline{\text{analytic}}\text{ in a strip}:\ \ 4 < \mathit{Re}\, d < 4+2n\ \ .$$

Therefore it can be defined by analytic continuation in the complex d-plane. The analytic continuation can be obtained by partial integration:

$$\int_0^\infty d\omega\omega^{d-5} f(\hat p,\hat k,\omega) = \frac{\omega^{d-4}}{d-4} f(\hat p,\hat k,\omega)\Big|_0^\infty - \int_0^\infty d\omega\frac{\omega^{d-4}}{d-4}\frac{\partial}{\partial\omega} f(\hat p,\hat k,\omega)$$

The first term vanishes in $4 < Re\, d < 4+2n$ the second term (integral) is convergent for $3 < Re\, d < 4+2n$ with a <u>*pole*</u> *at $d=4$! Using*

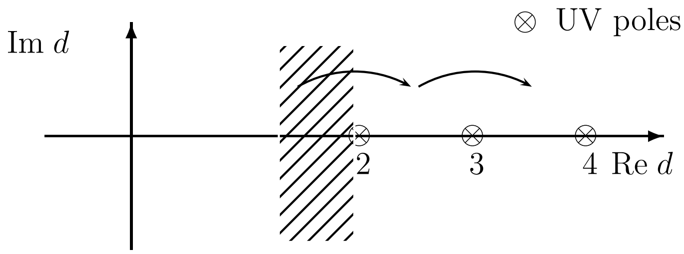
$$\frac{S_{d-4}}{d-4} = \frac{2\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d-4}{2}+1)}\ ;\ \ (d-4)\Gamma(\frac{d-4}{2}) = 2\Gamma(\frac{d-4}{2}+1)$$

[17] These infrared (IR) divergences have nothing to do with the IR divergences known to arise from massless particle propagators. They are the consequence of the analytical continuation to negative dimensions of the $d-4$ dimensional complementary space of the 4 dimensional physical space. The radial integration measure then assumes negative powers.

p-fold partial integration yields

$$I_\Gamma^d = \frac{2\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d-4}{2}+p)} \int d^4\hat{k} \int_0^\infty d\omega\omega^{d-5+2p} \left(-\frac{\partial}{\partial\omega^2}\right)^p f(\hat{p},\hat{k},\omega) \tag{96}$$

where the integral is convergent in $4-2p < \mathrm{Re}\, d < 2n-m = 4-d^{(4)}(\Gamma) \geq 2$ *.*
For a renormalizable theory at most 2 partial integrations are necessary to define the theory.



One problem case remains. For $n=1$ *the integral* $\int d^dk\ \frac{1}{k^2}$ *diverges for* <u>*any*</u> *d! and hence must be regularized differently*
1. *either by an IR regulator i.e. finite mass* m *or*
2. *by an UV cut off* Λ*.*
For simplicity of notation we may consider the problem in Euclidean space (see below): $k \to \underline{k}$ *etc. In the first case we obtain*

$$\begin{aligned}\frac{1}{(2\pi)^d}\int d^d\underline{k}\frac{1}{\underline{k}^2+m^2} &= \frac{S_d}{(2\pi)^d}\int_0^\infty d\omega\frac{\omega^{d-1}}{\omega^2+m^2} \ \text{ convergent for Re } d<2 \\ &= \frac{S_d}{(2\pi)^d}\frac{\Gamma(\frac{d}{2})\,\Gamma(1-\frac{d}{2})}{2}\,m^{d-2} \ \text{ analytical in d .}\end{aligned}$$

For Re $d>2$ *the limit* $\lim_{m\to 0}\equiv 0$ *exists and thus as an analytical function:*

$$\int d^dk\ \frac{1}{k^2} \equiv 0 \ \ \forall\, d\ .$$

In the second case we find

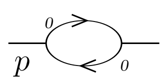
$$\begin{aligned}\frac{1}{(2\pi)^d}\int_{|\underline{k}|<\Lambda} d^d\underline{k}\ \frac{1}{\underline{k}^2} &= \frac{S_d}{(2\pi)^d}\int_0^\Lambda d\omega\omega^{d-3} \ \text{ convergent for Re } d>2 \\ &= \frac{S_d}{(2\pi)^d}\frac{1}{d-2}\,\Lambda^{d-2} \ \text{ analytical in d .}\end{aligned}$$

For Re $d<2$ *the limit* $\lim_{\Lambda\to\infty}\equiv 0$ *exists and thus as an analytical function:*

$$\int d^dk\ \frac{1}{k^2} \equiv 0 \ \ \forall\, d\ .$$

Notice that 1. and 2. yield the same unambiguous result![18] *The result also conforms with naive expectations based on dimensional analysis. Since a dimensionful parameter like a mass is not available to carry the dimension of the object it has to be zero for all $d > 2$.*

A somewhat different problematic (non-standard) situation one has with the integral $\int \frac{d^dk}{k^4}$ which again is divergent for any d. We get it when considering the massless loop integral



$$= \quad I(\underline{p}^2) = \frac{\mu_0^\varepsilon}{(2\pi)^d} \int d^d\underline{k} \frac{1}{\underline{k}^2} \frac{1}{(\underline{k}+\underline{p})^2}$$

at zero external momentum (i.e., on the mass-shell of the massless particle). While for $p^2 \neq 0$ dimensional regularization works in the standard fashion a direct evaluation at $p^2 = 0$ requires a more careful analysis, because at $d = 4$, the integral is logarithmically divergent at the same time in the UV and the IR region.

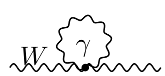
If we follow the reasoning presented for the integral $\int \frac{d^dk}{k^2}$ we would introduce an IR regularization, a mass or, equivalently, an IR cut-off $\omega_{\min}$ such that the integral is well defined for Re $d < 4$:[19]

$$\frac{\mu_0^\varepsilon}{(2\pi)^d} \int_{\omega_{\min}<|\underline{k}|} d^d\underline{k}\, \frac{1}{\underline{k}^4} = \mu_0^\varepsilon \frac{S_d}{(2\pi)^d} \int_{\omega_{\min}}^{\infty} d\omega \omega^{d-5} \; = -\mu_0^\varepsilon \frac{S_d}{(2\pi)^d} \frac{\omega_{\min}^{d-4}}{d-4}$$

which is analytic in d. For $\mathrm{Re}\ d > 4$ *the limit $\lim_{\omega_{\min}\to 0} \equiv 0$ exists and thus as an analytical function:*

$$\int d^dk\, \frac{1}{k^4} \equiv 0 \;\; \forall\, d\,.$$

[18]Remark: In the electroweak SM such integrals directly show up e.g. in the W self-energy via the diagram



with an electromagnetic (photon e.g. in Feynman gauge) "seagull" attached to the (external) W line. Remember that such self contractions are present due to the fact that one is working with the classical and not with the Wick ordered form of the Lagrangian. Working with a Wick ordered Lagrangian $:\mathcal{L}(x):$ would spoil the classical form of the ST- and/or WT-identities. Via the proper definition of such integrals, in fact, corresponding self-contractions yield a vanishing contribution for massless particles like the photon.

[19]The pre-factor $\mu_0^\varepsilon \frac{S_d}{(2\pi)^d}$, with $S_d = 2\pi^{d/2}/\Gamma(d/2)$ and $\Gamma(x)$ Euler's Γ-function, may be expanded in $\varepsilon = 4 - d$ as follows:

$$\begin{aligned} \mu_0^\varepsilon \frac{S_d}{(2\pi)^d} &= \frac{2}{(4\pi)^2} \exp(\frac{\varepsilon}{2}\ln\mu_0^2) \exp(\frac{\varepsilon}{2}\ln 4\pi) \frac{1}{\Gamma(d/2)} \\ &\simeq \frac{2}{(4\pi)^2} \left\{1 + \frac{\varepsilon}{2}\left[1 - \gamma + \ln 4\pi + \ln\mu_0^2\right]\right\} + O(\varepsilon^2) \end{aligned}$$

where we used $\Gamma(d/2) = (1-\frac{\varepsilon}{2})\,\Gamma(1-\frac{\varepsilon}{2}) \simeq (1-\frac{\varepsilon}{2})\,(1+\gamma\frac{\varepsilon}{2}) \simeq (1-\frac{\varepsilon}{2}\,(1-\gamma))$ where $\gamma = -\Gamma'(1)$ is Euler's constant.

If we expand the integral at finite $\omega_{\rm min}$ *in* $\varepsilon_{\rm UV} = 4 - d$ *we find*

$$\frac{\mu_0^\varepsilon}{(2\pi)^d}\int_{\omega_{\rm min}<|\underline{k}|} d^d\underline{k}\,\frac{1}{\underline{k}^4} \;\;\simeq\;\; \frac{1}{(4\pi)^2}\left\{\frac{2}{\varepsilon_{\rm UV}} + \left[1-\gamma+\ln 4\pi + \ln\mu_0^2\right] - \ln\omega_{\rm min}^2\right\} + O(\varepsilon)$$

Now, however, we cannot take any longer the limit $\lim_{\omega_{\rm min}\to 0}$ *because we have lost analyticity in* d *by the truncation of the series expansion. We may nullify the integral, however, by the choice*

$$\ln\omega_{\rm min}^2 = \left\{\frac{2}{\varepsilon_{\rm UV}} + \left[1-\gamma+\ln 4\pi+\ln\mu_0^2\right]\right\} \ .$$

Alternatively, we could introduce an UV cut-off $\omega_{\rm max}$ *such that the integral is well defined for* Re $d > 4$*:*

$$\frac{\mu_0^\varepsilon}{(2\pi)^d}\int_{|\underline{k}|<\omega_{\rm max}} d^d\underline{k}\,\frac{1}{\underline{k}^4} = \mu_0^\varepsilon\,\frac{S_d}{(2\pi)^d}\int_0^{\omega_{\rm max}} d\omega\omega^{d-5} \;=\mu_0^\varepsilon\,\frac{S_d}{(2\pi)^d}\frac{\omega_{\rm max}^{d-4}}{d-4}$$

which is analytic in d*. For* Re $d < 4$ *the limit* $\lim_{\omega_{\rm max}\to\infty} \equiv 0$ *exists and thus as an analytical function:*

$$\int d^dk\,\frac{1}{k^4} \equiv 0 \;\;\forall\, d\ ,$$

consistent with the previous result. Here, we may expand the result at finite $\omega_{\rm max}$ *in* $\varepsilon_{\rm \;IR} = d - 4$ *and find*

$$\frac{\mu_0^\varepsilon}{(2\pi)^d}\int_{|\underline{k}|<\omega_{\rm max}} d^d\underline{k}\,\frac{1}{\underline{k}^4} \;\;\simeq\;\; \frac{1}{(4\pi)^2}\left\{\frac{2}{\varepsilon_{\rm \;IR}} - \left[1-\gamma+\ln 4\pi + \ln\mu_0^2\right] + \ln\omega_{\rm max}^2\right\} + O(\varepsilon)$$

Again, we cannot take any longer the limit $\lim_{\omega_{\rm max}\to\infty}$ *but can make the integral vanish by choosing*

$$\ln\omega_{\rm max}^2 = \left\{\frac{-2}{\varepsilon_{\rm \;IR}} + \left[1-\gamma+\ln 4\pi+\ln\mu_0^2\right]\right\}$$

So far our argumentation was quite formal, and in fact the assignment $\int d^dk\,\frac{1}{k^4} \equiv 0$ *is somewhat misleading and could lead to wrong results. At* $d = 4$ *the integral is dimensionless and the absence of a dimensionful parameter does not help to support the above "result". It means that an UV divergence is canceled by an IR singularity. The two kind of singularities, however, physics-wise have nothing to do with each other*[20]*. We therefore have to look at what is happening more closely. If we regularize*

[20] Note that in case of the integral $\int d^dk\,\frac{1}{k^2} \equiv 0$ the situation is different. At $d = 4$ the integral is quadratically UV divergent but IR finite and the reason for its vanishing follows from the dimensional argument

the integral by introducing an explicite UV cut-off $\omega_{\max}$ as well as an explicite IR cut-off $\omega_{\min}$:

$$\begin{aligned}
\frac{\mu_0^\varepsilon}{(2\pi)^d}\int_{\omega_{\min}<|\underline{k}|<\omega_{\max}} d^d\underline{k}\;\frac{1}{\underline{k}^4} &= \mu_0^\varepsilon\,\frac{S_d}{(2\pi)^d}\int_{\omega_{\min}}^{\omega_{\max}} d\omega\omega^{d-5} \;=\mu_0^\varepsilon\;\frac{S_d}{(2\pi)^d}\frac{\omega^{d-4}}{d-4}\Bigg|_{\omega_{\min}}^{\omega_{\max}} \\
&\overset{\varepsilon\to 0}{\simeq} (4\pi)^{-2}\left\{\frac{-2}{\varepsilon}-\Big[1-\gamma+\ln 4\pi+\ln\mu_0^2\Big]+\ln\omega^2\right\}\Bigg|_{\omega_{\min}}^{\omega_{\max}}+O(\varepsilon) \\
&= (4\pi)^{-2}\;\ln\frac{\omega^2_{\max}}{\omega^2_{\min}} \\
&= (4\pi)^{-2}\;\{(Reg)_{\rm UV}-(Reg)_{\;\rm IR}\} = (4\pi)^{-2}\;\left\{\frac{2}{\tilde{\varepsilon}_{\rm UV}}-\frac{2}{\tilde{\varepsilon}_{\;\rm IR}}\right\}\;,
\end{aligned}$$

with the definitions

$$\begin{aligned}
(Reg)_{\rm UV} &= \frac{2}{\varepsilon}-\gamma+\ln 4\pi+\ln\mu^2_{\rm UV}\equiv\frac{2}{\tilde{\varepsilon}_{\rm UV}} \\
(Reg)_{\;\rm IR} &= \frac{2}{\varepsilon}-\gamma+\ln 4\pi+\ln{\mu^2}_{\rm IR}\equiv\frac{2}{\tilde{\varepsilon}_{\;\rm IR}}
\end{aligned}$$

as the UV- and the IR-regulators, respectively, and

$$\begin{aligned}
\ln\mu^2_{\rm UV} &= \ln\mu_0^2-\ln\omega^2_{\min} \\
\ln{\mu^2}_{\rm IR} &= \ln\mu_0^2-\ln\omega^2_{\max}\;.
\end{aligned}$$

We have $\mu^2_{\rm UV}\geq\mu^2_{\rm IR}$ as $\omega^2_{\max}\geq\omega^2_{\min}$. Note that the result at $d=4$ is identical to the one obtained by taking the difference between two subtraction points:

$$I(p^2=\omega^2_{\max})-I(p^2=\omega^2_{\min})=(4\pi)^{-2}\;\ln\frac{\omega^2_{\max}}{\omega^2_{\min}}\;.$$

Thus, in fact the integral is an arbitrary constant. The frequently adopted assignment

$$F.P.\int d^dk\;\frac{1}{(k^4)}\equiv 0$$

corresponds to the special choice $\tilde{\varepsilon}_{\;\rm IR}=\tilde{\varepsilon}_{\rm UV}$ or $\omega_{\max}=\omega_{\min}$. Such a convention, in general, may be adopted without problem when considering physical (renormalized) quantities which are UV finite and must be IR finite as well, if both kind of singularities are simultaneously regularized by dimensional regularization in the sense just discussed. It may be dangerous, however, if one is using such tricks in the derivation of renormalization group coefficients (which are not observables) where the UV part only contributes. In such a case one would obtain wrong results after mixing up IR with UV poles in ε or, equivalently, would identify ${\mu^2}_{\rm IR}$ with $\mu^2_{\rm UV}$.

In cases of doubts, one has to remember that in any renormalizable QFT the basic objects in the first place are the off-shell (time-ordered) Green functions, which are well defined and finite after renormalization. On-shell limits are more problematic in cases where massless particles are involved. The naive construction of the S-matrix elements then often does not work (IR problems in QED or QCD). In the case just discussed it thus may be a good idea to start from the off-shell integral $I(p^2)$ and take on-shell limits for quantities only for which they are well defined.

The considerations presented may be extended (with the proviso discussed for $n = 2$) to

$$\int d^dk \; (k^2)^m \equiv 0 \; ; \quad \int d^dk \; \frac{1}{(k^2)^n} \equiv 0$$

$$\int d^dk k^{\mu_i} \cdots k^{\mu_m} \equiv 0 \; . \tag{97}$$

If integrals are not dimensionless one can justify the results by dimensional arguments. This is equivalent to the rule:

$$F.P. \int_0^\infty d\omega \; \omega^\alpha = 0 \text{ any } \alpha \tag{98}$$

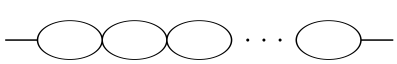
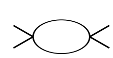
where again $\alpha = -1$ is a boarder case. These results are equivalent to the prescription: *if possible (if not see previous discussion)* **perform partial integrations until the integral converges for** $d \leq 4 - \varepsilon$; $\varepsilon > 0$ *infinitesimal*, **and ignore boundary terms**.

We may summarize the results as follows:

- In DR divergent Feynman integrals can be represented by integrals converging in the strip $3 < Re\, d < 4$.
- The analytic continuation to this strip is obtained by partial integration and ignoring the boundary terms.
- The integrals are meromorphic functions in d with poles at certain $d = n$ integer.
- The poles at $d = 4$ can only be removed by renormalization.

We add two remarks concerning higher orders and infrared problems.

Higher orders: The order of the poles is given by the order of the perturbation expansion (number of loops)

 $\propto \varepsilon^{-n}$ since  $\propto \varepsilon^{-1}$

Infrared problems: $m_i = 0$ integrals

a) <u>One-Loop</u>: the worst case is $I_\Gamma^{\mu_1 \cdots \mu_m}$ for $m = 0$. For <u>off-shell</u> momenta (i.e. p_i generic) the integral

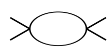
$$\int d^dk \; \frac{1}{\prod_{i=1}^n ((k+p_i)^2 + i\varepsilon)} = \int d^dk \; \frac{1}{k^2 + i\varepsilon} \frac{1}{(k+p_1')^2 + i\varepsilon} \; \cdots$$

has the domain of convergence

$$\begin{array}{ccccc} 2 & < & d & < & 2n \\ & \uparrow & & \uparrow & \\ & \text{for} & & \text{for} & \\ & IR & & UV & \text{convergence} \\ \Rightarrow & \multicolumn{4}{l}{\text{no problem for } n > 1!} \end{array}$$

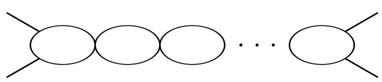
The only problem case (n=1) has been discussed already before.
b) *<u>Higher orders</u>:*

1)1 − loop :



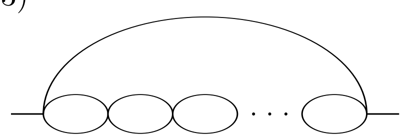
$$\propto \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{k^2} \, \frac{1}{(k+p)^2} \propto (p^2)^{\frac{d-4}{2}}$$

2)n − loop :



$$\propto (p^2)^{n\frac{d-4}{2}}$$

3)

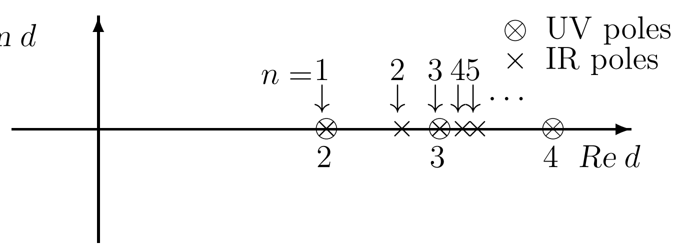


$$\propto \int \frac{d^d k}{(2\pi)^d} \, (k^2)^{n\frac{d-4}{2}} \, \frac{1}{(k+p)^2}$$

This integral is UV convergent for $d < 4$ and IR convergent for $d > \frac{4n}{n+1} \overset{n\to\infty}{\to} 4$!
Thus, in $n+1$-loop order the convergence domain is

$$\frac{4n}{n+1} < d < 4$$

and shrinks to zero as $n \to \infty$.



Result: *For generic <u>off-shell</u> momenta time ordered Green functions exist in DR also for the massless case (on-shell is another story!).*

5. 3. Tools for evaluation of Feynman integrals

1. *<u>$\varepsilon = 4 - d$ expansion, $\varepsilon \to +0$</u>.*

For the expansion of integrals near $d = 4$ we need some asymptotic expansions of Γ-functions:

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}\ ;\quad \Gamma(\frac{1}{2}) = \sqrt{\pi}\ ;\quad \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \tag{99}$$

$$\begin{aligned} \psi(1+x) &= \frac{d}{dx}\ln\Gamma(1+x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)} \overset{|x|<1}{=} -\gamma + \sum_{n=2}^{\infty}(-1)^n\zeta(n)x^{n-1} \\ \Gamma(1+x) &= \exp\left[-\gamma\, x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta(n)x^n\right] \end{aligned}$$

where $\zeta(n)$ denotes Riemann's Zeta function.

$$\begin{aligned} \Gamma'(1) &= -\gamma\ ;\quad \gamma = 0.577215\cdots \text{ Euler's constant} \\ \Gamma''(1) &= \gamma^2 + \zeta(2)\ ;\quad \zeta(2) = \frac{\pi^2}{6} = 1.64493\cdots \end{aligned}$$

As a result for later use we keep in mind

$$\Gamma(1+\frac{\varepsilon}{2}) = 1 - \frac{\varepsilon}{2}\gamma + (\frac{\varepsilon}{2})^2\frac{1}{2}(\gamma^2+\zeta(2)) + \cdots \tag{100}$$

Further relations

$$\begin{aligned} \Gamma(x)\,\Gamma(1-x) &= \frac{\pi}{\sin\pi x} \\ \Gamma(\frac{1}{2}+x)\,\Gamma(\frac{1}{2}-x) &= \frac{\pi}{\cos\pi x} \end{aligned}$$

2. *Bogolubov-Schwinger parametrization.*

Suppose we choose for each propagator an independent momentum and take into account momentum conservation at the vertices by δ-functions. Then, for $d = n$ integer, we use

i)

$$\frac{i}{p^2 - m^2 + i\varepsilon} = \int_0^{\infty} d\alpha e^{-i\alpha(m^2-p^2+i\varepsilon)}$$

ii)

$$\delta^{(d)}(k) = \frac{1}{(2\pi)^d}\int_{-\infty}^{+\infty} d^dx e^{ikx}$$

and find that all momentum integrations are of Gaussian type. The Gaussian integrals yield

$$\int d^dkP(k)e^{ia(k^2+2b(k\cdot p))} = P\left(\frac{-i}{2ab}\frac{\partial}{\partial p}\right)\left(\frac{\pi}{ia}\right)^{d/2} e^{-iab^2p^2}$$

for any polynomial P. The resulting form of the Feynman integral is the so-called Bogolubov-Schwinger representation.

3. *Feynman parametric representation.*

Transforming pairs of α-variables in the above Bogolubov-Schwinger parametrization according to (l is denoting the pair (i,k))

$$(\alpha_i,\alpha_k) \to (\xi_l,\alpha_l) : (\alpha_i,\alpha_k) = (\xi_l\alpha_l,(1-\xi_l)\alpha_l)$$

$$\int_0^\infty\int_0^\infty d\alpha_i d\alpha_k \cdots = \int_0^\infty d\alpha_l\alpha_l \int_0^1 d\xi_l \cdots \quad ,$$

the integrals are successively transformed into $\int_0^1 d\xi \cdots$ integrals and at the end there remains one α-integration only which can be performed using

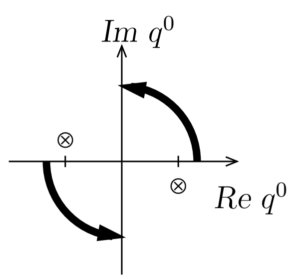
$$\int_0^\infty d\alpha\alpha^a e^{-\alpha x} = \Gamma(a+1)x^{-(a+1)} \quad .$$

The result is the Feynman parametric representation. If L is the number of lines of a diagram, the Feyman integral is $L-1$ dimensional (all other integrations being "trivial").

4. *Euclidean region, Wick rotations.*

Time ordered Green functions may be continued analytically in the complex p^0 (x^0) plane. Crucial is the $i\varepsilon$-prescription in the propagators:

$$\frac{1}{p^2-m^2+i\varepsilon} = \frac{1}{p^0-\sqrt{\vec{p}^{\,2}+m^2-i\varepsilon}}\,\frac{1}{p^0+\sqrt{\vec{p}^{\,2}+m^2-i\varepsilon}}$$



We can thus perform a rotation by $\pi/2$

$$p \to \underline{p}\ ; \quad p^0 \to ip^0 = p^4\ ; \quad p^2 \to -\underline{p}^2$$

without crossing any pole. The euclidean propagators

$$\frac{1}{p^2-m^2+i\varepsilon} \to -\frac{1}{\underline{p}^2+m^2}$$

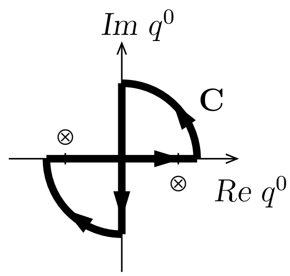
are positive (discarding the overall sign) and any Feyman amplitude in Minkowski space may be obtained via

$$I_M(p) = (-i)^{N_{\text{int}}}(-i)^{V-1}\, I_E(\underline{p})\,|_{p^4=ip^0\ ;\ m^2\to m^2-i\varepsilon} \tag{101}$$

from its euclidean version. Here, N_{int} denotes the number of internal lines (propagators) and V the number of vertices if we use the substitutions (convention dependent, see below)

$$\frac{1}{p^2-m^2+i\varepsilon} \to \frac{1}{\underline{p}^2+m^2}\ ;\ \ ig_i \to i\,(ig_i) = -g_i\ ;\ \ \int d^dk \to \int d^d\underline{k}$$

to define the euclidean Feynman amplitudes.
The basic property which allows us to perform a Wick rotation is analyticity which derives from the causality of a relativistic QFT. Since the amplitudes are analytical within the domain covered by the closed contour $\mathbf{C}$ *(*$\mathrm{Im}\,p^0 = f_{\mathrm{C}}\,(\mathrm{Re}\,p^0)$*)*



in the complex p^0-plane, with arc segments of radius R, we know that for any analytic function F(p)

$$\oint dp^0\, F(p^0,\vec{p}\,) = 0$$

and thus

$$\left(\int\limits_{-R}^{+R} d(\mathrm{Re}\,p^0) + \ i \oint\ d\varphi\, p^0 + \int\limits_{+iR}^{-iR} d(i\mathrm{Im}\,p^0) \right)\ F(p^0,\vec{p}\,) = 0\ \ .$$

On the arc segments we used that $p^0 = Re^{i\varphi}$ such that

$$dp^0 = \frac{dp^0}{d\varphi}\,d\varphi = iRe^{i\varphi}d\varphi = ip^o d\varphi\ \ .$$

Provided the three integrals converge in the limit $R \to \infty$ and the integrand falls off at infinity fast enough such that the contribution from the arc segments vanish

$$\lim_{R\to\infty} \oint d\varphi\, p^0\, F(p^0 = \mathrm{R}e^{i\varphi}, \vec{p}\,) = \lim_{R\to\infty} \int\limits_{0\leq\varphi\leq 90,\; 270\geq\varphi\geq 180} d\varphi\, p^0\, F(p^0 = \mathrm{R}e^{i\varphi}, \vec{p}\,) = 0$$

we obtain

$$\int\limits_{-\infty}^{+\infty} d(\mathrm{Re}\, p^0)\, F(p^0, \vec{p}\,) = \int\limits_{-i\infty}^{+i\infty} d(i\mathrm{Im}\, p^0)\, F(p^0, \vec{p}\,) = \int\limits_{-\infty}^{+\infty} dp^4\;\; F(p^0, \vec{p}\,)\,\Big|_{p^0=-ip^4} \quad .$$

For the dimensionally regularized amplitudes, where potentially divergent integrals are defined via analytic continuation from regions in the complex d-plane where integrals are manifestly convergent, the terms from the arc segments can always be dropped. Also note that dimensional regularization and the power counting rules (superficial degree of divergence etc.) hold irrespective of whether we work in d dimensional Minkowski space-time or in d-dimensional Euclidean space. The metric is obviously not important for the UV-behavior of the integrals.

Euclidean QFT may be defined via the path-integral with positive Euclidean action $\int d^d\underline{x}\, \mathcal{L}_E(\underline{x})$ determining the weight

$$\exp\; - \int d^d\underline{x}\, \mathcal{L}_E(\underline{x})$$

of each field configuration. This compares with the Minkowski version where the action $\int d^dx\, \mathcal{L}_M(x)$ enters as

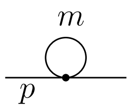
$$\exp\; i \int d^dx\, \mathcal{L}_M(x)$$

(which as a complex quantity does not allow for a probabilistic interpretation) in the corresponding relativistic QFT path-integral. The two are not just related by a substitution $x^0 \to -ix^d$ since the integration path must be deformed in addition. If we take the above just as a convention: then in any case in perturbation theory, obtained by expanding the exponential for the interaction part, a diagram exhibiting V vertices contributes with a factor $(-)^V$ to the Euclidean amplitude, while it contributes with a factor $(i)^V$ to the Minkowski amplitude. Thus
Relativistically invariant amplitudes depend on external momenta, q say, via Minkowskian scalar products: $F(\cdots) = F(p^2, pq, q^2)$ and upon a Wick rotation in all momenta we obtain $F(p^2, pq, q^2) = F(-\underline{p}^2, -\underline{pq}, -\underline{q}^2)$ in terms of Euclidean momenta. Usually, one chooses a different **convention** *to represent Euclidean quantities, e.g., $F(\cdots) \to \underline{F}(\underline{p}^2, \underline{pq}, \underline{q}^2)$ such that we have to take care of extra signs or phase factors in the relationship between F and $\underline{F}$. The precise rule of translation for our conventions has been specified in (101).*

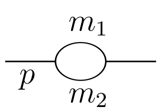
The relationship between Euclidean and Minkowski quantum field theory is not only a very basic and surprising general feature of any local relativistic field theory but is a property of central practical importance for the non-perturbative approach to QFT via the Euclidean path-integral (e.g., lattice QCD). In a QFT satisfying the Wightman axioms the continuation of the vacuum expectation values of time-ordered products of local fields (the time-ordered Green functions) from Minkowski space to four-dimensional Euclidean space is always possible[51]. Conversely, the Osterwalder-Schader theorem ascertains that the Euclidean correlation functions of fields can be analytically continued to Minkowski space, provided we have a local action which satisfies the so-called reflection positivity condition[52]. Accordingly, the full Minkowski QFT including its S-matrix, if it exists, can be reconstructed from the knowledge of the Euclidean correlation functions and from a mathematical point of view the Minkowski and the Euclidean version of a QFT are completely equivalent.

Scalar one-loop integrals *(Euclidean).*

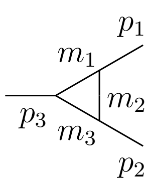
Here we apply our tools to the simplest scalar one-loop integrals (p.i.= partial integration).



$$\begin{aligned}
&= \frac{\mu^{4-d}}{(2\pi)^d}\int d^d\underline{k}\,\frac{1}{\underline{k}^2+m^2} = \mu^{4-d}(4\pi)^{-d/2}\int_0^\infty d\alpha\,\alpha^{-d/2}e^{-\alpha m^2} \\
&\qquad \text{convergent for } d<2 \quad *** \text{ [21]} \\
&\overset{p.i.}{=} -\frac{2m^2}{d-2}\mu^{4-d}(4\pi)^{-d/2}\int_0^\infty d\alpha^{1-d/2}e^{-\alpha m^2} \\
&\qquad \text{convergent for } d<4 \\
&= -2m^2(4\pi)^{-d/2}\frac{\Gamma(2-d/2)}{d-2}\left(\frac{m^2}{\mu^2}\right)^{d/2-2} \\
&= -2m^2(4\pi)^{-2}\frac{2}{\varepsilon}\Gamma(1+\tfrac{\varepsilon}{2})\frac{1}{2-\varepsilon}e^{\frac{\varepsilon}{2}(\ln 4\pi-\ln\frac{m^2}{\mu^2})} \\
&\overset{\varepsilon\to+0}{\simeq} m^2(4\pi)^{-2}\left\{\frac{2}{\varepsilon}-\gamma+1+\ln 4\pi-\ln\frac{m^2}{\mu^2}\right\}+O(\varepsilon)
\end{aligned}$$



$$\begin{aligned}
&= \frac{\mu^{4-d}}{(2\pi)^d}\int d^dk\,\frac{1}{\underline{k}^2+m_1^2}\,\frac{1}{(\underline{k}+\underline{p})^2+m_2^2} \\
&= \mu^{4-d}(4\pi)^{-d/2}\int_0^\infty d\alpha_1 d\alpha_2(\alpha_1+\alpha_2)^{-d/2}e^{-(\alpha_1 m_1^2+\alpha_2 m_2^2+\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\underline{p}^2)} \\
&\qquad \alpha_1 = x\lambda\,;\quad \alpha_2=(1-x)\lambda \\
&= \mu^{4-d}(4\pi)^{-d/2}\Gamma(2-\tfrac{d}{2})\int_0^1 dx(xm_1^2+(1-x)m_2^2+x(1-x)\underline{p}^2))^{d/2-2} \\
&\qquad \text{convergent for } d<4 \\
&= (4\pi)^{-2}\frac{2}{\varepsilon}\Gamma(1+\tfrac{\varepsilon}{2})e^{\frac{\varepsilon}{2}\ln 4\pi}\int_0^1 dx e^{-\frac{\varepsilon}{2}\ln\frac{xm_1^2+(1-x)m_2^2+x(1-x)\underline{p}^2}{\mu^2}} \\
&\overset{\varepsilon\to+0}{\simeq} (4\pi)^{-2}\left\{\frac{2}{\varepsilon}-\gamma+\ln 4\pi-\int_0^1 dx\ln\frac{xm_1^2+(1-x)m_2^2+x(1-x)\underline{p}^2}{\mu^2}\right\}+O(\varepsilon)
\end{aligned}$$



$$\begin{aligned}
&= \frac{\mu^{4-d}}{(2\pi)^d}\int d^d\underline{k}\,\frac{1}{\underline{k}^2+m_1^2}\,\frac{1}{(\underline{k}+\underline{p}_1)^2+m_2^2}\,\frac{1}{(\underline{k}+\underline{p}_1+\underline{p}_2)^2+m_3^2} \\
&\qquad \text{convergent for } d=4 \\
&\overset{\varepsilon\to+0}{\simeq} (4\pi)^{-2}\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\frac{1}{(\alpha_1+\alpha_2+\alpha_3)^2}e^{-(\alpha_1 m_1^2+\alpha_2 m_2^2+\alpha_3 m_3^2)} \\
&\quad \times e^{-\frac{\alpha_1\alpha_2\underline{p}_1^2+\alpha_2\alpha_3\underline{p}_2^2+\alpha_3\alpha_1\underline{p}_3^2}{\alpha_1+\alpha_2+\alpha_3}} \\
&\quad \alpha_1 = xy\lambda\,;\quad \alpha_2 = x(1-y)\lambda\,;\quad \alpha_3=(1-x)\lambda\,;\quad \alpha_1+\alpha_2+\alpha_3=\lambda \\
&= (4\pi)^{-2}\int_0^1 dydxx\frac{1}{N}
\end{aligned}$$

where

$$N = x^2y\,(1-y)\underline{p}_1^2+x\,(1-x)(1-y)\underline{p}_2^2+x\,(1-x)\,y\underline{p}_3^2+xym_1^2+x\,(1-y)\,m_2^2+(1-x)\,m_3^2$$

---

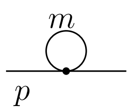
[21] A direct integration here yields

$$m^2(4\pi)^{-d/2}\Gamma(1-d/2)\left(\frac{m^2}{\mu^2}\right)^{d/2-2}$$

which by virtue of $\Gamma(1-d/2)=-2\Gamma(2-d/2)/(d-2)$ is the same analytic function as the one obtained via the partial integration method.

We summarize these results by listing the

Standard scalar one-loop integrals *($m^2 \hat{=} m^2 - i\varepsilon$).*



$$= \mu_0^\varepsilon \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2} \doteq -\frac{i}{16\pi^2} A_0(m) \tag{102}$$

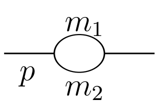
where

$$A_0(m) = -m^2(Reg + 1 - \ln m^2) \ . \tag{103}$$

By Reg we denote the UV regulator

$$Reg = \frac{2}{\varepsilon} - \gamma + \ln 4\pi + \ln \mu_0^2 \equiv \ln \mu^2 \tag{104}$$

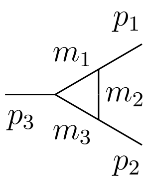
where the last identification defines the $\overline{MS}$ **scheme of minimal subtraction**.



$$= \mu_0^\varepsilon \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - m_1^2)\,((k+p)^2 - m_2^2))} \doteq \frac{i}{16\pi^2} B_0(m_1, m_2; p^2) \tag{105}$$

where

$$B_0(m_1, m_2; s) = Reg - \int_0^1 dz \ln(-sz(1-z) + m_1^2(1-z) + m_2^2 z - i\varepsilon) \ . \tag{106}$$



$$\begin{aligned} &= \mu_0^\varepsilon \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - m_1^2)\,((k+p_1)^2 - m_2^2)\,((k+p_1+p_2)^2 - m_3^2)} \\ &= -\frac{i}{16\pi^2} C_0(m_1, m_2, m_3; p_1^2, p_2^2, p_3^2) \end{aligned} \tag{107}$$

where

$$C_0(m_1, m_2, m_3; s_1, s_2, s_3) = \int_0^1 dx \int_0^x dy \frac{1}{ax^2 + by^2 + cxy + dx + ey + f} \tag{108}$$

with

$$\begin{array}{ll} a = s_2 & d = m_2^2 - m_3^2 - s_2 \\ b = s_1 & e = m_1^2 - m_2^2 + s_2 - s_3 \\ c = s_3 - s_1 - s_2 & f = m_3^2 - i\varepsilon \ \ . \end{array}$$

The UV-singularities (poles in ε at d=4) give raise to finite extra contributions *when they are multiplied with d (or functions of d) which arise from contractions like $g_\mu^\mu = d$, $\gamma^\mu \gamma_\mu = d$ etc. For $d \to 4$ we obtain:*

$$dA_0(m) = 4A_0(m) + 2m^2 \ , \ \ dB_0 = 4B_0 - 2 \ . \tag{109}$$

The explicit evaluation of the scalar integrals (up to the scalar four-point function) is discussed in Ref. [50].

5. 4. Tensor integrals (one-loop)

Integrals of the form

$$I^{\mu_1\cdots\mu_m}(p_1,\cdots) = \int \frac{d^dk}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_m}}{(k^2-m_1^2)((k+p_1)^2-m_2^2)((k+p_1+p_2)^2-m_3^2)\cdots}$$

can be reduced to <u>*scalar*</u> *one-loop integrals. In DR transformation of variables and partial fraction decomposition hold true independent of the convergence of the integral. The reduction of tensor integrals to scalar integrals may be achieved by the following steps:*

i) *Covariant decomposition:*

$$I^{\mu\cdots\mu_m}(p_1,p_2,\cdots) = p_1^{\mu_1}\cdots p_1^{\mu_m} I_{m1}(p_1^2,p_1p_2,p_2^2,\cdots) + \cdots$$

in terms of an appropriately symmetrized tensor basis formed with the linearly independent momenta and $g_{\mu\nu}$.

ii) *Contraction with* $g_{\mu\nu}$:

$$\frac{k^2}{k^2-m^2} = \frac{(k^2-m^2)+m^2}{k^2-m^2} = 1 + \frac{m^2}{k^2-m^2}$$

iii) *Contraction with* $p_{i\mu}$:

$$2kp_1 = ((k+p_1)^2-m_2^2) - (k^2-m_1^2) - (p_1^2-m_2^2+m_1^2)$$
$$\tfrac{2kp_1}{(1)(2)} = \tfrac{1}{(1)} - \tfrac{1}{(2)} - (p_1^2-m_2^2+m_1^2)\tfrac{1}{(1)(2)}$$

etc., until all I_{mi}*'s are given as linear combination of scalar integrals. By* $1/(i)$ *we denote the scalar propagator with mass* m_i*. In the following we work out*

Some basic tensor integrals

A factor $\frac{i}{16\pi^2}$ *is taken out for simplicity of notation, i.e.*

$$\int_k \cdots = \frac{16\pi^2}{i}\int \frac{d^dk}{(2\pi)^d}\cdots \quad .$$

In order to conform with the Passarino-Veltman convention in Ref. [49], we define the invariant functions $I_{m1},\cdots$ *using a factor* $(-1)^n$ *in front of the integrals, where* n *is the number of propagators, and a factor* (-1) *in front of the* $g^{\mu_i\mu_k}$*'s appearing in the kinematical tensors of the covariant decomposition. Accordingly, we consider integrals of the form*

$$\int_k \frac{k^{\mu_1}\cdots k^{\mu_m}}{(k^2-m_1^2)\cdots((k+p_1+p_2\cdots+p_{n-1})^2-m_n^2)} = (-1)^n\left(p_1^{\mu_1}\cdots p_1^{\mu_m} I_{m1} + \cdots\right) \quad .$$

in the following, and calculate the I_{mi}'s in terms of scalar functions. We now discuss tadpoles, self-energies and form-factors in turn.

1. *Tadpoles*

By performing a shift $k \to k+p$ of the integration variable we easily find the following results:

$$\int_k \frac{1}{(k+p)^2 - m^2} = \int_k \frac{1}{k^2 - m^2} = -A_0(m) \tag{110}$$

$$\int_k \frac{k^\mu}{(k+p)^2 - m^2} = \int_k \frac{k^\mu - p^\mu}{k^2 - m^2} = \underbrace{\int_k \frac{k^\mu}{k^2 - m^2}}_{0} - p^\mu \underbrace{\int_k \frac{1}{k^2 - m^2}}_{-A_0(m)} = p^\mu A_0(m) \tag{111}$$

$$\begin{aligned} \int_k \frac{k^\mu k^\nu}{(k+p)^2 - m^2} &= -p^\mu p^\nu A_{21} + g^{\mu\nu} A_{22} \\ = \int_k \frac{(k-p)^\mu (k-p)^\nu}{k^2 - m^2} &= -p^\mu p^\nu A_0(m) + \int_k \frac{k^\mu k^\nu}{k^2 - m^2} \end{aligned} \tag{112}$$

Using

$$g_{\mu\nu} \int_k \frac{k^\mu k^\nu}{k^2 - m^2} = d A_{22} = \int_k \frac{k^2}{k^2 - m^2} = \underbrace{\int_k 1}_{0} + m^2 \underbrace{\int_k \frac{1}{k^2 - m^2}}_{-A_0}$$

we find

$$d A_{22} = -m^2 A_0(m) \; ; \quad A_{21} = A_0(m)$$

and expanding in $d = 4 - \varepsilon$, $\varepsilon \to 0$ we get

$$\begin{aligned} 4 A_{22} &= -m^2 A_0(m) + \varepsilon A_{22} \\ &\simeq m^4 \cdot \frac{2}{\varepsilon} + \text{ finite} \end{aligned}$$

which implies

$$\varepsilon A_{22} \simeq \frac{2m^4}{4} + 0(\varepsilon)$$

and thus as a final answer

$$\begin{aligned} A_{21} &= A_0(m) \\ A_{22} &= -\frac{m^2}{d} A_0(m) \overset{\varepsilon \to 0}{\simeq} -\frac{m^2}{4} A_0(m) + \frac{m^4}{8} \end{aligned} \tag{113}$$

2. *Self-energies*

$$\int_k \frac{1}{k^2-m_1^2}\frac{1}{(k+p)^2-m_2^2} = B_0(m_1,m_2;p^2) \tag{114}$$

$$\int_k \frac{k^\mu}{(1)(2)} = p^\mu B_1(m_1,m_2;p^2) \tag{115}$$

where contraction with p_μ and using

$$2pk = (p+k)^2 - m_2^2 - (k^2-m_1^2) - (p^2+m_1^2-m_2^2)$$

yields

$$\begin{array}{rcl} 2p^2 B_1 &=& \int_k \frac{2pk}{(1)(2)} = \int_k \frac{1}{(1)} - \int_k \frac{1}{(2)} - (p^2+m_1^2-m_2^2)\int_k \frac{1}{(1)(2)} \\ &=& -A_0(m_1) + A_0(m_2) - (p^2+m_1^2-m_2^2)B_0 \end{array}$$

$$\| \quad B_1(m_2,m_2;p^2) = \tfrac{1}{2p^2}\left\{-A_0(m_1)+A_0(m_2)-(p^2+m_1^2-m_2^2)B_0\right\} \tag{116}$$

$$\int_k \frac{k^\mu k^\nu}{(1)(2)} = p^\mu p^\nu B_{21} - g^{\mu\nu}B_{22} \tag{117}$$

Contraction with p_ν gives

$$\begin{array}{rcl} 2p^\mu(p^2B_{21}-B_{22}) &=& \int_k \frac{k^\mu(2pk)}{(1)(2)} \\ &=& \int_k \frac{k^\mu}{(1)} - \int_k \frac{k^\mu}{(2)} - (p^2+m_1^2-m_2^2)\int_k \frac{k^\mu}{(1)(2)} \\ &=& -p^\mu A_0(m_2) - p^\mu(p^2+m_1^2-m_1^2)B_1(m_1,m_2;p^2) \end{array}$$

$$\triangleright \quad \underline{2(p^2B_{21}-B_{22}) = -A_0(m_2)-(p^2+m_1^2-m_2^2)B_1}$$

while contraction with $g_{\mu\nu}$ yields

$$p^2B_{21} - dB_{22} = \int_k \frac{k^2}{(1)(2)} = \int_k \frac{1}{(2)} + m_1^2\int_k \frac{1}{(1)(2)}$$

$$\triangleright \quad \underline{p^2B_{21} - dB_{22} = -A_0(m_2) + m_1^2 B_0(m_1,m_2;p^2)}$$

We thus find for arbitrary dimension d

$$\left\| \begin{array}{rcl} B_{21} &=& \frac{1}{(d-1)\,p^2}\left\{(1-d/2)A_0(m_2) - d/2(p^2+m_1^2-m_2^2)B_1 - m_1^2B_0\right\} \\ B_{22} &=& \frac{1}{2(d-1)}\left\{A_0(m_2)-(p^2+m_1^2-m_2^2)B_1 - 2m_1^2B_0\right\} \quad . \end{array}\right. \tag{118}$$

In order to expand in $d = 4 - \varepsilon, \varepsilon \to 0$ we consider

$$\begin{aligned} 2p^2 B_{21} - 2B_{22} &= -A_0(m_2) - (p^2 + m_1^2 - m_2^2) B_1 \\ p^2 B_{21} - 4B_{22} &= -A_0(m_2) + m_1^2 B_0 - \varepsilon B_{22} \end{aligned}$$

and obtain

$$B_{22} = \frac{1}{6} \left\{ A_0(m_2) - 2m_1^2 B_0 - (p^2 + m_1^2 - m_2^2) B_1 + 2\varepsilon B_{22} \right\} \quad .$$

The UV singular part (ε pole term) is

$$B_{22}^{sing} = -\frac{1}{4}(m_1^2 + m_2^2 - p^2/3)\frac{2}{\varepsilon}$$

and hence

$$2\varepsilon B_{22} = -(m_1^2 + m_2^2 - p^2/3) + O(\varepsilon) \quad .$$

This result also determines

$$B_{21} = \frac{1}{3p^2} \left\{ -A_0(m_2) - 2(p^2 + m_1^2 - m_2^2) B_1 - m_1^2 B_0 + \varepsilon B_{22} \right\}$$

and leads us to the final answer

$$\begin{aligned} B_{21} &= \tfrac{-1}{3p^2} \left\{ A_0(m_2) + 2(p^2 + m_1^2 - m_2^2) B_1 + m_1^2 B_0 + 1/2(m_1^2 + m_2^2 - p^2/3) \right\} \\ B_{22} &= \tfrac{1}{6} \left\{ A_0(m_2) - (p^2 + m_1^2 - m_2^2) B_1 - 2m_1^2 B_0 - (m_1^2 + m_2^2 - p^2/3) \right\} \end{aligned} \quad (119)$$

where the arguments of the B-functions are obvious.

Similarly:

$$\int_k \frac{k^\mu k^\nu k^\alpha}{(1)(2)} = p^\mu p^\nu p^\alpha B_{31} - \{pg\}^{\mu\nu\alpha} B_{32} \quad (120)$$

with $\{pg\}^{\mu\nu\alpha} \equiv p^\mu g^{\nu\alpha} + p^\nu g^{\mu\alpha} + p^\alpha g^{\mu\nu}$.
Contraction with p_α and applying the technique explained above a comparison of the coefficients of the kinematical tensors yields

$$\begin{aligned} B_{31} &= \tfrac{1}{p^4} \left\{ A_{22}(m_2) - A_{22}(m_1) - (p^2 + m_1^2 - m_2^2) B_{22} \right\} + \tfrac{1}{2p^2} \left\{ A_{21}(m_2) - (p^2 + m_1^2 - m_2^2) B_{21} \right\} \\ B_{32} &= \tfrac{1}{2p^2} \left\{ A_{22}(m_2) - A_{22}(m_1) - (p^2 + m_1^2 - m_2^2) B_{22} \right\} \quad . \end{aligned}$$

3. Form factors

In the simplest cases we define the following invariant amplitudes

$$\int_k \frac{1}{k^2 - m_1^2} \frac{1}{(k + p_1)^2 - m_2^2} \frac{1}{(k + p_1 + p_2)^2 - m_3^2} \doteq -C_0(m_1, m_2, m_3; p_1^2, p_2^2, p_3^2) \quad (121)$$

$$\int_k \frac{k^\mu}{(1)(2)(3)} \doteq -p_1^\mu C_{11} - p_2^\mu C_{12} \tag{122}$$

$$\int_k \frac{k^\mu k^\nu}{(1)(2)(3)} \doteq -p_1^\mu p_1^\nu C_{21} - p_2^\mu p_2^\nu C_{22} - (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) C_{23} + g^{\mu\nu} C_{24} \tag{123}$$

where $p_3 \doteq -(p_1 + p_2)$.
The C_{1i}*'s can be found using all possible independent contractions. This leads to the equations*

$$\underbrace{\begin{pmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{pmatrix}}_{X} \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

with

$$\begin{array}{rcl} R_1 & = & \frac{1}{2}(B_0(m_2, m_3; p_2^2) - B_0(m_1, m_3; p_3^2) \\ & & - (p_1^2 + m_1^2 - m_2^2) C_0) \\ R_2 & = & \frac{1}{2}\,(B_0(m_1, m_3; p_3^2) - B_0(m_1, m_2; p_1^2) \\ & & + (p_1^2 - p_3^2 - m_2^2 + m_3^2) C_0) \quad . \end{array}$$

The inverse of the kinematical matrix of the equation to be solved is

$$X^{-1} = \frac{1}{DetX} \begin{pmatrix} p_2^2 & -p_1 p_2 \\ -p_1 p_2 & p_1^2 \end{pmatrix} \;, \quad DetX \doteq p_1^2 p_2^2 - (p_1 p_2)^2$$

and the solution reads

$$\begin{array}{rcl} C_{11} & = & \dfrac{1}{DetX} \left\{ p_2^2 R_1 - (p_1 p_2) R_2 \right\} \\ C_{12} & = & \dfrac{1}{DetX} \left\{ -(p_1 p_2) R_1 + p_1^2 R_2 \right\} \quad . \end{array} \tag{124}$$

The same procedure applies to the more elaborate case of the C_{2i}*'s where the solution may be written in the form*

$$C_{24} = -\frac{m_1^2}{2} C_0 + \frac{1}{4} B_0(2,3) - \frac{1}{4}(f_1 C_{11} + f_2 C_{12}) + \frac{1}{4} \tag{125}$$

$$\begin{pmatrix} C_{21} \\ C_{23} \end{pmatrix} = X^{-1} \begin{pmatrix} R_3 \\ R_5 \end{pmatrix} ; \begin{pmatrix} C_{23} \\ C_{22} \end{pmatrix} = X^{-1} \begin{pmatrix} R_4 \\ R_6 \end{pmatrix} \tag{126}$$

with

$$\begin{array}{rcl} R_3 & = & C_{24} - \frac{1}{2}\,(f_1 C_{11} + B_1(1,3) + B_0(2,3)) \\ R_5 & = & -\frac{1}{2}\,(f_2 C_{11} + B_1(1,2) - B_1(1,3)) \\ R_4 & = & -\frac{1}{2}\,(f_1 C_{12} + B_1(1,3) - B_1(2,3)) \\ R_6 & = & C_{24} - \frac{1}{2}\,(f_2 C_{12} - B_1(1,3)) \end{array}$$

and

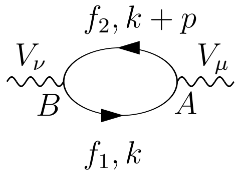
$$f_1 = p_1^2 + m_1^2 - m_2^2 \ ; \ \ f_2 = p_3^2 - p_1^2 + m_2^2 - m_3^2 \ \ .$$

The notation used for the B-functions is as follows: $B_0(1,2)$ *denotes the two point function obtained by dropping propagator* $\frac{1}{(3)}$ *from the form factor i.e.* $\int_k \frac{1}{(1)(2)}$ *and correspondingly for the other cases.*

6. Applications

1. Virtual fermion contributions to gauge boson self-energies

$$-i\Pi^{\mu\nu}(p) \doteq$$



$$= -g_A g_B \int \frac{d^d k}{(2\pi)^d} Tr \left\{ \frac{\not{k} + m_1}{k^2 - m_1^2} \gamma^\nu (a_B + b_B \gamma_5) \frac{\not{p} + \not{k} + m_2}{(p+k)^2 - m_2^2} \gamma^\mu (a_A + b_A \gamma_5) \right\}$$

For the different self-energies the couplings are given by:

1. $W^+W^- : g_A = g_B = \frac{M_W}{2v}$; $a_A = a_B = a = 1$, $b_A = b_B = b = -1$

   $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ doublet $m_1 \neq m_2$ — V_{12} assumed

2. $ZZ : g_A = g_B = \frac{M_Z}{2v}$; $a_A = a_B = a = 4Q_f \sin^2\Theta_W - 2I_{3f}$, $b_A = b_B = b = -2I_{3f}$

   $f_2 = f_1 = f$ single fermion — $2I_{3f} = \pm 1$, $m_1 = m_2 = m_f$

3. $Z\gamma : g_A = eQ_f, g_B = \frac{M_Z}{2v}$; $a_A = 1, a_B = a = 4Q_f \sin^2\Theta_W - 2I_{3f}$, $b_A = 0; b_B = b = -2I_{3f}$

   $f_2 = f_1 = f$ — $m_1 = m_2 = m_f$

4. $\gamma\gamma : g_A = g_B = eQ_f$; $a_A = a_B = 1$, $b_A = b_B = 0$

   $f_2 = f_1 = f$ — $m_1 = m_2 = m_f$

Calculation of the trace of Dirac matrices:

$$\begin{aligned}
& (\not{k} + m_1)\gamma^\nu (a + b\gamma_5)_B (\not{p} + \not{k} + m_2)\gamma^\mu (a + b\gamma_5)_A \\
= \; & \not{k}\gamma^\nu (a + b\gamma_5)_B (\not{p} + \not{k})\gamma^\mu (a + b\gamma_5)_A \\
& + m_1 m_2 \gamma^\nu (a + b\gamma_5)_B \gamma^\mu (a + b\gamma_5)_A \\
& + \text{terms with odd number of } \gamma'\text{s (have Tr () } = 0) \\
= \; & \not{k}\gamma^\nu (\not{p} + \not{k})\gamma^\mu (a_A a_B + b_A b_B + (a_A b_B + a_B b_A)\gamma_5) \\
& + m_1 m_2 \gamma^\nu \gamma^\mu (a_A a_B - b_A b_B - (a_A b_B - a_B b_A)\gamma_5) \\
& + \cdots
\end{aligned}$$

Now, $Tr\gamma^\nu\gamma^\mu\gamma_5 = 0$ *and* $Tr\not{k}\gamma^\nu(\not{p} + \not{k})\gamma^\mu\gamma_5 = 4i\epsilon^{\alpha\nu\beta\mu}k_\alpha(\not{p} + \not{k})_\beta$ *after integration* $\int_k k_\alpha \cdots = p_\alpha \cdots$ *cannot contribute since* $\epsilon^{\alpha\beta\nu\mu}p_\alpha p_\beta \equiv 0$.

The trace of the remaining terms is given by:

$$\begin{aligned}
Tr(\cdots) &= 4(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})k_\alpha(p+k)_\beta(a_Aa_B + b_Ab_B) \\
&+ 4m_1m_2g^{\mu\nu}(a_Aa_B - b_Ab_B) \\
&= 4(a_Ab_B + b_Ab_B)\left\{k^\nu(p+k)^\mu + k^\mu(p+k)^\nu - g^{\mu\nu}(k^2+kp)\right\} \\
&+ 4(a_Aa_B - b_Ab_B)m_2m_2g^{\mu\nu}
\end{aligned}$$

We now may evaluate the integrals. We use the notations $(1) = k^2 - m_1^2 + i\varepsilon$, $(2) = (p+k)^2 - m_2^2 + i\varepsilon$ *and* $\int_k \cdots \doteq \frac{16\pi^2}{i}\int\frac{d^dk}{(2\pi)^d}\cdots$ *and use the definitions*

$$\begin{aligned}
&\textstyle\int_k \frac{k^\mu k^\nu}{(1)(2)} \doteq p^\mu p^\nu B_{21} - g^{\mu\nu}B_{22} \\
&\textstyle\int_k \frac{k^\mu}{(1)(2)} \doteq p^\mu B_1 \\
&\textstyle\int_k \frac{1}{(1)(2)} = B_0 = B_0(m_1, m_2; p^2) \\
&\textstyle\int_k \frac{1}{(1)} = -A_0(m_1);\quad \int_k \frac{1}{(2)} = -A_0(m_2)\ .
\end{aligned}$$

We write $\Pi^{\mu\nu}$ *in the form*

$$\begin{aligned}
\Pi^{\mu\nu}(p) &= g^{\mu\nu}\Pi_1(p^2) + p^\mu p^\nu\Pi_2(p^2) \\
&= (g^{\mu\nu} - \tfrac{p^\mu p^\nu}{p^2})\Pi_1 + \tfrac{p^\mu p^\nu}{p^2}(p^2\Pi_2 + \Pi_1)\ .
\end{aligned}$$

Only the <u>transverse</u> *amplitude* Π_1 *contributes to* S*-matrix elements, e.g. if contracted with a polarization vector the* Π_2 *amplitude drops out due to* $\varepsilon_\mu(p,\lambda)p^\mu = 0$. *In general* Π_2 *cancels against ghost amplitudes.*
The relevant integrals we need are given by:

$$\begin{aligned}
2\textstyle\int_k \frac{k^\mu k^\nu}{(1)(2)} &= p^\mu p^\nu 2B_{21} - g^{\mu\nu}2B_{22} \\
2\textstyle\int_k \frac{p^\mu k^\nu}{(1)(2)} &= p^\mu p^\nu 2B_1 \\
\textstyle\int_k \frac{k^2+pk}{(1)(2)} &= \tfrac{1}{2}\left(\textstyle\int_k \frac{1}{(2)} + \int_k \frac{1}{(1)} + (m_1^2 + m_2^2 - p^2)\int_k \frac{1}{(1)(2)})\right) \\
&= \tfrac{1}{2}(-A_0(m_1) - A_0(m_2) + (m_1^2 + m_2^2 - p^2)B_0)
\end{aligned}$$

where, for the last integral, we have used the decomposition

$$k^2 + pk = \frac{1}{2}(2k^2 + 2pk) = \frac{1}{2}(\underbrace{k^2 - m_1^2}_{(1)} + \underbrace{(k+p)^2 - m_2^2}_{(2)} - (p^2 - m_1^2 - m_2^2))$$

We then obtain the result

$$\begin{aligned}
\Pi_1 \quad = \quad & (-i)\tfrac{i}{16\pi^2}4g_Ag_B\cdot \\
& \{\ (a_Aa_B + b_Ab_B)\left(-2B_{22} + \tfrac{1}{2}A_0(m_1) + \tfrac{1}{2}A_0(m_2) + \tfrac{1}{2}(p^2 - m_1^2 - m_2^2)B_0\right) \\
& +(a_Aa_B - b_Ab_B)m_1m_2B_0\} \\
\Pi_2 \quad = \quad & (-i)\tfrac{i}{16\pi^2}4g_Ag_B\cdot \\
& \ (a_Aa_B + b_Ab_B)\,(2B_{21} + 2B_1)
\end{aligned}$$

with

$$
\begin{aligned}
B_1 &= \tfrac{1}{2p^2}\left(-A_0(m_1)+A_0(m_2)-(p^2+m_1^2-m_2^2)B_0\right)\\
B_{21} &= \tfrac{1}{3p^2}\left(-A_0(m_2)-2(p^2+m_1^2-m_2^2)B_1-m_1^2B_0\right.\\
&\qquad\left.-\,1/2(m_1^2+m_2^2-p^2/3)\right)\\
B_{22} &= \tfrac{1}{6}\left(A_0(m_2)-(p^2+m_1^2-m_2^2)B_1-2m_1^2B_0\right.\\
&\qquad\qquad\left.-\,(m_1^2+m_2^2-p^2/3)\right)\quad .
\end{aligned}
$$

Inserting the latter expressions yields the final result:

$$
\boxed{
\begin{aligned}
\Pi_1 &= \tfrac{1}{16\pi^2}4g_Ag_B\cdot\Big\{\tfrac{1}{3}(a_Aa_B+b_Ab_B)\ \ [(m_1^2+m_2^2-p^2/3)\\
&+\ A_0(m_1)+A_0(m_2)-\tfrac{m_1^2-m_2^2}{2p^2}(A_0(m_1)-A_0(m_2))\\
&+\ \tfrac{2p^4-p^2(m_1^2+m_2^2)-(m_1^2-m_2^2)^2}{2p^2}\ B_0(m_1,m_2;p^2)]\\
&\quad +(a_Aa_B-b_Ab_B)m_1m_2B_0(m_1,m_2;p^2)\Big\}
\end{aligned}
}
\tag{127}
$$

We now specialize the result for the different self-energy functions:
1. W self-energy (contribution of a fermion doublet)

$$
\begin{aligned}
\Pi_1^{WW} &= \tfrac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\cdot\tfrac{4}{3}\Big\{m_1^2+m_2^2-p^2/3+A_0(m_1)+A_0(m_2)\\
&\quad -\tfrac{m_1^2-m_2^2}{2p^2}(A_0(m_1)-A_0(m_2))\\
&\quad +\tfrac{2p^4-p^2(m_1^2+m_2^2)-(m_1^2-m_2^2)^2}{2p^2}B_0(m_1,m_2;p^2)\Big\}
\end{aligned}
\tag{128}
$$

For the evaluation of $\Pi_1^{WW}(0)$ *we use the following relations:*

$$
\begin{aligned}
B_0(m_1,m_2;p^2) &= B_0(m_1,m_2;0)+p^2\dot{B}_0(m_1,m_2;0)+\cdots\\
B_0(m_1,m_2;0) &= -\frac{A_0(m_1)-A_0(m_2)}{m_1^2-m_2^2}\\
2(m_1^2-m_2^2)^2\dot{B}_0(m_1,m_2;0) &= A_0(m_1)+A_0(m_2)+(m_1^2+m_2^2)[1+B_0(m_1,m_2;0)]\\
A_0(m) &= -m^2+m^2\ln\frac{m^2}{\mu^2}
\end{aligned}
$$

where, writing the bare quantities in the $\overline{MS}$*-scheme,* $\ln\mu^2=\frac{2}{\varepsilon}-\gamma+\ln 4\pi$*. We find*

$$
\begin{aligned}
\Pi_1^{WW}(0) &= \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\Big\{m_1^2+m_2^2+A_0(m_1)+A_0(m_2)\\
&\qquad +\frac{(m_1^2+m_2^2)}{(m_1^2-m_2^2)}(A_0(m_1)-A_0(m_2))\Big\}\\
&= \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\left\{2\frac{m_1^4\ln\frac{m_1^2}{\mu^2}-m_2^4\ln\frac{m_2^2}{\mu^2}}{m_1^2-m_2^2}-(m_1^2+m_2^2)\right\}
\end{aligned}
\tag{129}
$$

2. Z self-energy (contribution of a single fermion)

$$\begin{aligned}\Pi_1^{ZZ} &= \tfrac{\sqrt{2}G_\mu M_Z^2}{16\pi^2}\tfrac{1}{3}\Big\{(a_f^2+b_f^2)\Big[2m_f^2-p^2/3+2A_0(m_f)\\ &\quad +(p^2-m_f^2)B_0(m_f,m_f;p^2)\Big]\\ &\quad +(a_f^2-b_f^2)3m_f^2B_0(m_f,m_f;p^2)\Big\}\end{aligned} \tag{130}$$

For the evaluation of $\Pi_1^{ZZ}(0)$ *we use:*

$$\begin{aligned}B_0(m_f,m_f;0) &= -1-\frac{A_0(m_f)}{m_f^2}\\ A_0(m_f) &= -m_f^2+m_f^2\ln\frac{m_f^2}{\mu^2}\end{aligned}$$

and find

$$\begin{aligned}\Pi_1^{ZZ}(0) &= \tfrac{\sqrt{2}G_\mu M_Z^2}{16\pi^2}\Big\{2b_f^2(m_f^2+A_0(m_f))\Big\}\\ &= \tfrac{\sqrt{2}G_\mu M_Z^2}{16\pi^2}\Big\{2m_f^2\ln\tfrac{m_f^2}{\mu^2}\Big\}\end{aligned} \tag{131}$$

a contribution which is purely axial (proportional to b_f*).*
3. $Z\gamma$-mixing ($b_Ab_B=0$ *pure vector contribution)*

$$\begin{aligned}\Pi_1^{Z\gamma} &= \tfrac{eQ_f(\sqrt{2}G_\mu)^{1/2}M_Z}{16\pi^2}\cdot a_f\cdot\tfrac{2}{3}\Big\{2m_f^2-p^2/3+2A_0(m_f)\\ &\quad +(p^2+2m_f^2)B_0(m_f,m_f;p^2)\Big\}\\ \Pi_1^{Z\gamma}(0) &= 0\\ \Pi_1^{Z\gamma} &\underset{m_f^2\ll|p^2|}{\simeq} -\tfrac{eQ_fa_f(\sqrt{2}G_\mu)^{1/2}M_Z}{16\pi^2}\tfrac{2}{3}p^2\Big\{\ln\tfrac{p^2}{\mu^2}-\tfrac{5}{3}\Big\}\end{aligned} \tag{132}$$

4. γ *self-energy (photon vacuum polarization)*

$$\begin{aligned}\Pi_1^{\gamma\gamma} &= \tfrac{e^2Q_f^2}{16\pi^2}\tfrac{4}{3}\Big\{2m_f^2-p^2/3+2A_0(m_f)+(p^2+2m_f^2)B_0(m_fm_f;p^2)\Big\}\\ &= -\tfrac{e^2Q_f^2}{16\pi^2}\tfrac{4}{3}p^2\Big[\ln\tfrac{m_f^2}{\mu^2}-\tfrac{5}{3}-y_f-2(1+\tfrac{y_f}{2})(y_f-1)G(y_f)\Big]\end{aligned} \tag{133}$$

where we defined $y_f=\frac{4m_f^2}{p^2}$ *and* $G(y_f)=\frac{1}{2\beta_f}\ln\frac{\beta_f+1}{\beta_f-1}$ *with* $\beta_f=\sqrt{1-y_f}$.
As it should be, we find $\Pi_1^{\gamma\gamma}(0)=0$ *i.e. the photon remains massless (unrenormalized). This remains true also at higher orders in the perturbation expansion.*
For $\Pi_1^{'\gamma\gamma}(p^2)=\frac{\Pi_1^{\gamma\gamma}}{p^2}$ *we find the following asymptotic values*

$$\begin{array}{lcl} \Pi_1^{'\gamma\gamma}(0) & = & -\frac{e^2}{16\pi^2}\frac{4}{3}Q_f^2 \ln\frac{m_f^2}{\mu^2} \\ \Pi_1^{'\gamma\gamma}(p^2) & \simeq & -\frac{e^2}{16\pi^2}\frac{4}{3}Q_f^2 \left(\ln\frac{p^2}{\mu^2} - \frac{5}{3}\right) \; . \\ & m_f^2 \ll \mid p^2 \mid & \end{array} \tag{134}$$

We may apply these results to calculate quantities which show up in the calculation of electroweak parameter shifts to be discussed in Secs. IV and V.

1. ρ-parameter

The ρ-parameter is defined as the neutral to charged current ratio, which within the SM is a finite gauge invariant calculable quantity. For the fermion contributions we obtain

$$\begin{array}{lcl} \rho & = & \frac{G_{NC}}{G_\mu} = 1 + \Delta\rho \\ \Delta\rho & = & \frac{\Pi_1^{ZZ}(0)}{M_Z^2} - \frac{\Pi_1^{WW}(0)}{M_W^2} \\ & = & \frac{\sqrt{2}G_\mu}{16\pi^2} N_{cf} \left(m_1^2 + m_2^2 + \frac{2m_1^2 m_2^2}{m_1^2 - m_2^2} \ln\frac{m_2^2}{m_1^2}\right) \\ & \simeq & \frac{\sqrt{2}G_\mu}{16\pi^2} N_{cf} \cdot \left\{ \begin{array}{lll} m_{heavy}^2 & ; & m_{heavy} \gg m_{light} \\ 0 & ; & m_1 = m_2 \end{array} \right\} \end{array}$$

(Veltman 1977). For the known fermion doublets only the top-bottom doublet has a large mass splitting. A very heavy top yields a contribution

$$\Delta\rho^{top} \simeq \frac{\sqrt{2}G_\mu}{16\pi^2} 3m_t^2; \;\; m_t \gg m_b \;\; .$$

2. Δr

Within the SM the Fermi constant G_μ can be calculated in terms of α, M_W and M_Z. It thus appears as a correction the the μ-decay amplitude

$$\sqrt{2}G_\mu = \frac{\pi\alpha}{M_W^2(1 - \frac{M_W^2}{M_Z^2})} (1 + \Delta r) \; .$$

One may write Δr in the form

$$\Delta r = \Delta\alpha - \frac{\cos^2\Theta_W}{\sin^2\Theta_W}\Delta\rho + \Delta r_{rem}$$

where Δr_{rem} collects the numerically small terms ($\sim$ 0.6%). The large term $\Delta\alpha$ is due to the photon vacuum polarization

$$\begin{array}{lcl} \Delta\alpha & = & \Pi_1^{'\gamma\gamma}(0) - \Pi_1^{'\gamma\gamma}(M_Z^2) \\ & = & \frac{\alpha}{3\pi} \sum_f Q_f^2 N_{cf} (\ln\frac{M_Z^2}{m_f^2} - \frac{5}{3}) \simeq 0.06 \; . \end{array}$$

The numerical value is given for the sum of the contributions from the light fermions e, μ, τ, u, d, c, s, b (see Sec. IV). Since the W-mass is not yet very precisely known we may use this result to predict M_W in terms of α, G_μ and M_Z. By solving the defining equation for M_W we obtain

$$M_W^2 = \frac{\rho M_Z^2}{2}\left(1+\sqrt{1-\frac{4A_0^2}{\rho M_Z^2}\frac{1}{1-\Delta\alpha}(1+\Delta r_{rem})}\,\right)$$

where $\rho = \frac{1}{1-\Delta\rho}$ *and*

$$A_0^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu} \quad .$$

3. NC couplings near the Z peak

The $Zf\bar{f}$*-vertex to lowest order is given by*

$$(\sqrt{2}G_\mu)^{1/2} 2M_Z\gamma^\mu(-Q_f\sin^2\Theta_W + (1-\gamma_5)\frac{I_{3f}}{2})$$

and higher order effects (radiative corrections) may be included by using renormalized effective couplings:

$$\begin{array}{lcl} G_\mu & \to & \rho_f G_\mu = G_{NCf}(M_Z^2) \\ \sin^2\Theta_W & \to & \kappa_f\sin^2\Theta_W = \sin^2\Theta_f \end{array}$$

Since α, G_μ *and* M_Z *are given, we may calculate* $\sin^2\Theta_f$ *using*

$$\sqrt{2}G_\mu M_z^2 \cos^2\Theta_f \sin^2\Theta_f = \frac{\pi\alpha}{1-\Delta r_f}$$

where Δr_f *has the form*

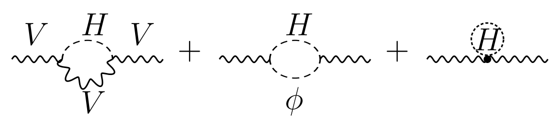
$$\Delta r_f = \Delta\alpha - \Delta\rho + \Delta_{f,rem} \ ,$$

a relation similar to the expression given for Δr*, however with a* $\Delta\rho$ *contribution which is by a factor* $\sin^2\Theta_W/\cos^2\Theta_W$ *smaller.*

We finally calculate another interesting type of diagrams, namely, those exhibiting a virtual Higgs particle.

2. **Virtual Higgs contributions to gauge boson self-energies**

$$-i\Pi^{\mu\nu}(p) \quad = $$



$$\begin{aligned}
W: \quad = \quad & \left(\frac{v^2 g^2}{2}\right)^2 \int_k \left(\frac{(1-\xi)k^\mu k^\nu}{k^2-\xi M_W^2+i\varepsilon} - g^{\mu\nu}\right) \frac{1}{k^2-M_W^2+i\varepsilon}\frac{1}{(k+p)^2-m_H^2+i\varepsilon} \\
& + \left(\frac{g}{2}\right)^2 \int_k \frac{1}{k^2-\xi M_W^2+i\varepsilon}\frac{1}{(p+k)^2-m_H^2+i\varepsilon}(2k+p)^\mu(2k+p)^\nu \\
& - \frac{g^2}{2}\frac{1}{2}\int_k \frac{g^{\mu\nu}}{k^2-m_H^2+i\varepsilon}
\end{aligned}$$

$$Z: \qquad g \to \frac{g}{\cos\Theta_W}, \quad M_W \to M_Z$$

We take the 't Hooft-Feynman gauge $\xi = 1$ *and use* $\frac{g^2}{4} = \frac{M_W^2}{v^2} = \sqrt{2}G_\mu M_W^2$ *obtaining*

$$\begin{aligned}
= \quad & \frac{i}{16\pi^2}\sqrt{2}G_\mu M_W^2 \cdot \\
& \{p^\mu p^\nu\left[B_0(M_W, m_H; p^2) + 4B_1 + 4B_{21}\right] \\
& \quad - g^{\mu\nu}\left[4M_W^2 B_0 + 4B_{22} + A_0(m_H)\right]\}
\end{aligned}$$

Thus we get the amplitudes

$$\begin{aligned}
\Pi_1^{WW}(p^2) = \quad & \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\{4B_{22} + 4M_W^2 B_0 + A_0(m_H)\} \\
\Pi_2^{WW}(p^2) = \quad & -\frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\{4B_{21} + 4B_1 + B_0\} \quad .
\end{aligned}$$

In terms of the scalar one-loop integrals we then find for the physical transverse part ($V = W, Z$):

$$\boxed{\begin{aligned}
\Pi_1^{VV}(p^2) = \frac{\sqrt{2}G_\mu M_V^2}{3\cdot 16\pi^2} \quad & \Big\{A_0(M_V) + 4A_0(m_H) + \frac{M_V^2-m_H^2}{p^2}(A_0(M_V) - A_0(m_H)) \\
& + (p^2 + 10M_V^2 - 2m_H^2 + \frac{(M_V^2-m_H^2)^2}{p^2})B_0(M_V, m_H; p^2) \\
& - 2(M_V^2 + m_H^2 - p^2/3)\Big\}
\end{aligned}} \tag{135}$$

Proceeding as we did for the fermions we get

$$\begin{aligned}
\Pi_1^{VV}(0) \quad = \quad & \frac{\sqrt{2}G_\mu M_V^2}{2\cdot 16\pi^2}\left\{4A_0(m_H) - M_V^2 - m_H^2 - 6\frac{M_V^2}{M_V^2-m_H^2}(A_0(M_V) - A_0(m_H))\right\} \\
= \quad & \frac{\sqrt{2}G_\mu M_V^2}{2\cdot 16\pi^2}\Big\{5(M_V^2 - m_H^2) + 4m_H^2 \ln\frac{m_H^2}{\mu^2} - 6M_V^2 \ln\frac{M_V^2}{\mu^2} \\
& + 6\frac{M_V^2 m_H^2}{M_V^2-m_H^2}\ln\frac{m_H^2}{M_V^2}\Big\}
\end{aligned} \tag{136}$$

in the 't Hooft-Feynman gauge $\xi = 1$ *and the* $\overline{MS}$*-scheme.*

Using these results we may calculate the Higgs contributions to the parameter shifts.

1. ρ-parameter

$$\begin{aligned}\Delta\rho &= \frac{\Pi_1^{ZZ}(0)}{M_Z^2} - \frac{\Pi_1^{WW}(0)}{M_W^2} = \frac{\sqrt{2}G_\mu}{2\cdot 16\pi^2}\left\{-6(M_Z^2 \ln\frac{M_Z^2}{\mu^2} - M_W^2 \ln\frac{M_W^2}{\mu^2})\right.\\ &\left. +5(M_Z^2 - M_W^2) + 6\left(\frac{M_Z^2 m_H^2}{M_Z^2 - m_H^2}\ln\frac{m_H^2}{M_Z^2} - \frac{M_W^2 m_H^2}{M_W^2 - m_H^2}\ln\frac{m_H^2}{M_W^2}\right)\right\} \quad .\end{aligned}$$

In contrast to the fermion contributions, the Higgs contribution alone is neither gauge invariant nor UV-finite! Only the sum with the remaining (non-fermionic) contribution is finite and gauge independent. For $\mu = M_W$ an $\xi = 1$ one obtains a possible splitting of terms which exhibits the full m_H-dependence in any case.
We finally consider limiting cases: (setting $\mu = M_W$)

i) $m_H \ll M_V$:

$$\Delta\rho = \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\cdot 3\frac{\sin^2\Theta_W}{\cos^2\Theta_W}\left(\frac{\ln\cos^2\Theta_W}{\sin^2\Theta_W} + \frac{5}{6}\right)$$

ii) $m_H \gg M_V$:

$$\Delta\rho = -\frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\cdot 3\frac{\sin^2\Theta_W}{\cos^2\Theta_W}\left(\ln\frac{m_H^2}{M_W^2} - \frac{5}{6}\right) \quad .$$

Notice:

1. *For $g' \to 0$ $(M_Z \to M_W,\ \sin^2\Theta_W \to 0)$ we get $\Delta\rho^{\text{Higgs}} \equiv 0$.*
   $W^\pm, Z$ would be $SU(2)_R$ triplet of a global $SU(2)_R$ of $\mathcal{L}_{\text{Higgs}}$, i.e. $\Delta\rho^{\text{Higgs}}$ measures breaking of $SU(2)_R$ by the weak hypercharge.

2. *The limit $m_H \to 0$ exists and yields a small finite term.*

3. *There remain no m_H^2 terms for $m_H \to \infty$. Instead one observes a logarithmic Higgs mass dependence.*

Similarly one finds:

2. Δr, Δr_f

$$\begin{aligned}\Delta r^{\text{Higgs}} &\simeq \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\frac{11}{3}\left(\ln\frac{m_H^2}{M_W^2} - \frac{5}{6}\right)\\ \Delta r_f^{Higgs} &\simeq \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\frac{1 + 9\sin^2\Theta_W}{3\cos^2\Theta_W}\left(\ln\frac{m_H^2}{M_W^2} - \frac{5}{6}\right) \quad .\end{aligned}$$

III. RENORMALIZATION

So far, we have defined dimensionally regularized Green functions for complex space-time dimensions d with Re $d < 4$ which have poles in $\varepsilon = d - 4$. These bare *Green functions have been obtained by the perturbation expansion based on the splitting*

$$\mathcal{L}_b = \mathcal{L}_{b\,o} + \mathcal{L}_{b\,int} \tag{137}$$

of the full Lagrangian $\mathcal{L}_b$ into a free and an interaction part. This splitting is not UV-finite, and hence not physical, and makes sense only if we have regularized the theory. The bare perturbation expansion thus is regularization dependent. *In particular, using dimensional regularization it depends on an arbitrary scale parameter μ! For the bare Green functions the limit $\varepsilon \to 0$ does not exist! Green functions which allow to take the limit $\varepsilon \to 0$ require renormalization, which amounts to a reorganization of the formal perturbation series.*

The basic reason for the problem is the following: We have tried to solve the equations of motion of the system without imposing appropriate boundary conditions. Since our goal is to calculate scattering matrix elements, the physical boundary conditions are obvious: We have to introduce renormalized fields which describe, at asymptotic times, free physical scattering states. For the electron field, for example

$$\psi_e^{ren}(\vec{x},t) \stackrel{t\to -/+\infty}{\to} \psi_{e\;in/out}(\vec{x},t) \tag{138}$$

must describe a free electron of mass m_e. This is the so called LSZ asymptotic condition [53]

Since masses and the normalization of fields are altered by quantum effects (loops) the physical boundary conditions (renormalization conditions) must be enforced by renormalization. These boundary conditions are conditions on the mass-shell $p^2 = m^2$ of the external particles, therefore the corresponding renormalization procedure (renormalization scheme) is called on-shell scheme.

The independent parameters are the physical particle masses plus a coupling constant. A natural choice for the coupling is the universal (due to electromagnetic current conservation) fine structure constant α. This defines a QED-like on-shell renormalization scheme with independent parameters*:*

$$\alpha, M_W, M_Z, m_f, m_H\;. \tag{139}$$

*All other couplings are then fixed (*dependent parameters*) by the mass-coupling relations:*

$$\begin{aligned}
\sin^2\Theta_W &= 1 - \frac{M_W^2}{M_Z^2} \\
g = \frac{\sqrt{4\pi\alpha}}{\sin\Theta_W} \quad &, \quad g' = \frac{\sqrt{4\pi\alpha}}{\cos\Theta_W} \\
\sqrt{2}G_\mu &= \frac{1}{v^2} = \frac{\pi\alpha}{M_W^2\sin^2\Theta_W}.
\end{aligned} \tag{140}$$

The renormalization then may be performed in two steps:

1. Parameter renormalization

The parameters in the true bare Lagrangian are the bare parameters $\alpha_b, M_{Wb}, \cdots$. We reparametrize the bare Lagrangian in terms of the physical parameters (experimental input) $\alpha, M_W, \cdots$ by the following parameter renormalizations:

$$\begin{aligned}
M^2_{Vb} &= M^2_V + \delta M^2_V = M^2_V \ (1 + \frac{\delta M^2_V}{M^2_V}) \ ; V = W, Z \\
m_{fb} &= m_f + \delta m_f = m_f \ (1 + \frac{\delta m_f}{m_f}) \\
m^2_{Hb} &= m^2_H + \delta m^2_H = m^2_H \ (1 + \frac{\delta m^2_H}{m^2_H}) \\
\alpha_b &= \alpha + \delta\alpha = \alpha \ (1 + \frac{\delta\alpha}{\alpha})
\end{aligned} \tag{141}$$

which have to be performed for the dependent parameters (which serve as convenient abbreviations only) correspondingly :

$$\begin{aligned}
\sin^2\Theta_{Wb} &= \sin^2\Theta_W + \delta\sin^2\Theta_W = \sin^2\Theta_W \ (1 + \frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W}) \\
G_{\mu b} &= G_\mu + \delta G_\mu = G_\mu \ (1 + \frac{\delta G_\mu}{G_\mu})
\end{aligned} \tag{142}$$

where, to linear *order (suitable for one-loop calculations):*

$$\begin{aligned}
\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W} &= \cot^2\Theta_W (\frac{\delta M^2_Z}{M^2_Z} - \frac{\delta M^2_W}{M^2_W}) \\
\frac{\delta G_\mu}{G_\mu} &= 2\frac{\delta v^{-1}}{v^{-1}} = \frac{\delta\alpha}{\alpha} - \frac{\delta M^2_W}{M^2_W} - \frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W} \ .
\end{aligned} \tag{143}$$

It is important to notice that these parameter shifts do not alter the invariance properties of the Lagrangian. The ST-identities thus keep their bare *form. Since the bare parameters and the renormalized parameters (determined by S-matrix elements) are gauge invariant also the parameter counterterms are* gauge invariant*! This statement is true only if the tadpoles are treated properly. Since (momentum independent) tadpoles drop out from physical quantities we will not discuss them further (see e. g. [54]).*

2. Field renormalization (wave function renormalization)

In order that the fields describe properly normalized scattering states we must renormalize them such that the residue of the propagator pole is unity.

For simplicity we ignore the infrared problem *caused by soft photon effects. This problem has to be treated in the same way as in pure QED and we assume the* reader to be familiar with it. We shall use an infinitesimal photon mass m_γ as an infrared regulator at intermediate steps. For observable quantities the limit $m_\gamma \to 0$ must exist.

We then write for the physical fields:

$$\begin{aligned} V_{\mu\, b} &= \sqrt{Z_V} V_{\mu\, ren}\, ; V = A, Z, W^\pm \\ \psi_{f\, b} &= \sqrt{Z_f}\psi_{f\, ren} \\ H_b &= \sqrt{Z_H} H_{ren} \end{aligned} \tag{144}$$

and the Z-factors are fixed by the condition that propagators of the renormalized fields have residue one at the pole.

For unstable particles, like the vector bosons, the location and residue of the pole are complex. Unitarity requires the counterterms to be real. Therefore the counterterms are determined by the real parts of the location and residue of the pole, in this case.

It may be questioned whether independent field renormalizations are compatible with the local non-abelian gauge structure. In fact the canonical (= bare) form of the ST-identities only admits a renormalization factor for each field multiplet! The following remarks are important here:

- The Z-factors are gauge dependent and in order to get gauge invariant S-matrix elements there is no freedom in the choice of the wave function renormalization factors. Only the Z-factors fixed by the LSZ-conditions for the individual fields lead to the physical S-matrix [55] [56].

- The apparent conflict with the ST-identities is not as serious at it looks at first. From the path-integral representation of the generating functional

$$Z\{J,\bar\chi,\chi,\cdots\} = \int \mathcal{D}V_{\mu i}\mathcal{D}\psi\mathcal{D}\bar\psi e^{i\int(\mathcal{L}_{eff}+JV+\bar\chi\psi+\bar\psi\chi+\cdots)} \tag{145}$$

  we learn that a change of the integration variables, for example,

$$V_{\mu i\, b} \to \sqrt{Z_i} V_{\mu i\, ren}$$

  does not change the value of the integral! This means that if the fields $V_{\mu i}$ do not appear as external fields, all the Z-factors drop out completely in the interior of the Feynman diagrams. The Z-factors only affect the external legs (source terms in Eq. (145)) of a diagram, i.e. only external fields carry a normalization factor

$$V_{\mu i\, b} \to V_{\mu i\, ren} = \frac{1}{\sqrt{Z_i}} V_{\mu i\, b} \quad . \tag{146}$$

Consequently: If we perform individual field renormalizations in the bare ST-identities their renormalized forms are not altered by higher order corrections, although, now, *they have no longer a simple canonical form. Notice that, when written in terms of the physical fields, the ST-identities do not look very symmetric anyhow. Thus, in terms of the physical fields the bare froms of the ST-identities are beeing* deformed *into renormalized versions of them by the multiplicative field renormalizations of the individual physical fields. If one insists in preserving the bare form of the ST-identities one has to renormalize away only the singular ε-pole terms. These of course satisfy the bare ST-identities. This latter procedure is called* minimal subtraction MS *or* $\overline{MS}$ - scheme, *defined by the substitutions*

$$\begin{aligned} MS: &\quad \frac{2}{\varepsilon} + \ln \mu_0^2 = \ln \mu_{MS}^2 \\ \overline{MS}: &\quad \frac{2}{\varepsilon} - \gamma + \ln 4\pi + \ln \mu_0^2 = \ln \mu_{\overline{MS}}^2 \,, \end{aligned} \tag{147}$$

respectively. These subtractions correspond to a choice of counterterms

$$\begin{array}{rcl} (\delta M_V^2)_{\overline{MS}} & = & (\delta M_V^2)_{OS,UV} \\ \vdots & & \vdots \\ (\delta\alpha)_{\overline{MS}} & = & (\delta\alpha)_{OS,UV} \\ (\delta Z_i)_{\overline{MS}} & = & (\delta Z_i)_{OS,UV} \end{array} \tag{148}$$

as compared to the on-shell scheme (OS). By the index UV we indicated the ε-pole terms related to the UV-divergences. All the renormalization schemes used by different authors range from pure $\overline{MS}$ to pure OS and mixtures of them!

The irreducible vertices are obtained by amputation of the external legs (amputated legs correspond to scattering states!). Amputation means multiplication with the inverse propagator which carries a factor Z_i. Thus, field renormalization for the irreducible vertices amounts to multiplication of an external (amputated) field by $\sqrt{Z_i}$. To leading order $Z_i = 1$ and we may write

$$Z_i = 1 + \delta Z_i \;;\; \sqrt{Z_i} \simeq 1 + \frac{1}{2}\delta Z_i + \cdots \,. \tag{149}$$

The renormalization procedure for physical amplitudes may be summarized by the following simple rules: Performing the parameter shifts and the field renormalizations and expanding to linear order (appropriate for one-loop calculations) we get the simple substitutions (we abbreviate $\sin^2\Theta_W = s_W^2$*)*

$$\begin{array}{ccc} eQ_f\gamma^\mu & \to & eQ_f\gamma^\mu \cdot \left(1 + \frac{1}{2}\delta Z_\gamma + \delta Z_f + \frac{\delta e}{e}\right) \\[2ex] \frac{M_Z}{v}\gamma^\mu\left(T_{3f}(1-\gamma_5) - 2Q_f s_W^2\right) & \to & \frac{M_Z}{v}\gamma^\mu\left(T_{3f}(1-\gamma_5) - 2Q_f s_W^2\left(1 + \frac{\delta s_W^2}{s_W^2}\right)\right) \\ & & \cdot\left(1 + \frac{1}{2}\delta Z_Z + \delta Z_f + \frac{1}{2}\frac{\delta M_Z^2}{M_Z^2} + \frac{1}{2}\frac{\delta G_\mu}{G_\mu}\right) \\[2ex] \frac{M_W}{\sqrt{2}v}\gamma^\mu(1-\gamma_5) & \to & \frac{M_W}{\sqrt{2}v}\gamma^\mu(1-\gamma_5) \\ & & \cdot\left(1 + \frac{1}{2}\delta Z_W + \frac{1}{2}\delta Z_{f_1} + \frac{1}{2}\delta Z_{f_2} + \frac{1}{2}\frac{\delta M_W^2}{M_W^2} + \frac{1}{2}\frac{\delta G_\mu}{G_\mu}\right) \end{array}$$

and analogously for the other vertices.

3. Renormalization schemes

The notion "renormalization scheme" is used in two different senses of the word. Often the term is used in a more technical sense as

- *a specific way of performing renormalization at intermediate steps. This includes the choice of the regularization, the way field renormalizations and/or parameter renormalizations are organized. Some authors emphasize the use of renormalized Green-functions at intermediate steps others are interested in on-shell matrix-elements only. If the same physical quantity is calculated in terms of the same parameters to the same order in perturbation theory the result does not depend on the choice of the scheme. This first kind of distinction of different schemes is therefore not relevant for the physics.*

The second possible distinction of renormalization schemes is more physical, namely as characterizing

- *a specific choice of input parameters. Perturbative predictions in terms of different input parameter sets are scheme-dependent as we shall see below.*

We will use the term in general in this second sense. Before we are going to discuss the scheme dependence of physical predictions we briefly give an incomplete survey of different schemes used for one-loop calculations in electroweak theory by different groups:

1.	$\overline{\mathrm{MS}}$	:	*[58]*	
2.	*semi OS*	:	*[59, 60]*	*parameter counterterms OS* one *OS Z-factor per gauge multiplet*
3.	*full OS*	:	*[61]*	*leads to S-matrix elements* *in one step!*
4.	$\star$	:	*[62]*	*emphasize VB propagator effects and* *running parameters (bubble summation)*

The relation between the OS-scheme and the $\star$-scheme is briefly discussed in an Appendix at the end of this Section.

Notice: *If a physical transition matrix element is calculated in terms of a given set of physical input parameters the answer does not depend on the scheme used at intermediate steps (the schemes differ by the bookkeeping only). Evidently in all schemes the starting quantities are the bare or the equivalent $\overline{MS}$ quantities. If a particular value for μ is chosen one may give numerical values for $\overline{MS}$ quantities e. g. for $\alpha_{\overline{MS}}$, $sin^2\Theta_{\overline{MS}}(\mu = M_Z)$.*

A scheme dependence *of physical predictions shows up if different input parameters are used in a calculation. A specific choice of experimental data points used* *as an input parameter set defines a renormalization scheme (RS). Parametrizations frequently used are the following:*

1) A natural choice of "basic" parameters is the QED-like parametrization in terms of the fine structure constant α and the physical particle masses

$$\alpha, M_W, M_Z, m_f, m_H \qquad (I)$$

often referred to as the "on-shell scheme". We shall refer to it as the α-scheme. It allows for a natural separation of the QED part of the electroweak radiative corrections which is dominated often by large soft photon effects accompanying external charged particles.

2) In the Standard Model , which unifies weak and electromagnetic interactions, we can use as a coupling parameter as well the Fermi constant G_μ instead of α. We then have

$$G_\mu, M_W, M_Z, m_f, m_H \qquad (II)$$

as an independent set of parameters. This set is suitable for processes which are dominated by neutral (NC) or charged (CC) current transitions. An important property of G_μ is that it is not running from low energy up to the vector boson mass scale $M_W(M_Z)$. This G_μ-scheme thus is a genuine high energy scheme in the sense that no large logarithms show up in the calculation of vector boson processes in the LEP energy region (Z and W-pair production).

We know that the parameters of the two schemes are related by [17]

$$\sqrt{2}G_\mu = \frac{\pi\alpha}{M_W^2 \sin^2\Theta_W}\frac{1}{1-\Delta r}, \tag{150}$$

where Δr is the non-QED correction to μ-decay calculated in the α-scheme. If not stated otherwise, we use the definition

$$\sin^2\Theta_W = s_W^2 = 1 - \frac{M_W^2}{M_Z^2} \tag{151}$$

for the weak mixing angle.

A disadvantage of the parametrizations (I) and (II) is that they require a precise knowledge of M_W which will be measured precisely at LEP2 only. In order to keep the input parameter errors as small as possible we have to replace M_W by G_μ in (I).

3) The scheme to be used as a starting point for precise calculations of radiative corrections uses

$$\alpha, G_\mu, M_Z, m_f, m_H \qquad (III)$$

as input parameters, with M_Z measured from the Z line-shape at LEP1.

4) A similar parameter set using the W-mass instead of the Z-mass

$$\alpha, G_\mu, M_W, m_f, m_H \qquad (IV)$$

seems not particularly interesting, since the W-mass will never be known more accurately than the Z-mass.

5) Another interesting possibility would be to predict quantities in terms of the low energy parameters

$$\alpha, G_\mu, \sin^2\Theta_{\nu_\mu e}, m_f, m_H \qquad (V)$$

where $\sin^2\Theta_{\nu_\mu e}$ is determined from neutrino-electron scattering (by CHARM II for example).

Scheme-dependence can be investigated by predicting an observable in terms of different input parameter sets. Since not all the parameters are known to the same precision we proceed as follows: We first predict M_W and $\sin^2\Theta_{\nu_\mu e}$ in the scheme (III) and then take any 3 parameters which are independent at tree level to calculate quantities like the vector boson widths $\Gamma_{Zf\bar{f}}$, $\Gamma_{Wf\bar{f}'}$, or the cross-sections $\sigma(e^+e^- \to f\bar{f})$, $\sigma(e^+e^- \to W^+W^-)$ e.t.c.

Predictions of physical quantities of course should not depend on the specific choice of the input parameters and, in fact, they do not if we include all orders of the perturbation expansion. Actually, the reparametrization invariance is inferred by renormalization group invariance. However, practical perturbative calculations are approximations obtained by truncation of the perturbation series. The accuracy of the finite order approximations depends on the choice of the input parameters i.e. finite order results are scheme dependent [64].

Let us illustrate this point by an example: Suppose we compute a matrix element M in the α-scheme (I) to one-loop order yielding a result

$$M^{(1)} = \alpha^n C[1 + b\alpha].$$

Now, suppose we calculate the same quantity in the G_μ-scheme (II) which amounts to a replacement of $\alpha \simeq 137^{-1}$ by $\alpha' = \frac{\alpha}{1-\Delta r} \simeq 128^{-1}$ i.e. to one-loop order $\alpha' = \alpha[1+a\alpha]$ and

$$M'^{(1)} = \alpha'^n C[1 + b'\alpha'].$$

Inserting α' we get

$$M'^{(1)} = M^{(1)} + \delta M$$

with $b' = b - na$ and

$$\delta M = \alpha^n C \left[(\frac{n(n-1)}{2} a^2 + (n+1)ab')\alpha^2 + \cdots + a^{n+1} b' \alpha^{n+2} \right].$$

Thus the result differs by δM. If we do not actually calculate the higher orders

$$\delta M = M'^{(1)} - M^{(1)}$$

must be considered as uncertainty due to unknown higher order effects.

*For LEP experiments one-loop calculations are insufficient to get the precision of 0.1% and one has to go to resummation improved calculations by including leading higher order effects. The study of the scheme dependence of resummation improved results is a way to estimate missing higher order contributions (*educated guess*). Of course only an actual n-loop calculation can tell us what the full n-loop answer is.*

Let us summarize the content of this subsection by the following conclusions:

- *If a physical quantity is calculated with different input parameters the answer is the same if we calculate it to arbitrary high orders.*

However:

- *Calculating a quantity to a given order the omitted higher order terms differ for different parametrizations. This leads to a* scheme dependence *of the result (approximate) due to different truncation errors.*

- *Differences can also be due to different resummation prescriptions (see below).*

After these general considerations we now discuss one-loop renormalization in details.

4. One-loop renormalization

4.1. Feynman rules

Starting point is the classical gauge invariant Lagrangian

$$\mathcal{L}_{inv} = \mathcal{L}_{Yang-Mills} + \mathcal{L}_{matter} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} \; .$$

The quantization is obtained by adding the gauge-fixing (GF) and Faddeev-Popov (FP) terms in order to get the quasi-invariant effective action suitable for the path-integral quantization:

$$\mathcal{L}_{eff} = \mathcal{L}_{inv} + \mathcal{L}_{GF} + \mathcal{L}_{FP} \; .$$

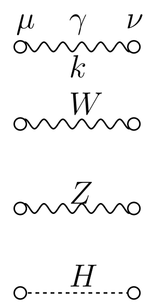
The correct Feynman rules for non-abelian Gauge theories have first been obtained by
't Hooft [7]. Here we restrict ourselves to write down, in Fig. 8, the Feynman rules in the Feynman - 't Hooft gauge for the physical fields. The gauge self-couplings are given in terms of the tensors (momenta incoming)

$$\begin{aligned} V^{\rho\sigma,\,\mu}(p) &= g^{\rho\sigma}\,(p_2-p_1)^\mu \;+\; g^{\rho\mu}\,(p_1-p_3)^\sigma \;+\; g^{\sigma\mu}\,(p_3-p_2)^\rho \\ T^{\mu\nu,\,\rho\sigma} &= 2\,g^{\mu\nu}g^{\rho\sigma} \;-\; g^{\mu\rho}g^{\nu\sigma} \;-\; g^{\mu\sigma}g^{\nu\rho} \end{aligned}$$

These Feynman rules are "complete" only in the unitary gauge. In this gauge

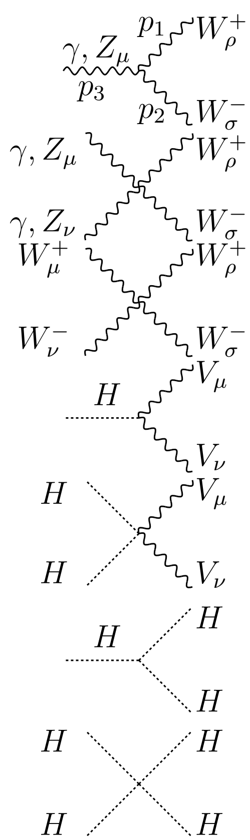
$$\underset{\text{Feynman-'t Hooft}}{-ig_{\mu\nu}\frac{1}{k^2-M^2}} \;\;\rightarrow\;\; \underset{\text{unitary (non-renormalizable)}}{-i\,(g_{\mu\nu}-\frac{k_\mu k_\nu}{M^2})\,\frac{1}{k^2-M^2}}$$

such that $\partial_\mu W_i^\mu = 0$ *on the mass-shell (i.e.* $k^\mu\,(\cdots) = 0$ *for* $k^2 = M^2$*).*



	$A^\mu \Box A_\mu$	$-i\ g^{\mu\nu}\frac{1}{k^2}$
$+$	$W^{+\mu}\left(\Box + M_W^2\right) W^-_\mu$	$-i\ g^{\mu\nu}\frac{1}{k^2-M_W^2}$
$+$	$Z^\mu\left(\Box + M_Z^2\right) Z_\mu$	$-i\ g^{\mu\nu}\frac{1}{k^2-M_Z^2}$
$+$	$H\left(\Box + m_H^2\right) H$	$\frac{i}{k^2-m_H^2}$

$+$ *interaction vertices :*



$\gamma,Z\,W^+W^-$	$ig_{\gamma,Z}V^{\rho\sigma,\mu}(p);$	$g_\gamma = e, \quad g_Z = g\cos\Theta_W$
$\gamma,Z\ \gamma,Z\,W^+W^-$	$-iC_iT^{\mu\nu,\rho\sigma};$	$C_{\gamma\gamma} = e^2$ $C_{\gamma Z} = eg\cos\Theta_W$ $C_{ZZ} = g^2\cos^2\Theta_W$
$W^+W^+W^-W^-$	$ig^2T^{\mu\nu,\rho\sigma};$	
HVV	$ig_VM_Vg^{\mu\nu}$	$V = Z, W$
$HHVV$	$i\frac{1}{2}g_V^2g^{\mu\nu}$	$g_Z = \frac{g}{\cos\Theta_W}, \quad g_W = g$
HHH	$-i3\frac{m_H^2}{v}$	
$HHHH$	$-i3\frac{m_H^2}{v}$	

Figure 8a: Feynman rules for $\mathcal{L}_{YM} + \mathcal{L}_{Higgs}$

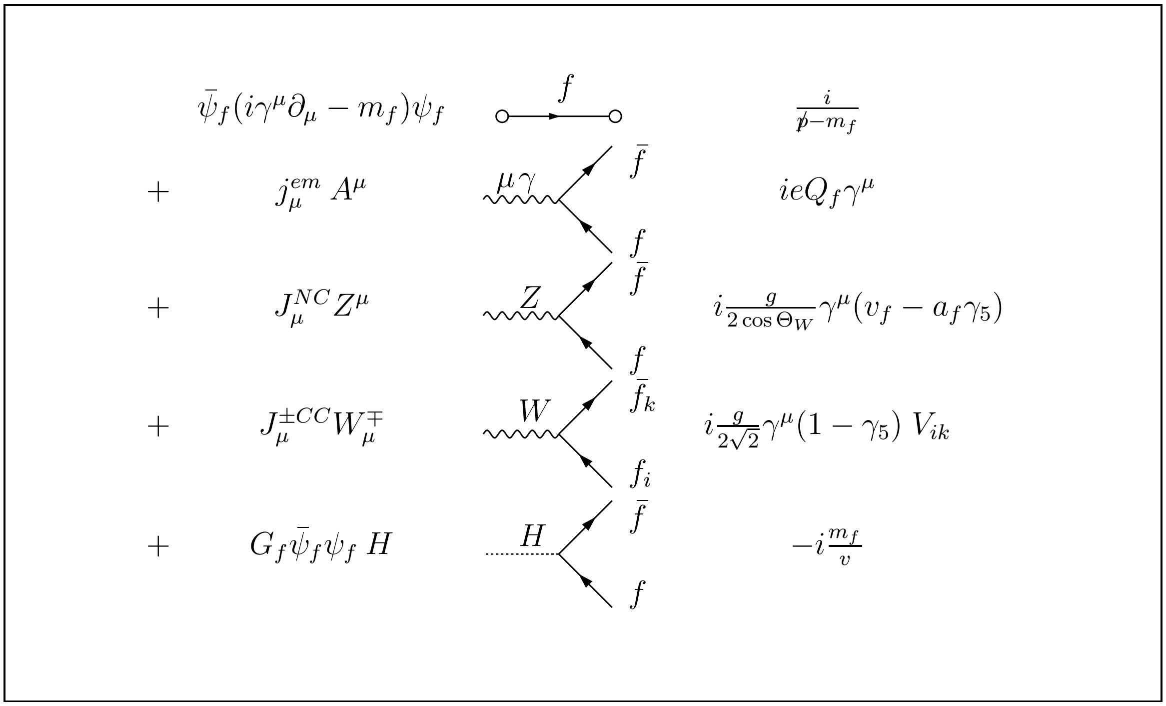


Figure 8b: Feynman rules for $\mathcal{L}_{matter} + \mathcal{L}_{Yukawa}$

Here and in the following we do not explicitly write the $i\varepsilon$-prescription for the Feynman propagators and include it in the mass. Thus M^2 always stands for $M^2 - i\varepsilon$.

As discussed in Sec. II the gauge boson propagators are only defined after fixing a gauge, because the 4-component field $W_{\mu i}$ describes only 3 physical degrees of freedom (2 transverse and 1 longitudinal). A convenient gauge is the general covariant and linear 't Hooft gauge also called "R_ξ- gauge" for which the massive vector boson propagators take the form

$$-i\,\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M^2}\right)\,\frac{1}{k^2 - M^2} \tag{152}$$

The prize we have to pay in going from the physical non-renormalizable unitary gauge to a renormalizable gauge is that we have to take into account ghosts: the 3 Higgs ghosts $\phi^\pm$ and ϕ and 4 Faddeev-Popov ghosts $\zeta^\pm$, ζ and $\aleph$, which have 39 additional interaction vertices. The existence of the 't Hooft gauge is conceptually very important because it allows interpolating in a continuous way between a renormalizable gauge like the Feynman-'t Hooft *gauge with $\xi = 1$ (simple propagators, unphysical polarization, ghosts) and the unitary gauge reached as $\xi \to \infty$ (no ghosts, Lee-Yang terms, UV-behavior of off-shell quantities bad). For the gauge invariant (ξ-independent) physical quantities this infers at the same time renormalizability* and *unitarity.*

After these introductory remarks, we are going to discuss renormalization in more detail. In order to be able to control the UV-divergences, we have to use a renormalizable gauge (validity of power counting arguments). In order to keep notation as handy as possible we use the Feynman-'t Hooft gauge. We have to inspect those Green functions which are superficially divergent, propagators, form-factors and four-point functions.

4.2 VB propagator corrections

Since, in physical matrix elements (on-shell quantities), the longitudinal parts of the VB propagators cancel against ghost amplitudes, as a consequence of the Slavnov-Taylor identities, we need to consider only the transverse part in the following. In order to see how the splitting into transverse and longitudinal parts works, we introduce the projectors

$$\underset{\text{transverse projector}}{T_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}} \quad , \quad \underset{\text{longitudinal projector}}{L_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}}$$

which satisfy

$$\begin{aligned} T^\mu_\nu + L^\mu_\nu &= \delta^\mu_\nu \\ T^\mu_\rho T^\rho_\nu = T^\mu_\nu \quad &, \quad L^\mu_\rho L^\rho_\nu = L^\mu_\nu \\ T^\mu_\rho L^\rho_\nu = 0 \quad &, \quad L^\mu_\rho T^\rho_\nu = 0 \end{aligned}$$

and write a VB-propagator in the form

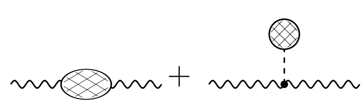
$$\begin{aligned} D_{\mu\nu}(k) &= -i\, \Big(T_{\mu\nu} \cdot \Pi_1(k^2) + L_{\mu\nu} \cdot \Pi_2(k^2)\Big) \\ &= -i\, \Big(g_{\mu\nu} \cdot \Pi_1(k^2) + k_\mu k_\nu \cdot \hat{\Pi}_2(k^2)\Big) \end{aligned} \tag{153}$$

with $\Pi_2 = k^2\hat{\Pi}_2 + \Pi_1$. Thus the transverse amplitude Π_1 is uniquely given by the $g_{\mu\nu}$-term in the propagator and Π_2 does not mix with the transverse part. The index 1 will be omitted in the following

4.2.1 The W-propagator

Diagrammatically the W-propagator is given in Fig. 9a. Since the $g_{\mu\nu}$-tensor in front of the transverse self-energy acts as a unit tensor, we may omit it for notational convenience. Thus $-i/(k^2 - M_W^2)$ represents the free transverse VB-propagator and

$$-i\ g_{\mu\nu}\ \Pi_W(k^2) \equiv \tag{154}$$



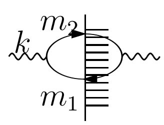
defines the self-energy function as the propagator with amputated legs, given by the sum of one-particle irreducible (1pi) diagrams. These are the graphs which cannot be cut into two disjoint parts by cutting one line. The tadpole graphs (2nd group *in the figure above) play a special role. They must be included if one wants to have gauge invariant mass counterterms. They cancel however in physical quantities and will be omitted henceforth. At one loop order the propagator is then given by*

$$-i\ D_W(k^2) = \frac{-i}{k^2 - M_W^2}\left(-i\ \Pi_W(k^2)\right)\frac{-i}{k^2 - M_W^2}$$

The full or dressed propagator is given by the geometrical progression (Dyson summation)
Let us briefly discuss some important properties of Π_W*:*
1) $\Pi_W(k^2)$ *is complex, when* $k^2 > (m_1 + m_2)^2$

$$\Pi_W = Re\,\Pi_W + i\ Im\,\Pi_W$$

m_1 *and* m_2 *are the masses of the particles into which the* W *can decay. For example* W^- *can decay into* $\bar{\nu}_e e^-$ *and we have* $m_1 = m_{\nu_e} = 0$ *and* $m_2 = m_e$ *so* $Im\Pi_W \neq 0$ *if* $k^2 > m_e^2$*. As a rule, a cut diagram*



contributes to the imaginary part if the cut diagram kinematically allows physical intermediate states. The W *is an unstable particle and on the mass-shell* $k^2 = M_W^2$ *of the* W *we have*

$$Im\,\Pi_W(k^2 = M_W^2) = M_W\ \Gamma_W \neq 0 \tag{155}$$

defining the finite width Γ_W *of the* W*-particle. The real part* $Re\,\Pi_W$ *is UV-divergent and requires renormalization: At lowest order the propagator is*

$$D_W = \frac{1}{k^2 - M_W^2}$$

which has a pole $k^2 = M_W^2$ *with residue one. In higher orders we define the mass (and the width) from the location of the pole of the propagator, which for* unstable particles *lies in the complex* k^2*-plane. We define the pole to lie at*

$$(k^2)_{pole} = M_W^2 - i\ M_W\ \Gamma_W \equiv \tilde{M}_W^2 \tag{156}$$

thus we have the correspondence

physical mass	$\Longleftrightarrow$	*real part of location of propagator pole*
width	$\Longleftrightarrow$	*imaginary part of the location of the pole .*

By our derivation above we obtained

$$D_W = \frac{1}{k^2 - M_W^2 + \Pi_W(k^2)}$$

with Re $\Pi_W(M_W^2) \neq 0$*, which tells us that the location of the pole gets shifted by radiative corrections . Consequently,* M_W *in the previous equation cannot be the physical mass of the* W *but it is the* bare *mass. Thus*

$$M_W^2 \to M_{Wb}^2 = M_W^2 + \delta M_W^2$$

where δM_W^2 *is the mass counterterm fixed by the condition:*

$$\begin{gathered} Re\,[k^2 - M_W^2 - \delta M_W^2 + \Pi_W(k^2)]\Big|_{k^2=\tilde{M}_W^2} = 0 \\ \Updownarrow \\ \delta M_W^2 = Re\Pi_W(\tilde{M}_W^2) \simeq Re\Pi_W(M_W^2) \end{gathered} \tag{157}$$

this removes the quadratically divergent term from the W *self-energy. Since* $\Gamma_W/M_W = O(\alpha)$*, we may use* $\tilde{M}_W^2 \simeq M_W^2$ *in the one-loop approximation. Now, after one subtraction,*

$$D_W = \frac{1}{k^2 - M_W^2 + \Big(\Pi_W(k^2) - Re\Pi_W(\tilde{M}_W^2)\Big)}$$

is logarithmically divergent, only. Thus it still has poles in $\varepsilon = d - 4$*. If the* W *is not an external particle (describing a scattering state) we may use minimal subtraction here by applying the substitution Eq. (147). This procedure preserves the bare form of the Slavnov-Taylor identities. For an external* W *we have to proceed differently and perform on-shell wave function renormalization: It is fixed by the condition*

- *the real part of the residue of the propagator pole must be normalized to one.*

Because the W *is a charged particle the on-shell residue of the pole does not exist for massless photons (QED infrared problem). As mentioned earlier we use an infinitesimal photon mass in this case in order to be able to proceed in the canonical way which, in a strict sense, applies to neutral particles only. After these remarks, we go on with the determination of the residue of the pole. If we expand the self-energy at the pole*

$$\Pi_W(k^2) \simeq \Pi_W(\tilde{M}_W^2) + (k^2 - \tilde{M}_W^2)\,\frac{d\Pi_W}{dk^2}(\tilde{M}_W^2) + \cdots\;;\; k^2 \to \tilde{M}_W^2$$

we obtain, using $\delta M_W^2 = Re\,\Pi(\tilde{M}_W^2)$*,* $M_W\Gamma_W = Im\,\Pi(\tilde{M}_W^2)$*,*

$$\begin{aligned} D_W &= \frac{1}{k^2 - M_W^2 + \Big(\Pi_W(k^2) - Re\,\Pi_W(\tilde{M}_W^2)\Big)} \\ &= \frac{1}{k^2 - \tilde{M}_W^2}\,\frac{1}{1 + \frac{d\Pi_W}{dk^2}(\tilde{M}_W^2)} + O(k^2 - \tilde{M}_W^2) \end{aligned}$$

and the residue of the pole can be read off. If we now perform the field renormalization Eq. (144) and consider the propagator of the renormalized field $D_{W\,ren} = Z_W^{-1}\; D_{W\,bare}$ *i. e.*

$$\frac{1}{k^2 - M_W^2 + \Pi_{W\,ren}(k^2)} = \frac{1}{Z_W} \cdot \frac{1}{k^2 - M_W^2 + \Big(\Pi_W(k^2) - Re\,\Pi_W(\tilde{M}_W^2)\Big)} \tag{158}$$

which is required to have residue one and thus

$$Z_W = Re\,\left[1 + \frac{d\Pi_W}{dk^2}(\tilde{M}_W^2)\right]^{-1} \ . \tag{159}$$

If we expand to linear order (suitable for 1-loop calculations)

$$\delta Z_W = Z_W - 1 \simeq -Re\,\frac{d\Pi_W}{dk^2}(M_W^2) \tag{160}$$

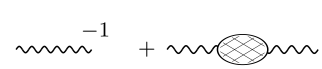
and the renormalized self-energy function reads

$$\Pi_{W\,ren}(k^2) \;\; = \;\; \Pi_W(k^2) - Re\,\Pi_W(M_W^2) - (k^2 - M_W^2)\; Re\,\frac{d\Pi_W}{dk^2}(M_W^2)\,. \tag{161}$$

The wave function renormalization also affects the imaginary part and hence the width by a next order term. Denoting by $\Delta\Gamma_W^{(1)}$ *the next order corrections not considered here, the corrected width reads*

$$\Gamma_W^{(1)} = \Big(\Gamma_W^{(0)} + \Delta\Gamma_W^{(1)}\Big) \,/\, \left(1 + Re\,\frac{d\Pi_W}{dk^2}(M_W^2)\right) \ . \tag{162}$$

We finally notice that the inverse bare propagator

$$-i\; g_{\mu\nu}\; D_W^{-1} = -i\; g_{\mu\nu}\; \Big(k^2 - M_W^2 + \Pi_W(k^2)\Big) =$$ 

is given by the irreducible self-energy diagrams.

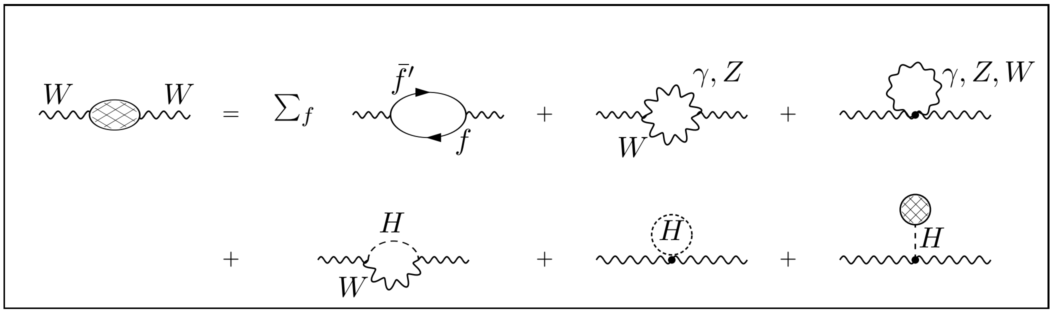


Figure 9a: W self-energy diagrams

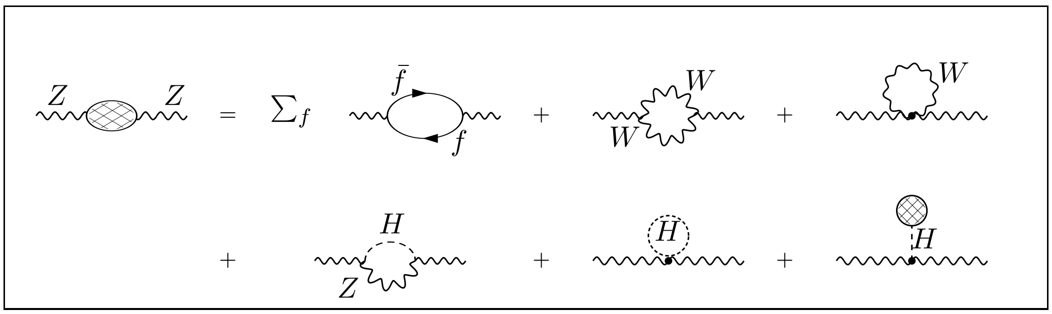


Figure 9b: Z self-energy diagrams

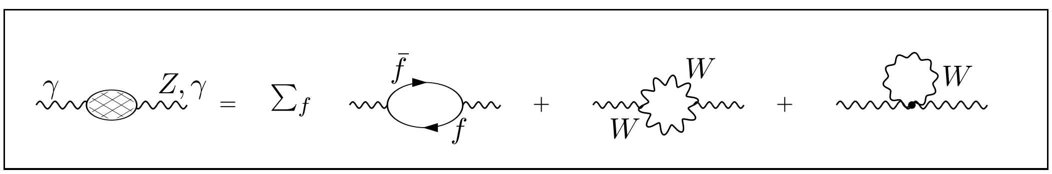


Figure 9c: γ and γZ self-energy diagrams

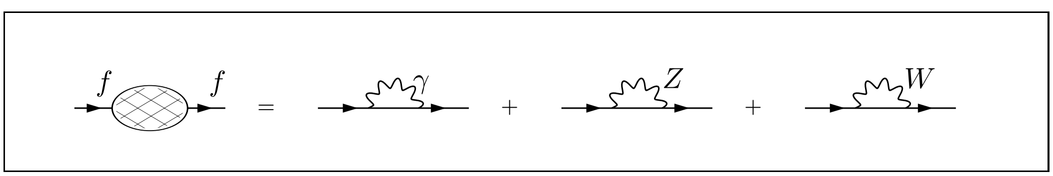


Figure 10: Fermion self-energy diagrams

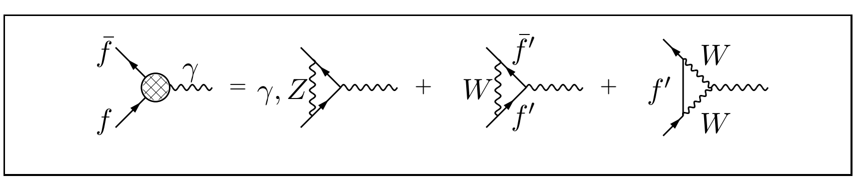
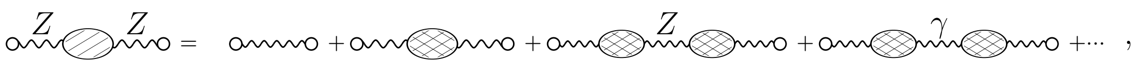


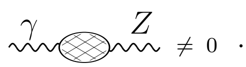
Figure 11: Electromagnetic vertex diagrams

4.2.2 The (Z, γ)-propagator

The renormalization of the Z-propagator



proceeds similarly to the W-propagator, however, the situation is complicated by $\gamma - Z$ mixing



Due to mixing one cannot treat the Z and the γ propagators separately. They rather form a 2×2 -matrix propagator. The simplest way to treat this problem is to start from the inverse propagator given by the irreducible self-energies (sum of 1pi diagrams). Again we restrict ourselves to a discussion of the transverse part and we take out a trivial factor $-i\, g_{\mu\nu}$ in order to keep notation as simple as possible. With this convention we have for the inverse $\gamma - Z$ propagator the symmetric matrix

$$\hat{D}^{-1} = \begin{pmatrix} k^2 + \Pi_{\gamma\gamma}(k^2) & \Pi_{\gamma Z}(k^2) \\ \Pi_{\gamma Z}(k^2) & k^2 - M_Z^2 + \Pi_{ZZ}(k^2) \end{pmatrix} \tag{163}$$

Using 2×2 matrix inversion

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Rightarrow M^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

we find for the propagators

$$\begin{aligned}
D_{\gamma\gamma} &= \frac{1}{k^2 + \Pi_{\gamma\gamma}(k^2) - \frac{\Pi^2_{\gamma Z}(k^2)}{k^2 - M_Z^2 + \Pi_{ZZ}(k^2)}} \simeq \frac{1}{k^2 + \Pi_{\gamma\gamma}(k^2)} \\
D_{\gamma Z} &= \frac{-\Pi_{\gamma Z}(k^2)}{(k^2 + \Pi_{\gamma\gamma}(k^2))(k^2 - M_Z^2 + \Pi_{ZZ}(k^2)) - \Pi^2_{\gamma Z}(k^2)} \simeq \frac{-\Pi_{\gamma Z}(k^2)}{k^2\,(k^2 - M_Z^2)} \\
D_{ZZ} &= \frac{1}{k^2 - M_Z^2 + \Pi_{ZZ}(k^2) - \frac{\Pi^2_{\gamma Z}(k^2)}{k^2 + \Pi_{\gamma\gamma}(k^2)}} \simeq \frac{1}{k^2 - M_Z^2 + \Pi_{ZZ}(k^2)}\ .
\end{aligned} \tag{164}$$

These expressions sum correctly all the reducible bubbles. The approximations indicated are the one-loop results. The extra terms are higher order contributions. For precision physics at LEP they have to be taken into account because, as we shall see later, one-loop approximations are insufficient. Of course we have to proceed order by order in perturbation theory and we only discuss the one-loop case here. At one-loop order the Z propagator is renormalized in the same way as the W propagator. Thus with $M^2_{Zb} = M^2_Z + \delta M^2_Z$ and $Z_{\mu b} = \sqrt{Z_Z} Z_{\mu ren}$

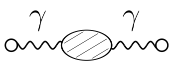
$$\delta M_Z^2 = Re\,\Pi_{ZZ}(M_Z^2)\ , \quad Z_Z = Re\,\left[1 + \frac{d\Pi_{ZZ}}{dk^2}(M_Z^2)\right]^{-1}\ . \tag{165}$$

[22] *Diagrammatically the Z-propagator is given in Fig. 9b.*

For the photon propagator the unbroken local $U(1)_{em}$-invariance (conservation of the electromagnetic current) implies

$$\Pi_{\gamma\gamma}(k^2) = k^2\, \Pi'_{\gamma\gamma}(k^2) \tag{166}$$

and hence (ignoring the higher order mixing term (see below))



$$= -i\, g_{\mu\nu}\, \frac{1}{k^2}\, \frac{1}{1+\Pi'_{\gamma\gamma}(k^2)} = -i\, g_{\mu\nu}\, D_\gamma(k^2) \tag{167}$$

and thus the pole is strictly at $k^2 = 0$. No photon mass term is generated by higher order effects and there is no photon mass renormalization. Like in QED, the photon

[22] We should mention that the definition of the physical vector boson masses M_W and M_Z is not unique because of the instability of these particles. Usually they are defined by the real parts of the locations of the poles of the transverse parts of the W and Z propagators:

$$\begin{aligned} D_W(s) &= \frac{1}{s - M_W^2 - \delta M_W^2 + \Pi_W(s)} \\ D_Z(s) &= \frac{1}{s - M_Z^2 - \delta M_Z^2 + \Pi_{ZZ}(s) - (\Pi_{\gamma Z}(s))^2/(s + \Pi_{\gamma\gamma}(s))} \;. \end{aligned}$$

To the order $O(\alpha)$ (neglecting the mixing term in the Z propagator), M^2 is the physical mass if we fix the mass counterterm δM^2 by $\delta M^2 = \mathrm{Re}\Pi(M^2)$. The total width Γ is determined by the imaginary part of the self-energy function Π according to $M\Gamma = \mathrm{Im}\,\Pi(M^2)$.
A subtlety comes in, if we want to include higher order effects, because the vector bosons are unstable particles such that the poles of the propagators are located at complex values $s_0 = M^2 - iM\Gamma$ of s. To our knowledge, all LEP physics calculations , which intend to include higher order effects systematically, have been using the "physical" masses defined by the location of the propagator pole in the zero width approximation such that

$$\begin{aligned} \delta M_W^2 &= \mathrm{Re}\Pi_W(M_W^2) \\ \delta M_Z^2 &= \mathrm{Re}(\Pi_{ZZ}(M_Z^2) - (\Pi_{\gamma Z}(M_Z^2))^2/(M_Z^2 + \Pi_{\gamma\gamma}(M_Z^2))) \end{aligned}$$

are the mass counterterms. Since, near the resonance, the imaginary part of Π is linear in s to a very good approximation, $\mathrm{Im}\,\Pi(s) \simeq s\Gamma/M$ [65], the real part of the location of the pole is not M^2 but $M'^2 = M^2 - \Gamma^2$ (by insertion of s_0 given above in $\mathrm{Im}\,\Pi(s)$) (see Consoli and Sirlin in Ref. [65]). Thus, there is a difference between the two definitions of the mass given by $M - M' = \frac{1}{2}\Gamma^2/M$. The "true" mass M' to the order $O(\alpha^2)$ coincides with the "reduced" mass introduced by Bardin et al. [67], which is obtained if one redefines the mass and the width in such a way, that the s dependence of the width in the propagator disappears near resonance:

$$\sigma_{peak} \propto \frac{s}{(s-M^2)^2 + s^2\frac{\Gamma^2}{M^2}} = \frac{1}{1+\gamma^2}\;\frac{s}{(s-M'^2)^2 + M'^2\Gamma'^2}$$

with $\gamma = \Gamma/M$, $M' = M/\sqrt{1+\gamma^2}$ and $\Gamma' = \Gamma/\sqrt{1+\gamma^2}$. So M_Z-$M_Z^{'}$ is about 35 MeV and, depending on the top mass, M_W-$M_W^{'}$ is about 30 or 40 MeV.
It should be stressed that this higher order ambiguity in the definition of the vector boson masses does not mean that $O(\alpha^2)$ effects are not taken care off correctly in the standard approach. The two definitions just lead to a different bookkeeping of the higher order effects.

wave function renormalization is given by

$$Z_\gamma = \left[1 + \Pi'_{\gamma\gamma}(0)\right]^{-1} \simeq 1 - \Pi'_{\gamma\gamma}(0) \; . \tag{168}$$

The mixing amplitude has to be renormalized as well. The proper renormalized photon and Z fields must be determined such that the (γ, Z)-propagator has the correct particle pole structure. To this end we have to guarantee that the renormalized propagator matrix is diagonal at the photon pole $k^2 = 0$ and at the Z-pole $k^2 = \tilde{M}_Z^2 \simeq M_Z^2$. This is satisfied precisely if the $\gamma - Z$ mixing amplitude vanishes at both poles (see Eq. (164)). Thus the renormalized mixing self-energy must be

$$\Pi_{\gamma Z\, ren}(k^2) = \Pi_{\gamma Z}(k^2) - \Pi_{\gamma Z}(0) - \frac{k^2}{M_Z^2}\left(Re\,\Pi_{\gamma Z}(M_Z^2) - \Pi_{\gamma Z}(0)\right) \; . \tag{169}$$

This can be achieved by two subsequent transformations of the bare fields:

i) Infinitesimal (perturbative) rotation

$$\left(\begin{array}{c} A_b \\ Z_b \end{array}\right) = \left(\begin{array}{cc} 1 & -\Delta_0 \\ \Delta_0 & 1 \end{array}\right)\left(\begin{array}{c} A' \\ Z' \end{array}\right) \tag{170}$$

diagonalizing the mass matrix at one-loop (n+1-loop) order given that the mass matrix has been diagonalized at tree (n-loop) level.

ii) Upper diagonal matrix wave function renormalization inducing a kinetic mixing term (this cannot be done by an orthogonal transformation)

$$\left(\begin{array}{c} A' \\ Z' \end{array}\right) = \left(\begin{array}{cc} \sqrt{Z_\gamma} & -\Delta_Z \\ 0 & \sqrt{Z_Z} \end{array}\right)\left(\begin{array}{c} A_r \\ Z_r \end{array}\right) \tag{171}$$

which allows to normalize the residues to one for the γ and Z-propagator and to shift to zero the mixing propagator at the Z-pole.

Thus the relationship between the bare and the renormalized (LSZ) fields is (expanded to linear order)

$$\begin{aligned} A_b &= \sqrt{Z_\gamma} A_r - \left(\Delta_Z + \Delta_0\right) Z_r \\ Z_b &= \sqrt{Z_Z} Z_r + \Delta_0\, A_r \; , \end{aligned} \tag{172}$$

generalizing (118). The counterterms Δ_0 and Δ_Z are determined by the condition (144)

$$\begin{aligned} \Delta_0 &= \frac{\Pi_{\gamma Z}(0)}{M_Z^2} \\ \Delta_Z &= \frac{Re\,\Pi_{\gamma Z}(M_Z^2) - \Pi_{\gamma Z}(0)}{M_Z^2} \end{aligned} \tag{173}$$

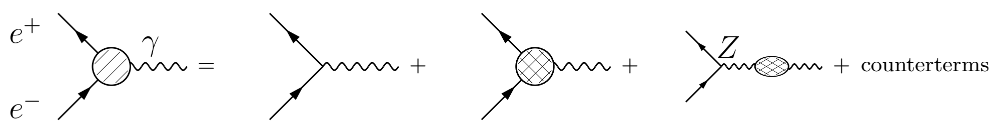
Of course, the field transformations induce mixing counterterms at the vertices. Again, this non-symmetric transformation only affects the bookkeeping such that the propagator pole structure becomes obvious. It does not change the value of the functional integral i.e. the mixing counterterms cancel in the interior of Feynman diagrams.

4.3 Charge renormalization

In electroweak theory charge renormalization looks formally pretty much the same as in pure QED. There are of course additional Feynman diagrams contributing. In particular there are new γZ mixing contributions. The fermion propagators are renormalized in the same way as the electron propagator in QED. However unlike in QED the right-handed and left-handed fields are renormalized in a different way such that

$$\delta Z_f = z_{vf} + z_{af}\gamma_5 \, . \tag{174}$$

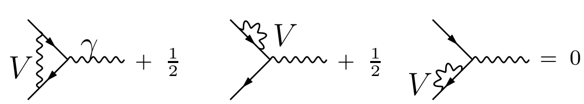
Finally, we have to determine the counterterm for the electric charge. The condition is that



evaluated in the Thomson limit ($k^2 = 0$, $E_\gamma \to 0$) gives the renormalized charge e. Thus

$$\begin{aligned} - \;\; & i\, e \left\{ \gamma^\mu \left(1 + \frac{\delta e}{e} + \frac{1}{2}\delta Z_\gamma + z_{ve} + A_1^{\gamma ee} - \frac{v_e}{2\sin\Theta_W \cos\Theta_W}\frac{\Pi_{\gamma Z}}{M_Z^2} \right.\right. \\ & \left.\left. +(z_{ae} + A_2^{\gamma ee} - \frac{a_e}{2\sin\Theta_W\cos\Theta_W}\frac{\Pi_{\gamma Z}}{M_Z^2})\gamma_5 \right) - i\sigma^{\mu\alpha}\frac{k_\alpha}{2m_e}A_3^{\gamma ee} \right\} \\ \to \;\; - \;\; & ie\gamma^\mu \;\; \text{in the Thomson limit} \end{aligned} \tag{175}$$

where $A_i^{\gamma ee}$ are vertex corrections and $\Pi_{\gamma Z}$ is the $\gamma - Z$ mixing term. By the electromagnetic Ward-Takahashi identity ($\partial_\mu j^\mu_{em} = 0$) some of the diagrams cancel. For example, we have ($V = \gamma, Z$)



The diagrams with the loops sitting on the external legs are contributions to the wave function renormalization and the factor $\frac{1}{2}$ has its origin in Eq. (149).

While in pure QED

$$\frac{\delta e}{e} = -\frac{1}{2}\delta Z_\gamma = \frac{1}{2}\Pi'_\gamma(0)$$

in the Standard Model we find

$$\frac{\delta e}{e} = \frac{1}{2}\Pi'_\gamma(0) - \frac{1-4s_W^2}{4s_W c_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} - A_1^{\gamma ee}(0) - z_{ve} = \frac{1}{2}\Pi'_\gamma(0) + 2K s_W^2 L\ . \qquad (176)$$

where $K = \frac{\alpha}{4\pi s_W^2}$, $L = \ln\frac{M_W^2}{\mu^2}$. *The last term is the non-abelian contribution from bosonic loops in the* $\overline{MS}$ *scheme and the Feynman-'t Hooft gauge. Since*

$$2\frac{s_W}{c_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} = 4K s_W^2 L$$

we may write

$$\frac{\delta\alpha}{\alpha} = 2\frac{\delta e}{e} = \Pi'_\gamma(0) + 2\frac{s_W}{c_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2}\ . \qquad (177)$$

The fermionic contributions $\Pi^f_{\gamma Z}(0) = 0$ *vanish at zero momentum transfer. By the e.m. Ward-Takahashi identity we have*

$$A_2^{\gamma ee} + z_{ae} - \frac{1}{4s_W c_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} = 0\ . \qquad (178)$$

With δe*, the mass counterterms and the wave function renormalization factors we have a complete set of counterterms which allow to renormalize all other divergent quantities. The Feynman diagrams for the vector boson self-energies are depicted in Fig. 9. Since the tadpoles drop out in renormalized quantities we will not consider them. The fermion self-energies are needed for the determination of the wave function renormalization factors only. The diagrams for the fermion self energies and the electromagnetic vertex are shown in Figs. 10 and 11, respectively. Graphs involving ghost fields and graphs which vanish in the limit of vanishing fermion masses are not shown.*

Appendix: $\star$*-scheme and* $\overline{MS}$*-scheme versus on-shell scheme*

The $\star-$ *scheme is equivalent to* $\overline{\mathrm{MS}}$ *together with propagator resummation and a particular choice of physical boundary conditions. VB self-energies plus certain universal vertex and box contributions are incorporated in running "bare" parameters*

$$\begin{aligned}
\frac{1}{e_*^2(q^2)} &= \frac{1}{e_b^2(\mu^2)} - Re\,\Pi'_{QQ}(q^2,\mu^2) \\
\frac{1}{g_*^2(q^2)} &= \frac{1}{g_b^2(\mu^2)} - Re\,\Pi'_{3Q}(q^2,\mu^2) \\
\frac{1}{4\sqrt{2}G_{\mu\star}(q^2)} &= \frac{1}{4\sqrt{2}G_{\mu b}(\mu^2)} - Re\,(\Pi_\pm - \Pi_{3Q})(q^2,\mu^2) \\
\frac{1}{4\sqrt{2}G_{\mu\star}\rho_\star(q^2)} &= \frac{1}{4\sqrt{2}G_{\mu b}\rho_b(\mu^2)} - Re\,(\Pi_{33} - \Pi_{3Q})(q^2,\mu^2)\ .
\end{aligned}$$

Here, the reduced self-energy amplitudes have been defined by

$$\begin{aligned}
\Pi_{\gamma\gamma} &= e_b^2\,\Pi_{QQ} \\
\Pi_{\gamma Z} &= \frac{e_b^2}{s_b c_b}\left(\Pi_{3Q} - s_b^2\Pi_{QQ}\right) \\
\Pi_{ZZ} &= \frac{e_b^2}{s_b^2 c_b^2}\left(\Pi_{33} - 2s_b^2\Pi_{3Q} + s_b^4\Pi_{QQ}\right) \\
\Pi_{WW} &= \frac{e_b^2}{s_b^2}\,\Pi_{\pm}
\end{aligned}$$

with $s_b^2 = e_b^2/g_b^2$ *and* $c_b^2 = 1 - s_b^2$. *Such reduced self-energy functions have been used before in Refs. [72, 73]. Notice that at this point the running parameters do not satisfy the appropriate physical boundary conditions. For example, for fixed bare parameters,*

$$e_\star^2(q^2) \overset{q^2\to 0}{\not\to} e^2 = 4\pi\alpha \quad .$$

In order to fulfill the physical renormalization conditions the bare parameters must be tuned *appropriately. The* $\star$*-scheme uses matching conditions for* α, G_μ *and* M_Z

$$\begin{aligned}
e^2 &= e_\star^2(0) = 4\pi\alpha^{exp} \\
G_\mu &= G_{\mu\star}(0) = G_\mu^{exp} \\
\rho &= \rho_\star(0) = \rho_{\nu N}^{exp} \\
M_Z^2 &= \left.\frac{e_\star^2}{s_\star^2 c_\star^2}\frac{1}{4\sqrt{2}G_{\mu\star}\rho_\star}\right|_{M_Z^2} = M_Z^{exp2}
\end{aligned}$$

With the definition

$$g^2 = g_\star^2(M_Z^2) = 2\sqrt{2}G_{\mu\star}\rho_\star M_Z^2\left\{1 + \sqrt{1 - \frac{e_\star^2}{\sqrt{2}G_{\mu\star}\rho_\star M_Z^2}}\right\}$$

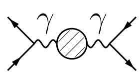
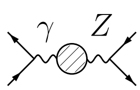
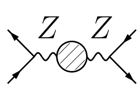
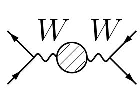
the running of the parameters is determined by

$$\begin{aligned}
e_\star^2(q^2) &= \frac{e^2}{1 - e^2\Delta_Q(q^2)} \\
g_\star^2(q^2) &= \frac{g^2}{1 - g^2\Delta_{3Q}(q^2)} \\
G_{\mu\star}(q^2) &= \frac{G_\mu}{1 - 4\sqrt{2}G_\mu\Delta_\pm(q^2)} \\
G_{\mu\star}(q^2)\rho_\star(q^2) &= \frac{G_\mu\rho}{1 - 4\sqrt{2}G_\mu\rho\Delta_3(q^2)}
\end{aligned}$$

where

$$
\begin{aligned}
\Delta_Q(q^2) &= Re\left\{\Pi'_{QQ}(0) - \Pi'_{QQ}(q^2)\right\} \\
\Delta_{3Q}(q^2) &= Re\left\{\Pi'_{3Q}(M_Z^2) - \Pi'_{3Q}(q^2)\right\} \\
\Delta_{\pm}(q^2) &= Re\left\{\Pi_{\pm}(q^2) - \Pi_{3Q}(q^2) - \Pi_{\pm}(0)\right\} \\
\Delta_3(q^2) &= Re\left\{\Pi_3(q^2) - \Pi_{3Q}(q^2) - \Pi_{33}(0)\right\} \quad .
\end{aligned}
$$

Evaluated at the vector boson mass scale, these running parameters have been used in Ref. [73], with the exception that G_μ, which does not run up to the vector boson mass scale, was kept fixed. After having imposed the matching conditions for given α, G_μ and M_Z, all quantities in the standard OS-scheme have equivalent representations in the $\star$-scheme. Let $\hat{\Pi}$ denote the renormalized self-energies expressed in terms of α, G_μ and M_Z. For four-fermion processes with light fermions, suppressing the external fermion current matrix elements, we obtain the following correspondence:

	$=$	$\frac{e^2}{1+\hat{\Pi}'_{\gamma\gamma}(s)}$	$=$	$e_*^2 = \bar{e}^2$
	$:$	$s_W^2 + s_W c_W \frac{\hat{\Pi}'_{\gamma Z}(s)}{1+\hat{\Pi}'_{\gamma\gamma}(s)}$	$=$	$s_*^2 = \bar{s}^2$
	$=$	$\frac{e^2}{s_W^2 c_W^2}\,\frac{1}{s-M_Z^2+\hat{\Pi}_Z(s)}$	$=$	$\frac{e_*^2}{s_*^2 c_*^2}\,\frac{1}{s-\frac{e_*^2}{s_*^2 c_*^2}\frac{1}{4\sqrt{2}G_{\mu *}\rho_*}+i\sqrt{s}\Gamma_{*Z}(s)}$
	$=$	$\frac{e^2}{s_W^2}\,\frac{1}{s-M_W^2+\hat{\Pi}_W(s)}$	$=$	$\frac{e_*^2}{s_*^2}\,\frac{1}{s-\frac{e_*^2}{s_*^2}\frac{1}{4\sqrt{2}G_{\mu *}}+i\sqrt{s}\Gamma_{*W}(s)}$

The weak mixing angles $\bar{s}^2$ and $s_W^2 = \sin^2\Theta_W$ are determined from α, G_μ and M_Z using

$$
\bar{s}^2\bar{c}^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2}\frac{1}{1-\Delta\bar{r}}\;, \quad s_W^2 c_W^2 \;=\; \frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2}\frac{1}{1-\Delta r}
$$

and the W-mass is given by $M_W^2 = M_Z^2\cos^2\Theta_W$. The radiative corrections Δr and $\Delta\bar{r}$ will be given in detail in Secs. IV and V, respectively. The renormalized VB self-energies have been defined here as suitable for the study of four-fermion processes. Since there are no external vector bosons involved, the VB wave-function renormalization factors drop out from the matrix-elements (remember the discussion after Eq. (144) at the beginning of this Section). However, in order to get finite (renormalized) self-energy functions a second subtraction (besides mass renormalization) in necessary. The one chosen here is obtained in a natural way by starting from the *bare matrix-elements and rewriting them in terms of the renormalized parameters by means of the shifts Eqs. (141-143). The parameter counterterms then may be combined with the bare self energies, where they show up in form of wave function factors. One obtains*

$$
\begin{aligned}
\hat{\Pi}_W(s) &= \Pi_W(s) - \Pi_W(M_W^2) - (s - M_W^2)\left((\frac{\delta\alpha}{\alpha})' - (\frac{\delta \sin^2\Theta_W}{\sin^2\Theta_W})' \right) \\
\hat{\Pi}_Z(s) &= \Pi_Z(s) - \Pi_Z(M_Z^2) - (s - M_Z^2)\left((\frac{\delta\alpha}{\alpha})' - \frac{c_W^2 - s_W^2}{c_W^2}(\frac{\delta \sin^2\Theta_W}{\sin^2\Theta_W})' \right) \\
\hat{\Pi}_{\gamma Z}(s) &= \Pi_{\gamma Z}(s) - \Pi_{\gamma Z}(0) + \ s \ \left(\frac{s_W}{c_W}(\frac{\delta \sin^2\Theta_W}{\sin^2\Theta_W})' \right) \\
\hat{\Pi}_\gamma(s) &= \Pi_\gamma(s) - \ s \ \left((\frac{\delta\alpha}{\alpha})' \right)
\end{aligned}
$$

with

$$
\begin{aligned}
(\frac{\delta\alpha}{\alpha})' &= \frac{\delta\alpha}{\alpha} - 2\frac{\sin\Theta_W}{\cos\Theta_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} \\
&= \Pi_\gamma'(0) \\
(\frac{\delta \sin^2\Theta_W}{\sin^2\Theta_W})' &= \frac{\delta \sin^2\Theta_W}{\sin^2\Theta_W} + 2\frac{\cos\Theta_W}{\sin\Theta_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} \\
&= \frac{\cos^2\Theta_W}{\sin^2\Theta_W}\ Re\ \left\{ \frac{\Pi_Z(M_Z^2)}{M_Z^2} - \frac{\Pi_W(M_W^2)}{M_W^2} + 2\frac{\sin\Theta_W}{\cos\Theta_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} \right\}
\end{aligned}
$$

for the "renormalized" self-energies. Since the splitting into self-energy and vertex+box contributions is not gauge-invariant and finite terms proportional to $\Pi_{\gamma Z}(0)$ *have been subtracted from the self-energies and added to the vertex+box contributions such that the two groups of contributions are separately finite. We mention that* $\Pi_{\gamma Z}(0)$ *vanishes in the unitary gauge as well as in the* $\overline{MS}$ *scheme for* $\mu = M_W$. *In the 't Hooft-Feynman gauge the vertex+box contribution is numerically small, though not negligible.*

In the $\star$*-scheme the physical widths of the Z is determined from the imaginary part of the propagator by (see Eq. (162))*

$$\Gamma_Z = \frac{\Gamma_{*Z}(M_Z^2) + \Delta\Gamma_Z}{1 + \kappa_{*Z}}$$

where $1 + \kappa_{*Z}$ *is determined by the residue of the Z-propagator*

$$s - \frac{e_*^2}{s_*^2 c_*^2}\frac{1}{4\sqrt{2}G_{\mu *}\rho_*} \ = \ (s - M_Z^2)\ (1 + \kappa_{*Z})$$

and $\Delta\Gamma_Z$ *stands for additional corrections (vertex, QED and possible QCD corrections). For the W width corresponding equations hold.*

The effective weak mixing angle parametrizing the NC-couplings at LEP energies has been given different names by different authors. Up to numerically small contributions $(s')^2$ [72], $sin^2\Theta_{eff}$ [73], $s_\star^2$ [62], $\sin^2\bar{\Theta} = \bar{s}^2$ [74] and $\sin^2\Theta_{\overline{MS}} = \sin^2\hat{\Theta}$ [58] are equivalent, particularly, for what concerns the fermion contributions, the top-quark mass and the Higgs mass dependences.

The special treatment of the self-energies is justified because they include the large non-QED corrections (fermion loops) and can be used to get improved Born approximations, which take into account the numerically most relevant non-photonic corrections. Of course, in order to get fully corrected four-fermion amplitudes form-factor and box-diagram corrections must be added. In general only the full set of corrections is gauge-invariant and finite. Any kind of splitting into effective couplings plus remainders is ambiguous and only a matter of bookkeeping and should not affect physical predictions within the given precision of the perturbative approximation.

The resummation of the reducible blocks involved in the above treatment of the propagator corrections means that some higher order effects have been taken into account while others (e.g. two-loop irreducible contributions) have been omitted. The question is whether this partial resummation of higher order terms leads to a better approximation to the unknown full answer. For the gauge couplings e and g one can show that the propagator resummation is equivalent to solving the renormalization group (RG) for the running gauge couplings, which is a systematic resummation of the leading logarithmic corrections. For the other two parameters G_μ and ρ the summation of the reducible diagrams only does not properly include terms of leading size! i.e. the two-loop irreducible diagrams give contributions of the same order as the square of the one loop result included in the bubble summation. This will be discussed in detail in the next section.

The relationship between the standard OS-scheme and the $\overline{MS}$ scheme is relatively simple. For example, for the weak mixing angle the OS version $\sin^2\Theta_W$ is related to its bare counterpart by (116,117)

$$\begin{aligned}\sin^2\Theta_b &= \left(1 + \tfrac{\delta\sin^2\Theta_W}{\sin^2\Theta_W}\right)\sin^2\Theta_W \\ \tfrac{\delta\sin^2\Theta_W}{\sin^2\Theta_W} &= \tfrac{\cos^2\Theta_W}{\sin^2\Theta_W}\; Re\left\{\tfrac{\Pi_Z(M_Z^2)}{M_Z^2} - \tfrac{\Pi_W(M_W^2)}{M_W^2}\right\}\end{aligned}$$

while the $\overline{MS}$ version is defined by

$$\sin^2\Theta_b = \left(1 + (\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W})_{\overline{MS}(\mu=M_W)}\right)\sin^2\hat{\Theta}$$

where $(\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W})_{\overline{MS}(\mu=M_W)}$ only picks the UV singular term from $\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W}$. The choice $\mu = M_W$ for the scale is made here because we are interested in an effective $\sin^2\Theta$ at LEP energies. The relation between the two mixing angles thus reads, expanded to linear order,

$$\sin^2\hat{\Theta} = \left(1 + \frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W} - (\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W})_{\overline{MS}(\mu=M_W)}\right)\ \sin^2\Theta_W$$

which is finite, depends however on the particular choice of μ. The finite quantity

$$\Delta\rho = \left\{ \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} + 2\frac{\sin\Theta_W}{\cos\Theta_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} \right\}$$

exhibiting the leading heavy particle effects is present in $\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W}$ only

$$\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W} = \frac{\cos^2\Theta_W}{\sin^2\Theta_W}\Delta\rho + \cdots$$

but absent in $(\frac{\delta\sin^2\Theta_W}{\sin^2\Theta_W})_{\overline{MS}(\mu=M_W)}$. Hence the main difference is exhibited in

$$\sin^2\hat{\Theta} = \left(1 + \frac{\cos^2\Theta_W}{\sin^2\Theta_W}\Delta\rho\right)\ \sin^2\Theta_W = \sin^2\Theta_W + \Delta\rho\cos^2\Theta_W$$

and one may calculate $\hat{s}^2 = \sin^2\hat{\Theta}$ from α, G_μ and M_Z by

$$\hat{s}^2\hat{c}^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2}\frac{1}{1-\Delta\hat{r}}\ ,$$

where $\Delta\hat{r}$ is obtained from Δr, discussed in the next section, by replacing the OS counterterms by their $\overline{MS}$ counterparts. Corresponding, considerations apply to other quantities.

A final remark should be made. The advantage of the effective weak mixing parameters, or other running parameters, is that they are flavor independent and take into account the universal large fermion loop effects. The disadvantage is that they are theoretical constructs and do not simply relate to physical quantities, like for example $\sin^2\Theta_W$, which is determined by the physical VB mass ratio and is completely model independent. It is also clear from the many slightly different definitions that a natural definition accepted by everybody does not exist. Hence a precise comparison of different definitions always needs a lot of explanation, and the members of the radiative corrections community can keep busy by debating for their preferred parametrization. After all a properly done physical prediction to a given accuracy should not depend on such bookkeeping questions.

Appendix to section III. RENORMALIZATION at two loops

In this Appendix we extend the renormalization to two-loop order, which is somewhat more involved mainly because of the mixing effects. This can be achieved by two subsequent transformations of the bare fields: given by Eqs.(170) and (171) which combine to

$$\begin{pmatrix} A_b \\ Z_b \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \sqrt{Z_\gamma} & -\Delta_Z \\ 0 & \sqrt{Z_Z} \end{pmatrix} \begin{pmatrix} A_r \\ Z_r \end{pmatrix}$$

where we wrote the first matrix in form of a global rotation with $s_\theta \doteq \sin\theta$ *and* $c_\theta \doteq \cos\theta$ *in terms of a rotation angle* θ*. The product matrix has the form*

$$\boldsymbol{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c_\theta\sqrt{Z_\gamma} & -c_\theta\Delta_Z - s_\theta\sqrt{Z_Z} \\ s_\theta\sqrt{Z_\gamma} & -s_\theta\Delta_Z + c_\theta\sqrt{Z_Z} \end{pmatrix} .$$

The renormalized propagator is given by

$$D_r = \boldsymbol{R}^{-1} D_b (\boldsymbol{R}^{-1})^T$$

and thus

$$D_r^{-1} = \boldsymbol{R}^T D_b^{-1} \boldsymbol{R} \ .$$

Explicitely the renormalized inverse propagator then has the form

$$\begin{aligned} D_r^{-1} &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} k^2 + \Pi_{\gamma\gamma}(k^2) & \Pi_{\gamma Z}(k^2) \\ \Pi_{\gamma Z}(k^2) & k^2 - M_Z^2 - \delta M_Z^2 + \Pi_{ZZ}(k^2) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 D_{\gamma\gamma}^{-1} + 2ac D_{\gamma Z}^{-1} + c^2 D_{ZZ}^{-1} & ab D_{\gamma\gamma}^{-1} + (ad+bc)\, D_{\gamma Z}^{-1} + cd D_{ZZ}^{-1} \\ ab D_{\gamma\gamma}^{-1} + (ad+bc)\, D_{\gamma Z}^{-1} + cd D_{ZZ}^{-1} & b^2 D_{\gamma\gamma}^{-1} + 2bd D_{\gamma Z}^{-1} + d^2 D_{ZZ}^{-1} \end{pmatrix} . \end{aligned}$$

The mixing counterterms b *and* c*, which are functions of* Δ_Z *and* s_θ*, are fixed by the condition that* $D_r^{-1}(k^2)$ *is diagonal on the respective mass-shells of the photon and the* Z*:*

$$ab\left[k^2 + \Pi_{\gamma\gamma}(k^2)\right] + (ad+bc)\,\Pi_{\gamma Z}(k^2) + cd\left[k^2 - M_Z^2 - \delta M_Z^2 + \Pi_{ZZ}(k^2)\right] = 0$$

at $k^2 = 0$ *and at the the* Z *pole* $k^2 = s_P = M_Z^2 - iM_Z\Gamma_Z$*. Thus*

$$ab\Pi_{\gamma\gamma}(0) + (ad+bc)\,\Pi_{\gamma Z}(0) + cd\left[-M_Z^2 - \delta M_Z^2 + \Pi_{ZZ}(0)\right] = 0$$

and

$$ab\left[s_P + \Pi_{\gamma\gamma}(s_P)\right] + (ad+bc)\,\Pi_{\gamma Z}(s_P) + cd\left[s_P - M_Z^2 - \delta M_Z^2 + \Pi_{ZZ}(s_P)\right] = 0 \ .$$

Since the renormalization parameters are required to be real (unitarity), we must require

$$ab\left[M_Z^2 + Re\ \Pi_{\gamma\gamma}(s_P)\right] + (ad+bc)\ Re\ \Pi_{\gamma Z}(s_P) + cd\left[Re\ \Pi_{ZZ}(s_P) - \delta M_Z^2\right] = 0$$

and as a consistency check

$$ab\left[Im\ \Pi_{\gamma\gamma}(s_P) - M_Z\Gamma_Z\right] + (ad+bc)\ Im\ \Pi_{\gamma Z}(s_P) + cd\left[Im\ \Pi_{ZZ}(s_P) - M_Z\Gamma_Z\right] = 0$$

should be satisfied, where

$$\begin{aligned} Re\ \Pi_{..}(s_P) &= Re\ \Pi_{..}(M_Z^2) + M_Z\Gamma_Z Im\ \Pi'_{..}(M_Z^2) + \cdots \\ Im\ \Pi_{..}(s_P) &= Im\ \Pi_{..}(M_Z^2) - M_Z\Gamma_Z Re\ \Pi'_{..}(M_Z^2) + \cdots \quad . \end{aligned}$$

Note that $a, d = 1 + O(\alpha)$, $b, c = O(\alpha)$ while the irreducible self energy functions Π and their derivatives Π' as well as the width Γ_Z and the mass counerterm δM_Z^2 are $O(\alpha)$ with the exception $\Pi_{\gamma\gamma}(0) = O(\alpha^2)$ (since at one loop $\Pi^{(1)}_{\gamma\gamma}(0) = 0$). We thus can resolve iterratively for b and c the conditions

$$+cdM_Z^2 = (ad+bc)\,\Pi_{\gamma Z}(0) + ab\Pi_{\gamma\gamma}(0) + cd\left[\Pi_{ZZ}(0) - \delta M_Z^2\right]$$

and

$$-abM_Z^2 = (ad+bc)\ Re\ \Pi_{\gamma Z}(s_P) + abRe\ \Pi_{\gamma\gamma}(s_P) + cd\left[Re\ \Pi_{ZZ}(s_P) - \delta M_Z^2\right] \quad .$$

To two loops:

$$\begin{aligned} +c/a &= \frac{\Pi_{\gamma Z}(0)}{M_Z^2} + c/a\left[\frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\delta M_Z^2}{M_Z^2}\right] + \frac{bc}{ad}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} + b/d\frac{\Pi_{\gamma\gamma}(0)}{M_Z^2} \\ &= \frac{\Pi_{\gamma Z}(0)}{M_Z^2} + \frac{\Pi_{\gamma Z}(0)}{M_Z^2}\left[\frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\delta M_Z^2}{M_Z^2}\right] + O(\alpha^3) \end{aligned} \qquad (179)$$

$$\begin{aligned} -b/d &= \frac{Re\ \Pi_{\gamma Z}(s_P)}{M_Z^2} + b/d\frac{Re\ \Pi_{\gamma\gamma}(s_P)}{M_Z^2} + c/a\left[\frac{Re\ \Pi_{ZZ}(s_P)}{M_Z^2} - \frac{\delta M_Z^2}{M_Z^2}\right] + \frac{bc}{ad}\frac{Re\ \Pi_{\gamma Z}(s_P)}{M_Z^2} \\ &= \frac{Re\ \Pi_{\gamma Z}(M_Z^2)}{M_Z^2} - \frac{Re\ \Pi_{\gamma Z}(M_Z^2)}{M_Z^2}\frac{Re\ \Pi_{\gamma\gamma}(M_Z^2)}{M_Z^2} + \frac{\Gamma_Z}{M_Z} Im\ \Pi'_{\gamma Z}(M_Z^2) + O(\alpha^3) \end{aligned} \qquad (180)$$

With these values for b and c the renormalized propagator is diagonal at $k^2 = 0$ and at $k^2 = s_P$ and we can look at the diagonal terms:

$$\begin{aligned} &a^2\left[k^2 + \Pi_{\gamma\gamma}(k^2)\right] + 2ac\Pi_{\gamma Z}(k^2) + c^2\left[k^2 - M_Z^2 - \delta M_Z^2 + \Pi_{ZZ}(k^2)\right] = \\ &\quad a^2\Pi_{\gamma\gamma}(0) + 2ac\Pi_{\gamma Z}(0) + c^2\left[-M_Z^2 + \Pi_{ZZ}(0) - \delta M_Z^2\right] \\ &\quad + \left\{a^2\left[1 + \Pi'_{\gamma\gamma}(0)\right] + 2ac\Pi'_{\gamma Z}(0) + c^2\left[1 + \Pi'_{ZZ}(0)\right]\right\}k^2 + \cdots \\ &\quad = k^2 + O(k^4) \end{aligned}$$

and

$$\begin{aligned}
b^2\left[k^2+\Pi_{\gamma\gamma}(k^2)\right]+2bd\Pi_{\gamma Z}(k^2)+d^2\left[k^2-M_Z^2-\delta M_Z^2+\Pi_{ZZ}(k^2)\right]=\\
b^2\left[s_P+\Pi_{\gamma\gamma}(s_P)\right]+2bd\Pi_{\gamma Z}(s_P)+d^2\left[s_P-M_Z^2-\delta M_Z^2+\Pi_{ZZ}(s_P)\right]\\
+\left\{b^2\left[1+\Pi'_{\gamma\gamma}(s_P)\right]+2bd\Pi'_{\gamma Z}(s_P)+d^2\left[1+\Pi'_{ZZ}(s_P)\right]\right\}(k^2-s_P)+\cdots\\
=k^2-s_P+O((k^2-s_P)^2)
\end{aligned}$$

where the coefficient of the leading term on the r.h.s. is required to be unity (essentially the on-sell wavefunction renormalization condition). We thus obtain

$$\begin{aligned}
a^2\Pi_{\gamma\gamma}(0)+2ac\Pi_{\gamma Z}(0)+c^2\left[-M_Z^2+\Pi_{ZZ}(0)-\delta M_Z^2\right] &= 0\\
a^2\left[1+\Pi'_{\gamma\gamma}(0)\right]+2ac\Pi'_{\gamma Z}(0)+c^2\left[1+\Pi'_{ZZ}(0)\right] &= 1
\end{aligned}$$

and

$$\begin{aligned}
b^2\left[s_P+\Pi_{\gamma\gamma}(s_P)\right]+2bd\Pi_{\gamma Z}(s_P)+d^2\left[s_P-M_Z^2-\delta M_Z^2+\Pi_{ZZ}(s_P)\right] &= 0\\
b^2\left[1+\Pi'_{\gamma\gamma}(s_P)\right]+2bd\Pi'_{\gamma Z}(s_P)+d^2\left[1+\Pi'_{ZZ}(s_P)\right] &= 1
\end{aligned}$$

Again, since the renormalization constants should be real, we have two conditions from the real parts and two self consistency conditions (should be satisfied automatically once the real ones are satisfied) from the imaginary parts:

$$\begin{aligned}
b^2\left[M_Z^2+Re\,\Pi_{\gamma\gamma}(s_P)\right]+2bdRe\,\Pi_{\gamma Z}(s_P)+d^2\left[Re\,\Pi_{ZZ}(s_P)-\delta M_Z^2\right] &= 0\\
b^2\left[Im\,\Pi_{\gamma\gamma}(s_P)-M_Z\Gamma_Z\right]+2bdIm\,\Pi_{\gamma Z}(s_P)+d^2\left[Im\,\Pi_{ZZ}(s_P)-M_Z\Gamma_Z\right] &= 0\\
b^2\left[1+Re\,\Pi'_{\gamma\gamma}(s_P)\right]+2bdRe\,\Pi'_{\gamma Z}(s_P)+d^2\left[1+Re\,\Pi'_{ZZ}(s_P)\right] &= 1\\
b^2Im\,\Pi'_{\gamma\gamma}(s_P)+2bdIm\,\Pi'_{\gamma Z}(s_P)+d^2Im\,\Pi'_{ZZ}(s_P) &= 0
\end{aligned}$$

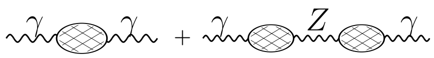
$$\begin{aligned}
a^2 &= \frac{1}{1+\Pi'_{\gamma\gamma}(0)+2c/a\Pi'_{\gamma Z}(0)+(c/a)^2(1+\Pi'_{ZZ}(0))}\\
&= \frac{1}{1+\Pi'_{\gamma\gamma}(0)+2\Pi'_{\gamma Z}(0)\frac{\Pi_{\gamma Z}(0)}{M_Z^2}+\left(\frac{\Pi_{\gamma Z}(0)}{M_Z^2}\right)^2+O(\alpha^3)}
\end{aligned}$$

$$\begin{aligned}
d^2 &= \frac{1}{1+Re\,\Pi'_{ZZ}(s_P)+2b/dRe\,\Pi'_{\gamma Z}(s_P)+(b/d)^2(1+Re\,\Pi'_{\gamma\gamma}(s_P))}\\
&= \frac{1}{1+Re\,\Pi'_{ZZ}(M_Z^2)+2Re\,\Pi'_{\gamma Z}(M_Z^2)\frac{Re\,\Pi_{\gamma Z}(M_Z^2)}{M_Z^2}+\left(\frac{Re\,\Pi_{\gamma Z}(M_Z^2)}{M_Z^2}\right)^2+M_Z\Gamma_Z Im\,\Pi''(M_Z^2)+O(\alpha^3)}
\end{aligned}$$

The photon mass on-shell conditions to two loops read

$$\Pi_{\gamma\gamma}(0)+\Pi_{\gamma Z}(0)\frac{\Pi_{\gamma Z}(0)}{M_Z^2}+O(\alpha^3)=0$$

which means that the full one photon irreducible (not the 1pi contribution itself) photon self-energy must vanish at zero momentum (masslessness of the photon):



$+\cdots \quad |_{k^2=0} \quad = \; 0 \quad .$

Finally, the mass counterterm of the Z is fixed by

$$\begin{aligned}\delta M_Z^2 &= Re\,\Pi_{ZZ}(s_P) + 2b/dRe\,\Pi_{\gamma Z}(s_P) + (b/d)^2 \left[M_Z^2 + Re\,\Pi_{\gamma\gamma}(s_P)\right] \\ &= Re\,\Pi_{ZZ}(M_Z^2) - Re\,\Pi_{\gamma Z}(M_Z^2)\frac{Re\,\Pi_{\gamma Z}(M_Z^2)}{M_Z^2} + M_Z\Gamma_Z Im\,\Pi'(M_Z^2) + O(\alpha^3)\end{aligned}$$

We are ready now to determine the parameters of the two renormalization matrices. Since we have used the short notation $a = c_\theta\sqrt{Z_\gamma}$, $b = -c_\theta\Delta_Z - s_\theta\sqrt{Z_Z}$, $c = s_\theta\sqrt{Z_\gamma}$ and $d = -s_\theta\Delta_Z + c_\theta\sqrt{Z_Z}$, we have to determine the mixing angle $s_\theta = \theta + O(\theta^3)$ and we adopt the notation $\theta = \Delta_0$ used before (Sec. 3).

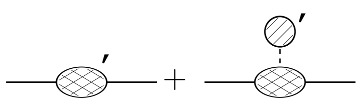
Renormalization of Dirac fields to two loops

The full bare inverse Dirac propagator for some fermion f is given by

$$S_{0f}^{\prime -1} = -i\ \{\not p - m_{0f} - \Sigma_f(p)\}$$

where

$$-i\Sigma_f(p) \equiv \qquad (181)$$



The covariant decomposition of the 1PI self-energy of a massive fermion $\Sigma_f(p)$ is given by

$$\Sigma(p) = \not p\ \Big(A(p^2, m_0, \cdots) + \gamma_5 C(p^2, m_0, \cdots)\Big) - m_0\ \Big(B(p^2, m_0, \cdots) + \gamma_5 D(p^2, m_0, \cdots)\Big)\ ,$$

where A, B, C and D are Lorentz scalar functions which depend on p^2 and on all parameters (indicated by the dots) of the SM. By hermiticity $\gamma_0\Sigma^+\gamma_0 = \Sigma$ we must have $D \equiv 0$[23] In order to find the pole of the propagator let us consider the inverse full propagator matrix

$$iS_0^{\prime -1}(p) = \not p - m_0 - \Sigma(p) = \not p\ (1 - A) - m_0\ (1 - B) - \not p\gamma_5\, C\ \ .$$

The pole is given by a matrix $\check{\not p}$ in spinor space

$$\not p - m_0 - \Sigma(p)|_{\not p = \check{\not p},\, p^2 = s_P} = 0\ ,$$

[23] This only holds when working with the chirality preserving anti-commuting γ_5. Note that $\gamma^\mu\gamma_5$ is not hermitian when γ_5 is not anti-commuting. In this case Ward-Takahashi identities must be restored by adding appropriate counter-terms before normal renormalization procedures can be applied.

which is only diagonal in the helicity representation of the γ-matrices. By s_P we again denote the location of the pole in the p^2-plane (see below). In terms of the invariant amplitudes the pole condition (182) reads

$$\check{\not p}\left(1-\check{A}-\check{C}\gamma_5\right)-m_0\;\left(1-\check{B}\right)=0 \tag{182}$$

where $\check{X}\equiv X(p^2,m_0^2,\cdots)|_{p^2=s_P}$ is an invariant amplitude evaluated on-shell at the pole position s_P[24]*. The pole solution may be worked out easily*[25]*. We find*

$$\check{\not p}=m_0\,\frac{1-\check{B}}{1-\check{A}-\check{C}\gamma_5}=m_0\;(a+b\gamma_5)$$

with

$$a=\frac{(1-\check{B})(1-\check{A})}{(1-\check{A})^2-\check{C}^2}\;,\quad b=\frac{(1-\check{B})\,\check{C}}{(1-\check{A})^2-\check{C}^2}\;\;.$$

The parity conjugate of $\check{\not p}$ we denote by

$$\hat{\not p}=m_0\;(a-b\gamma_5)$$

and we have

$$s_P=\check{\not p}\hat{\not p}=m_0^2\,(a^2-b^2)=m_0^2\,\frac{(1-\check{B})^2}{(1-\check{A})^2-\check{C}^2}$$

In fact the full Dirac propagator may be written as

$$-iS_0'(p)=\frac{\not p\;(1-A)+m_0\;(1-B)-\not p\gamma_5\,C}{p^2\;((1-A)^2-C^2)-m_0^2\;((1-B)^2)} \tag{183}$$

and the pole condition then reads

$$s_P-m_0^2-\Omega(s_P,m_0^2,\cdots)=0$$

where

$$\Omega(p^2,m_0^2,\cdots)\equiv p^2\;\left(2A-A^2+C^2\right)-m_0^2\;\left(2B-B^2\right)\;\;. \tag{184}$$

---

[24] Again s_P will be given as an iterative solution of the form (**??**). The invariant functions of course will be expanded accordingly: $\check{X}=X(s_P,m_0^2,\cdots)=X^{(1)}(m_0^2,m_0^2,\cdots)+X^{(2)}(m_0^2,m_0^2,\cdots)+(s_P-m_0^2)^{(1)}\,X^{(1)'}(m_0^2,m_0^2,\cdots)+\cdots$

[25] In vector-like theories like QED and QCD where $\check{\not p}$ is proportional to the unit matrix, one may define a complex "pole mass" by $\check{\not p}=\tilde{M}=M'-\frac{i}{2}\Gamma'$. However, for unstable particles ($\Gamma'\neq 0$) there is a mismatch with the usual definition as a pole in the complex p^2-plane: $\check{\not p}\check{\not p}=s_P=M^2-iM\Gamma=(M'-\frac{i}{2}\Gamma')^2$ such that one has to redefine $M=\sqrt{M'^2-\Gamma'^2/4}\;,\quad \Gamma=M'\Gamma'/M$. In the parity violating SM a pole mass definition via $\check{\not p}$ would be obscure because of the γ_5 terms.

One easily checks that the numerator matrix is non-singular at the zero of the denominator of the full Dirac propagator. Thus the solution takes the form (??) with Π *replaced by* Ω*. In terms of the invariant amplitudes* A, B *and* C *to two-loops the iterative solution reads*

$$\begin{aligned} s_P \;&=\; m_0^2 \left\{1 + 2\left(\bar{A}^{(1)} - \bar{B}^{(1)}\right) + 2\left(\bar{A}^{(2)} - \bar{B}^{(2)}\right) - (\bar{A}^{(1)})^2 + (\bar{B}^{(1)})^2 + (\bar{C}^{(1)})^2 \right. \\ &\qquad \left. + 4m_0^2\left(\bar{A}^{(1)} - \bar{B}^{(1)}\right)\left(\bar{A}^{(1)\prime} - \bar{B}^{(1)\prime}\right)\right\} \end{aligned} \tag{185}$$

where $\bar{X} \equiv X(p^2, m_0^2, \cdots)|_{p^2=m_0^2}$ *and* $\bar{X}' = dX(p^2, m_0^2, \cdots)/dp^2|_{p^2=m_0^2}$.
The renormalized propagator is obtained from the bare one by an appropriate wave function renormalization factor of the form $\sqrt{Z_2} = 1 + \alpha + \beta\gamma_5$[26]

$$S_{f\mathrm{r}} = \frac{1}{\sqrt{Z_2}}\, S_{f0}\, \frac{1}{\gamma^0\sqrt{Z_2}\gamma^0}\;, \quad or \quad \frac{1}{\not p - m_r - \Sigma_r} = \frac{1}{\sqrt{Z_2}(\not p - m_0 - \Sigma)\gamma^0\sqrt{Z_2}\gamma^0}\;.$$

In order to work out the on-shell wave function renormalization condition (LSZ asymptotic condition) we have to perform an expansion of the inverse bare propagator (182) about the "pole" $\check{\not p}$*. Again* X' *denotes the derivative with respect to* p^2 *of the invariant amplitude* X*. Using*

$$p^2 - s_P = (\not p - \check{\not p})(\not p + \hat{\not p}) \simeq (\not p - \check{\not p})(\check{\not p} + \hat{\not p}) = 2m_0\, a\, (\not p - \check{\not p}) \tag{186}$$

and the pole condition (182) we obtain

$$\not p - m_0 - \Sigma(p) = (\not p - \check{\not p})\left(1 - \check{F} - \check{G}\gamma_5\right)$$

where

$$1 - \check{F} - \check{G}\gamma_5 = 1 - \check{A} - \check{C}\gamma_5 - 2m_0^2\, a\,(a + b\gamma_5)\left(\check{A}' + \check{C}'\gamma_5\right) + 2m_0^2\, a\check{B}'\;.$$

Thus, the residue to the right of the bare propagator pole reads $1/(1 - \check{F} - \check{G}\gamma_5)$ *with*

$$F = A + 2m_0^2\, a\,[aA' + bC'] - 2m_0^2\, aB'\;,\;\; G = C + 2m_0^2\, a\,[aC' + bA']\;,$$

a, b *from (183) and the amplitudes to be evaluated at* $p^2 = s_P$*. Note that the residue matrix does not commute with the pole factor. Applying the wave function renormalization (186), the renormalized inverse propagator reads*

$$\sqrt{Z_2}\,(\not p - \check{\not p})\,(1 - \check{F} - \check{G}\gamma_5)\,\gamma^0\sqrt{Z_2}\gamma^0\;.$$

[26] In the unbroken phase of the SM the left-handed and the right-handed fermion fields get renormalized independently by c-number renormalization factors $\sqrt{Z_L}$ and $\sqrt{Z_R}$, respectively. In the broken phase, a Dirac field is renormalized by $\sqrt{Z_2} = \sqrt{Z_L}\,\Pi_- + \sqrt{Z_R}\,\Pi_+$ where $\Pi_\pm = \frac{1}{2}(1 \pm \gamma_5)$ are the chiral projectors. Hence, the wave function renormalization factor, becomes a matrix $\sqrt{Z_2} = 1 + \alpha + \beta\gamma_5$ and the bare fields are related to the renormalized one's by $\psi_0(x) = \sqrt{Z_2}\psi_r(x)$, which for the adjoint field reads $\bar{\psi}_0(x) = \bar{\psi}_r(x)\gamma^0\sqrt{Z_2}\gamma^0$.

We note that in contrast to the renormalization of boson fields or Dirac fields in parity conserving theories where wave function renormalization factors are just c-number functions, here we are confronted with non-commutative matrix renormalization of the from

$$\sqrt{Z_2} = 1 + \alpha + \beta\gamma_5 \;, \quad \gamma^0\sqrt{Z_2}\gamma^0 = 1 + \alpha - \beta\gamma_5 \;,$$

which leads to the effect that the matrix $\check{\not p}$*, solving the pole condition of the bare propagator, gets renormalized in a non-trivial manner by the wave function renormalization. The reason is that* $\not p$ *anti-commutes with* γ_5 *while* $\check{\not p}$ *is commuting.*
We may write

$$\sqrt{Z_2}\,(\not p - \check{\not p})\,(1 - \check{F} - \check{G}\gamma_5)\,\gamma^0\sqrt{Z_2}\gamma^0 = (\not p - \check{\not p}_r)\,\gamma^0\,(1 - \check{F} + \check{G}\gamma_5)\,Z_2\gamma^0$$

with

$$\check{\not p}_r = \check{\not p}\,\frac{\sqrt{Z_2}}{\gamma^0\sqrt{Z_2}\gamma^0}$$

and the LSZ condition reads

$$Z_2^{-1} = 1 - \check{F} + \check{G}\gamma_5 \;\; .$$

The "position of the pole" is shifted to

$$\check{\not p} \to \check{\not p}_r = \check{\not p}\,\frac{\sqrt{Z_2}}{\gamma^0\sqrt{Z_2}\gamma^0} = m_0\,(a + b\gamma_5)\,\frac{1+\alpha+\beta\gamma_5}{1+\alpha-\beta\gamma_5} \quad . \qquad (187)$$

However, we notice that $\check{\not p}\hat{\not p} = \check{\not p}_r\hat{\not p}_r$*, with* $\hat{\not p}_r = \check{\not p}_r\big|_{\gamma_5\to-\gamma_5}$*, remains invariant under this transformation such the* s_P*, the complex pole of the full Dirac propagator in the* p^2*-plane, indeed [and as expected] is the same for the bare and for the renormalized propagator.*
We finally expand Z_2^{-1} *up to two loops*

$$\begin{aligned} Z_2^{-1} \;\; = \;\; & 1 - \bar{A}^{(1)} - 2m_0^2\,(\bar{A}^{(1)'} - \bar{B}^{(1)'}) + \left(\bar{C}^{(1)} + 2m_0^2\,\bar{C}^{(1)'}\right)\gamma_5 \\ & -\bar{A}^{(2)} - 2m_0^2\,\left(\bar{A}^{(2)'} - \bar{B}^{(2)'} + 2(\bar{A}^{(1)} - \bar{B}^{(1)})\bar{A}^{(1)'} - (\bar{A}^{(1)} - \bar{B}^{(1)})\bar{B}^{(1)'} + \bar{C}^{(1)}\bar{C}^{(1)'}\right) \\ & + \left\{\bar{C}^{(2)} + 2m_0^2\,\left(\bar{C}^{(2)'} + \bar{A}^{(1)'}\bar{C}^{(1)} + 2(\bar{A}^{(1)} - \bar{B}^{(1)})\bar{C}^{(1)'}\right)\right\}\,\gamma_5 + \cdots \quad . \qquad (188) \end{aligned}$$

It is not recommended to try to work with the renormalization factors of the individual fields $\sqrt{Z_2}$ *since they are more complicated and in applications fermi field always appear in conjugate pairs. One can always achieve by simple algebra that* Z_2^{-1} *or* Z_2 *is the only object needed.*

IV. RENORMALIZATION OF MASS-COUPLING RELATIONS

The title of this Section could read as well: "Calculation of the muon decay constant G_μ in terms of α and the vector boson masses". By the relation Eq. (34) the parameters M_W, M_Z, α and G_μ are not independent. Here we calculate G_μ from α, M_W and M_Z (on-shell scheme):

$$G_\mu = \frac{\pi\alpha}{\sqrt{2}} \frac{1}{M_W^2 \sin^2\Theta_W} \frac{1}{1-\Delta r}$$

where $\Delta r \neq 0$ due to radiative corrections. Since the QED corrections have been already included in the definition of G_μ, we have to calculate the non-QED part of the μ decay transition amplitude for $k^2 \simeq 0$

$$-4\frac{G_\mu}{\sqrt{2}} J_\mu^{(\mu)} J^{(e)\mu} \quad .$$

Here, $J_\mu^{(\mu)} = \bar{u}_{\nu_\mu} [\gamma_\mu (1-\gamma_5)]\, u_\mu$ and $J_\mu^{(e)} = \bar{u}_e \ [\gamma_\mu (1-\gamma_5)]\, v_{\nu_e}$ denote the muon (μ) and the electron (e) charged current matrix elements, u and v are the external spinors. The different contributions are shown in Fig. 12.

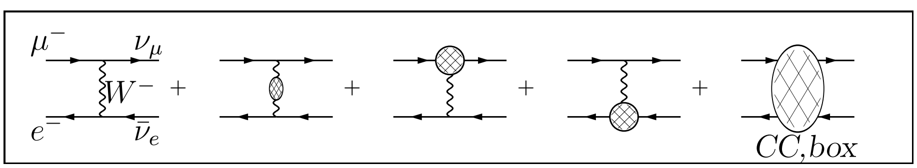


Figure 12a: Radiative corrections to μ-decay

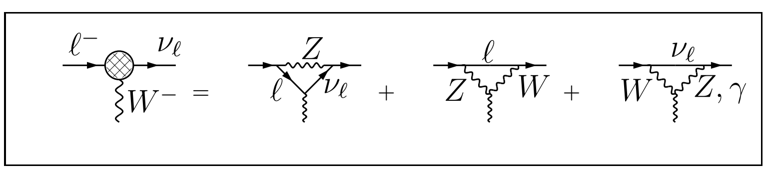


Figure 12b: CC vertex diagrams

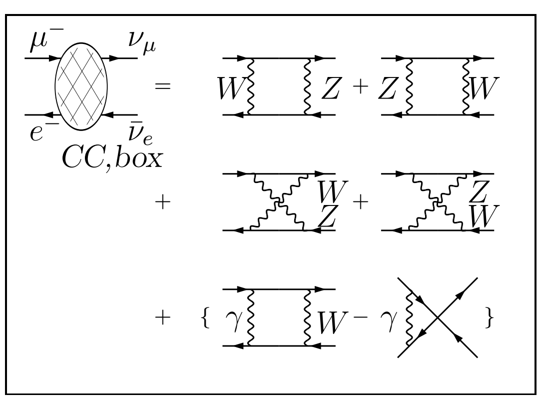


Figure 12c: CC box diagrams

At the tree level we read off

$$\frac{G_\mu}{\sqrt{2}} = \frac{e^2}{8M_W^2 \sin^2\Theta_W} = \frac{\pi\alpha}{M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right)} \,.$$

We may check the validity of this relation by using the experimental values for α, G_μ *and* $\sin^2\Theta_W = 0.231 \pm 0.006$, *obtained from deep inelastic* $\nu_\mu N$ *scattering, for a prediction of the vector boson masses which are given by*

$$M_W = \frac{A_0}{\sin\Theta_W}\ ,\ M_Z = \frac{M_W}{\cos\Theta_W}\ ;\ A_0 = \left(\frac{\pi\alpha}{\sqrt{2}G_\mu}\right)^{1/2} = 37.2802(3)\ \ GeV\,. \qquad (189)$$

Comparing the lowest order predictions $M_W = 77.57 \pm 1.01$ *and* $M_Z = 88.39 \pm 0.81$ *with the experimental values* $M_W^{exp} = 80.19 \pm 0.32$ *and* $M_Z^{exp} = 91.176 \pm 0.021$, *we see that the numbers are not in agreement with eachother. This disagreement illustrates the importance of radiative corrections .*

Including the one-loop radiative corrections we distinguish among 1) propagator (self-energy) corrections, 2) vertex corrections and 3) box contributions. We will neglect terms proportional to the light fermion masses, since for $m_f \ll M_W$, *they are numerically insignificant. This will lead to rather simple analytical expressions for the vertex and box contributions in the low energy limit.*
Using the bare parameter relations Eqs. (141-143) we get

$$\begin{aligned}
\frac{G_\mu}{\sqrt{2}} &= \frac{e_b^2}{8\sin^2\Theta_{Wb}M_{Wb}^2}\left\{1+\frac{\Pi_W(0)}{M_W^2}+\delta_{CC,vertex}+\delta_{CC,box}\right\} \\
&= \frac{e^2}{8\sin^2\Theta_W M_W^2}\left\{1+2\frac{\delta e}{e}-\frac{\cos^2\Theta_W}{\sin^2\Theta_W}\left(\frac{\delta M_Z^2}{M_Z^2}-\frac{\delta M_W^2}{M_W^2}\right)\right. \\
&\quad \left. -\frac{\delta M_W^2}{M_W^2}+\frac{\Pi_W(0)}{M_W^2}+\delta_{CC,vertex}+\delta_{CC,box}\right\} \\
&= \frac{\pi\alpha}{2\sin^2\Theta_W M_W^2}\left\{1+\Delta r\right\} \qquad (190)
\end{aligned}$$

The vertex and box diagrams are depicted in Figs. 12b and 12c.
The important quantity Δr *has been calculated first by Sirlin [31]. We read off the formal one-loop result from the foregoing expression. Collecting the self-energy terms in* Δr_{se} *we may write*

$$\begin{aligned}
\Delta r &= \Delta r(\alpha, M_W, M_Z, m_H, m_f) \\
&= \Delta r_{se} + \Delta r_{vertex+box}\ . \qquad (191)
\end{aligned}$$

and denoting $s_W^2 = \sin^2\Theta_W$ *and* $c_W^2 = \cos^2\Theta_W$ *we have*

$$\Delta r_{vertex+box} = \frac{\alpha}{4\pi s_W^2}\ (6+\frac{7-4s_W^2}{2s_W^2}\ln c_W^2\ ) \qquad (192)$$

which is the sum of the vertex, box and lepton wave-function contributions plus a γZ mixing term $2\frac{c_W}{s_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2}$, rendering the term ultraviolet finite (in the 't Hooft-Feynman gauge) [27] *. If we insert the expressions for the counter-terms and rewrite the result by splitting off the self-energies at $k^2 = 0$ as*

$$\Pi(k^2) \equiv \Pi(0) + k^2\,\Pi'(k^2)$$

the self-energy contributions read:

$$\begin{aligned}
\Delta r_{se} &= \Pi'_\gamma(0) - Re\,\Pi'_\gamma(M_Z^2) \qquad (193)\\
&-\frac{\cos^2\Theta_W}{\sin^2\Theta_W}\left\{\frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} + 2\frac{\sin\Theta_W}{\cos\Theta_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2}\right\}\\
&-Re\,\Pi'_W(M_W^2) + Re\,\Pi'_\gamma(M_Z^2) + \frac{\cos\Theta_W}{\sin\Theta_W}Re\,\Pi'_{\gamma Z}(M_Z^2)\\
&-\frac{\cos^2\Theta_W}{\sin^2\Theta_W}Re\,\left\{\Pi'_Z(M_Z^2) - \Pi'_W(M_W^2) + \frac{\sin\Theta_W}{\cos\Theta_W}\Pi'_{\gamma Z}(M_Z^2)\right\}
\end{aligned}$$

This is a representation of Δr_{se} in terms of the unrenormalized gauge boson self-energy functions. The form of this result exhibits the large and potentially large terms in Δr which we may write as

$$\Delta r = \Delta\alpha - \frac{\cos^2\Theta_W}{\sin^2\Theta_W}\Delta\rho + \Delta r_{rem} \qquad (194)$$

[27]Different from the NC processes (at one-loop order) , for the CC processes there is no natural separation into QED and "weak" part in the Standard Model. The QED corrections to μ-decay are not ultraviolet finite and they do not form a gauge invariant subset. This is in contrast also to the QED corrections for this process when modeled by an effective Fermi interaction, which can be transformed into a NC form via a Fierz transformation. The only trouble is caused by the photonic box diagram. After subtraction of the photonic four-fermion vertex correction, which has been included by convention in the QED correction factor of Eq. (36), an ultraviolet divergent and gauge dependent contribution R_w, as indicated in Fig. 12c, is left over which has to be included in Eq. (192).

We then have

$$\Delta r_{vertex+box} = 2\left(\frac{\delta e}{e}\right)_{vertex} + \delta_{CC,vertex} + \delta_{CC,box} + 2\frac{c_W}{s_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2}$$

where

$$\begin{aligned}
2\left(\frac{\delta e}{e}\right)_{vertex} &= -2A_1^{\gamma ee} + \frac{4s_W^2-1}{2s_Wc_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} = K\cdot 4s_W^2 L\\
\delta_{CC,vertex} &= \left(A_L^{W\mu\nu_\mu} + A_L^{We\nu_e}\right) = -K\cdot 2\left\{\left(2+\frac{s_W^2}{2}\right)L + \left(\frac{1}{2}-\frac{3}{s_W^2}\right)c_W^2\ln c_W^2 + \left(\frac{s_W^2}{4}-3\right)\right\}\\
\delta_{CC,box} &= A_{LCC}^{box} = -K\cdot\frac{1}{2s_W^2}\left(-3+6c_W^2+2c_W^4\right)\ln c_W^2 + R_w
\end{aligned}$$

where $K = \frac{\alpha}{4\pi s_W^2}$, $L = \ln\frac{M_W^2}{\mu^2}$ and $R_w = K\cdot\frac{s_W^2}{2}\,(2L+1)$. The amplitudes $A_.$ are normalized to the Born terms. We refer the reader to [68] for a more detailed discussion.

where

$$\begin{aligned}\Delta\alpha &= \Pi'_\gamma(0) - Re\,\Pi'_\gamma(M_Z^2) &(195)\\ \Delta\rho &= \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} + 2\frac{\sin\Theta_W}{\cos\Theta_W}\frac{\Pi_{\gamma Z}(0)}{M_Z^2}\ . &(196)\end{aligned}$$

are the large (due to fermion loop contributions) terms and Δr_{rem} *is the remainder. Though the latter term is numerically smaller by one order of magnitude it is an interesting term which includes contributions from gauge boson self-couplings and Higgs-vector boson interactions. We are now going to discuss the various terms in Eq. (194) in some detail.*

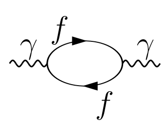
1. $\Delta\alpha$

$\Delta\alpha$ *is the photon vacuum polarization contribution which comes in through*

$$\begin{aligned}2\frac{\delta e}{e} &= \Pi'_\gamma(0) + \cdots\\ &= \Pi'_\gamma(0) - Re\,\Pi'_\gamma(M_Z^2) + \cdots + Re\,\Pi'_\gamma(M_Z^2)\\ &= \Delta\alpha + \cdots\end{aligned}$$

and is large due to the large change in scale going from zero momentum (Thomson limit) to the Z-mass scale $\mu = M_Z$*. Here, by zero momentum more precisely we mean the light fermion mass thresholds. The leading light fermion* ($m_f \ll M_W$) *contribution is given by*

$$\Delta\alpha = \sum_f$$



$$\begin{aligned}&= \frac{\alpha}{3\pi}\sum_f Q_f^2 N_{cf}(\ln\frac{M_Z^2}{m_f^2} - \frac{5}{3}\,)\\ &= \Delta\alpha_{leptons} + \Delta\alpha^{(5)}_{hadrons} + \Delta\alpha_{top}\ . &(197)\end{aligned}$$

Since the top quark is heavy we cannot use the light fermion approximation for it. A very heavy top-quark in fact gives no contribution since

$$\Delta\alpha_{top} \simeq -\frac{\alpha}{3\pi}\frac{4}{15}\frac{M_Z^2}{m_t^2} \to 0$$

when $m_t \gg M_Z$*.*

A serious problem is the low energy contributions of the five light quarks u,d,s,c and b which cannot be reliably calculated using perturbative QCD. Fortunately one can evaluate this hadronic term $\Delta\alpha^{(5)}_{hadrons}$ *from hadronic* e^+e^-*-annihilation data by using a dispersion relation. The relevant vacuum polarization amplitude satisfies the convergent dispersion relation*

$$Re\Pi'_\gamma(s) - \Pi'_\gamma(0) = \frac{s}{\pi}Re\int_{s_0}^{\infty} ds'\frac{Im\Pi'_\gamma(s')}{s'(s'-s-i\varepsilon)}$$

and using the optical theorem (unitarity) one has

$$Im\Pi'_\gamma(s) = \frac{s}{e^2}\sigma_{tot}(e^+e^- \to \gamma^* \to hadrons)(s)\ .$$

In terms of the cross-section ratio

$$R(s) = \frac{\sigma_{tot}(e^+e^- \to \gamma^* \to hadrons)}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}\ ,$$

where $\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$ *at tree level, we finally obtain*

$$\Delta\alpha^{(5)}_{hadrons}(M_Z^2) \;=\; -\frac{\alpha M_Z^2}{3\pi} Re \int_{4m_\pi^2}^{\infty} ds \frac{R(s)}{s(s - M_Z^2 - i\varepsilon)}\ . \qquad (198)$$

Using the experimental data for $R(s)$ *up to* $E_{cut} = 40$ *GeV (for larger energies* γZ *mixing would complicate the analysis) and perturbative QCD for the high energy tail we get (see Appendix to this section)*

$$\begin{aligned}\Delta\alpha^{(5)}_{hadrons}(s) \;=&\; 0.0282 \pm 0.0009 \\ &+0.002980 \cdot \{\ln(s/s_0) + 0.005696 \cdot (s_0/s - 1)\}\end{aligned} \qquad (199)$$

with $\sqrt{s_0} = 91.176$ *GeV [69]. In the range 50 GeV* $\leq \sqrt{s} \leq$ *200 GeV the above fit is "exact" as compared to the error. Alternatively, this result of the dispersion calculation can be reproduced by using perturbative QCD with the effective "quark masses"*

$$\begin{aligned} m_u &= 62 \text{ MeV}, \quad & m_d &= 83 \text{ MeV} \\ m_s &= 215 \text{ MeV}, \quad & m_c &= 1.5 \text{ GeV} \\ m_b &= 4.5 \text{ GeV} \end{aligned}$$

and a QCD correction factor $(1 + \alpha_{s,eff}/\pi)$ *with* $\alpha_{s,eff} = 0.133$ [28].
We should mention that a light fermion not only contributes to $\Delta\alpha$ *but also to* Δr_{rem}*:*

$$\Delta r^f_{rem} \simeq \frac{\alpha}{4\pi s_W^2}\,(1 - \frac{c_W^2}{s_W^2})\,\frac{N_{cf}}{6} K_{QCD} \ln c_W^2 .$$

This yields $\Delta r_{rem,leptons} \simeq 0.0015$ *and* $\Delta r^{(5)}_{rem,hadrons} \simeq 0.0040$.

Perturbative QCD corrections for light quarks (at some high energy scale) are taken care off by the factor $K_{QCD} = 1 + \delta_{QCD}$ *given by*

$$\delta_{QCD} = \frac{\alpha_s(M_Z^2)}{\pi} + 1.405\,\left(\frac{\alpha_s(M_Z^2)}{\pi}\right)^2 \qquad (200)$$

[28] *Warning*: Do not use these values for the quark masses for small space-like momenta (as needed in Bhabha scattering). These would give wrong results.

using [70]

$$\Lambda^{(5)}_{\overline{MS}} = 200^{+200}_{-100} \;\; MeV \text{ corresponding to } \alpha_s(M_Z^2) = 0.117 \pm 0.01 \; . \tag{201}$$

We first assume the top to be a "normal" not too heavy fermion and will discuss heavy top-quark effects in a second step. If there would not exist heavy unknown particles, Δr would be determined by the following typical contributions (m_t = 60 GeV, m_H = 100 GeV):

$$\begin{array}{lcccl} \Delta r_{leptons} & \simeq & 0.0315{+}0.0015 & = & 0.0330 \\ \Delta r_{hadrons} & \simeq & 0.0282{+}0.0040 & = & 0.0322 \pm 0.0009 \\ \Delta r_{top} & \simeq & 0.0025 & & (\text{depends on } m_t) \\ \Delta r_{bosons} & \simeq & 0.0033 & & (\text{depends on } m_H) \; . \end{array}$$

The term $\Delta r_{vertex+box}$ $\simeq$0.0064 is included in Δr_{bosons}. For the light fermions the individual contributions from $\Delta\alpha$ and Δr_{rem} are exhibited as a sum of two terms. The full analytic expression for a light top would be

$$\Delta r^{top} = \frac{\alpha}{3\pi}\frac{4}{3}\left(\ln\frac{M_Z^2}{m_t^2} - \frac{5}{3}\right) + \frac{\alpha}{16\pi s_W^2}\left(1 - \frac{c_W^2}{s_W^2}\right)\; 2\; \ln c_W^2 \tag{202}$$

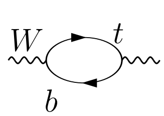
for $m_t \ll M_Z$.

Numerically the fermionic contributions dominate. The bosonic contributions are smaller by one order of magnitude but they are nevertheless non-negligible. The self-energy contributions are large and depend on unknown physics, like the top mass, the Higgs mass, on 4th family fermion masses etc. Next we consider what happens if the top is very heavy.

2. $\Delta\rho$

It has been observed first by Veltman [71] that fermion doublets with large mass splitting give large non-decoupling contributions to $\Delta\rho$ (large weak isospin breaking effects). We know that the top quark is unexpectedly heavy, $m_t \simeq$ 173 GeV, while $m_b \simeq$ 4.8 GeV is comparatively light.

The diagrams yielding leading doublet mass splitting effects are those which exhibit Wtb (CC) transitions and are quadratically divergent. The Ztt and Zbb (NC) vertices do not mix t and b and thus do not "feel" the mass splitting. In our case we are concerned with the finite part of the W self-energy diagram [29]



$$= \; -\frac{\alpha}{4\pi}\frac{1}{4s_W^2}N_c\frac{m_t^2}{M_W^2} + \cdots$$

[29] The UV singular terms are proportional to m_f^2 also for the Z self-energy and the latter must be taken into account to cancel the UV divergence of the W self-energy.

It yields a k^2-independent leading term which is (for dimensional reasons) quadratic in m_t. We thus obtain

$$\Delta\rho = \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} \simeq \frac{\alpha}{16\pi s_W^2} N_c \frac{m_t^2}{M_W^2} + \cdots \tag{203}$$

and this large contribution gets further enhanced *in Δr*

$$\Delta r|_{heavy} \quad = \quad -\frac{c_W^2}{s_M^2}\Delta\rho + \cdots$$

by an enhancement factor $\simeq$ 3.34 for s_W^2 = 0.23 .

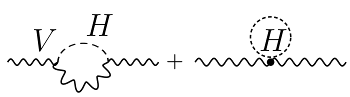
The remainder also contains logarithmic terms which are not negligible numerically. A heavy top would give the contribution

$$\Delta r^{top} = -\frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\left\{3\frac{c_W^2}{s_W^2}\frac{m_t^2}{M_W^2} + 2\left(\frac{c_W^2}{s_W^2} - \frac{1}{3}\right)\ln\frac{m_t^2}{M_W^2} + \frac{4}{3}\ln c_W^2 + \frac{c_W^2}{s_W^2} - \frac{7}{9}\right\} . \tag{204}$$

Let us mention finally that whereas $\Delta\alpha$ is unchanged by unknown physics, $\Delta\rho$ is sensitive to all kinds of $SU(2)_L$ multiplets which directly couple to the gauge bosons and exhibit large mass-splitting.

3. Higgs contribution

The Higgs contributions deserve our special attention. In the light fermion approximation only the vector-boson self-energy diagrams



$V = Z, W$

contribute. At one-loop order there is no quadratic Higgs mass dependence in $\Delta\rho$ and in Δr. The leading heavy Higgs contribution is logarithmic:

$$\begin{aligned}\Delta\rho^{Higgs} &\simeq -\frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\frac{s_W^2}{c_W^2}\left\{3\left(\ln\frac{m_H^2}{M_W^2} - \frac{5}{6}\right)\right\} \\ \Delta r^{Higgs} &\simeq \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\left\{\frac{11}{3}\left(\ln\frac{m_H^2}{M_W^2} - \frac{5}{6}\right)\right\} \quad (m_H \gg M_W).\end{aligned} \tag{205}$$

This is due to the accidental $SU(2)_R$ symmetry of the Higgs sector in the minimal Standard Model, which implies $\rho = 1$ at tree level (Veltman screening) [75]. More precisely, the theorem states that for vanishing fermion masses quadratic terms are absent. Furthermore, in $\Delta\rho$ also the logarithmic term disappears in the limit of vanishing $U(1)_Y$ coupling g'. The logarithmic term in the low energy observable $\Delta\rho$ is a consequence of the weak isospin breaking by hypercharge. On the other hand, in Δr, the coefficient of the logarithm does not depend on g'. Next we have to include the leading higher order effects.

4. Summation of leading higher order effects

Our one-loop calculation gave us the $O(\alpha)$ result

$$\sqrt{2}G_\mu = \frac{\pi\alpha}{\sin^2\Theta_W M_W^2}\left(1+\Delta r\right) .$$

Typically we get $\Delta r \simeq 0.07$ for M_Z=91 GeV, m_t=60 GeV and m_H=100 GeV. For the next order term we expected a contribution of the order $\Delta r^2 \simeq 0.005$. This would yield a shift in the prediction of the W mass (in terms of α, G_μ and M_Z) of $\delta M_W \simeq$ 190 MeV. Since M_W will be measured with an accuracy of $\delta M_W \simeq$ 70 MeV at LEP2, the $O(\alpha)$ result is insufficient for LEP experiments and we have to think about how to include the leading higher order terms.

a. Summation of leading logarithms.

The summation of leading logarithms is governed by the renormalization group. Since, in our case, the leading logs showed up in the QED vacuum polarization only, the leading log summation may be understood as the solution of the renormalization group equation for the $U(1)_{em}$ coupling constant (μ = renormalization scale)

$$\mu^2\frac{\partial}{\partial\mu^2}\alpha(\mu^2) = \frac{\beta(\alpha)}{2} = \frac{\alpha^2(\mu^2)}{3\pi}\sum_{m_f<\mu} N_{cf}Q_f^2$$

yielding the effective fine structure constant at scale M_Z

$$\alpha(M_Z) = \frac{\alpha}{1-\Delta\alpha} \tag{206}$$

where

$$\Delta r \simeq \Delta\alpha \simeq \frac{\alpha}{3\pi}\sum_{m_f<M_Z} N_{cf}Q_f^2 \ln\frac{M_Z^2}{m_f^2}$$

in this approximation. Thus Eq. (34) obtained from our one-loop result by the substitution

$$1+\Delta r \rightarrow \frac{1}{1-\Delta r}$$

represents the resummation of all powers of $(\alpha\ln\frac{M_Z^2}{m_f^2})$. It is important to notice that the leading log summation is scheme independent. This can be seen by writing, in leading log approximation,

$$\Delta\alpha^{-1} = \frac{1}{\alpha(0)} - \frac{1}{\alpha(\mu^2)} = \frac{1}{3\pi}\sum_{m_f<\mu} N_{cf}Q_f^2 \ln\frac{\mu^2}{m_f^2}\ ;\ \mu \le M_W$$

exhibiting that the r.h.s is independent of the electroweak couplings and hence of the parametrization used.

Including non-leading log terms we observe that the substitution

$$1+\Delta r = 1+\Delta\alpha+\Delta r_w \to \frac{1}{1-\Delta\alpha-\Delta r_w} = \frac{1}{1-\Delta r}$$

in fact only is correct if Δr_w is small. This would be the case only if the top would be light. As a next step we have to investigate what happens if $\Delta\rho$ is large.

b. Summation of large $\Delta\rho$ terms.

A careful analysis of the resummation of large $\Delta\rho$ terms [76] shows that Eq. (34) gets modified into

$$G_\mu = \frac{\pi\alpha}{\sqrt{2}M_W^2\sin^2\Theta_W}\left\{\frac{1}{1-\Delta\alpha}\frac{1}{1+\frac{\cos^2\Theta_W}{\sin^2\Theta_W}(\Delta\rho)_{irr}} + \Delta r_{rem}\right\}\ . \tag{207}$$

Here, $(\Delta\rho)_{irr}$ represents the leading irreducible contribution to the ρ parameter defined from the ratio of neutral current to charged current amplitudes at low energy, calculated in Ref. [77], i.e.

$$\frac{G_{NC}(0)}{G_{CC}(0)} = \rho = 1+(\Delta\rho)_{irr}+(\Delta\rho)_{irr}^2+\cdots = \frac{1}{1-(\Delta\rho)_{irr}}. \tag{208}$$

It is important to note that, in contrast to $\Delta\alpha$, which is not significantly modified by the inclusion of two loop irreducible contributions,

$$\Delta\alpha^{(1)}_{leptons} \to (1+\frac{3\alpha}{4\pi})\Delta\alpha^{(1)}_{leptons}$$

where $\Delta\alpha^{(1)}_{leptons}$ is the one-loop lepton contribution to $\Delta\alpha$, ρ as defined in Eq. (210), can sizably differ from the one loop result. In fact as shown in Ref. [76], by including the two loop irreducible terms calculated in Ref. [77], one finds

$$(\Delta\rho)_{irr} = N_{cf}x_f[1-(2\pi^2-19)x_f+\cdots]\ ,\ \ x_f = \frac{\Delta m_f^2}{8\pi^2}\frac{G_\mu}{\sqrt{2}}\ . \tag{209}$$

This means that low energy physics, is not sensitive to the bare *mass splitting (Δm_f^2), but rather to the effective quantity*

$$(\Delta m_f^2)_{eff} = \Delta m_f^2\left\{1-(2\pi^2-19)\frac{\Delta m_f^2 G_\mu}{8\pi^2\sqrt{2}}\right\}.$$

The screening effects, due to the Yukawa coupling with the scalar sector, may become large for a large mass splitting. This phenomenon, if confirmed from a closer inspection of the higher order terms in the perturbative expansion, may have far reaching consequences (possible restoration of decoupling) for our understanding of the Standard Model .

If we take the result of the full one loop calculation and include correctly the $\Delta\alpha$ and $\Delta\rho$ effects, resummed to all orders, we arrive at the final expression

$$M_W^2 = \frac{\rho M_Z^2}{2}\,(1+\sqrt{1-\frac{4A_0^2}{\rho M_Z^2}\,(\frac{1}{1-\Delta\alpha}+\Delta r_{rem})}\;). \tag{210}$$

Nonleading one-loop self-energy effects can be included by using Eq. (210) together with the replacements [76] [78]:

$$\begin{aligned}\Delta\alpha &\rightarrow \Delta e = \Pi'_\gamma(0) - \Pi'_W(M_W^2) + \frac{c_W}{s_W}\,\Pi'_{\gamma Z}(M_Z^2)\\ \Delta\rho &\rightarrow \Delta\hat{\rho} = \frac{\Pi_Z(M_Z^2)}{M_Z^2} - \frac{\Pi_W(M_W^2)}{M_W^2} + \frac{s_W}{c_W}\frac{\Pi_{\gamma Z}(M_Z^2)+\Pi_{\gamma Z}(0)}{M_Z^2}\,,\end{aligned} \tag{211}$$

where Π_Z includes γZ mixing terms as given in Eq. (139). We have checked that the above substitution reproduces correctly all self-energy terms up to $O(\alpha^2)$. Such a resummation could make sense for the fermion contributions, which form a gauge invariant subset. However, since terms like the irreducible contribution proportional to $\frac{\alpha}{4\pi}\sqrt{2}G_\mu m_t^2\ln(m_t^2/M_Z^2)$ are unknown, non leading terms and the vertex and box corrections, (contributing to Eq. (34)) should be added perturbatively i.e. included in Δr_{rem}.

Remarks on the resummation of "large" effects:
As we have seen large contributions often may be resummed which may lead to an improvement of the approximation and to a reduction of the error which is due to missing higher order terms. Thus one may improve existing results by resummation without doing a true next order calculation. However, resummations not necessarily lead to a better approximations in case there are missing terms of the same size which may largely compensate the ones accounted for by the resummation. An example of a justified resummation (i.e., one which one can prove to lead to an improvement) is the leading logs of the vacuum polarization: the leading two-loop terms proportional to $(\frac{\alpha}{\pi}\ln\frac{s}{m_f^2})^2$ comes from the reducible contribution the iterated one-loop result. The 1pi two-loop diagram only contributes a subleading term $(\frac{\alpha}{\pi})^2\ln\frac{s}{m_f^2}$. The effect is large because of the large change in scale from $2m_f$ to $\sqrt{s}\gg 2m_f$. The reducible term wins because it is enhanced by a large log $\ln\frac{s}{m_f^2}$ while the reducible and the 1pi are of the same perturbative order and thus potentially of similar size the first is enhances provided a large scale change is involved. The whole structure of leading, subleading, etc. logs and their resummation is governed by the RG. In contrast the large $G_\mu m_t^2 \propto y_t^2$ (y_t the top Yukawa coupling: $m_t = v y_t/\sqrt{2}$) term in $\Delta\rho$ stems from the large weak-isospin splitting by the Yukawa couplings of the top/ bottom quark doublet which manifests itself in the W self-energy. A heuristic way to understand the origin of these terms is the gauge fixing condition

$$-\partial_\mu W^{\mu\pm} \pm i\xi_W M_W\phi^\pm = 0$$

Above have calculated directly the Z and W self-energies. In order to understand the large top-quark mass limit another more elegant approach may be used. In the limit M_W, $M_Z \ll m_t$ and m_H arbitrary which is of interest here S-matrix elements are dominated by the longitudinal vector boson degrees of freedom and according to the equivalence theorem *[63]: with m_t as a high energy scale, one is allowed to replace (up to a phase and up to $O(M/m_t)$ corrections) a longitudinally polarized vector boson by its corresponding unphysical scalar. An equivalent relationship is obtained in the limit of vanishing gauge couplings, g', $g \to 0$, from the Ward-Takahashi identities which derive from the remaining global symmetry.*
By virtue of these Ward-Takahashi identities for the ρ-parameter, we may replace Eq. 196 (the last term is zero as fermions do not contribute) by

$$\Delta\rho \simeq \Pi'_{\varphi^\pm}(0) - \Pi'_{\varphi}(0) \ ,$$

where we have decomposed the Higgs ghost self energies as $\Pi_\varphi(q^2) = \Pi_\varphi(0) + q^2\Pi'_\varphi(q^2)$. This latter expression is simpler to calculate because the scalar vertices are simpler and the number of diagrams to be considered is reduced by roughly a factor of two. This representation also makes transparent where the Yukawa couplings come from, the scalars per definition couple via the Yukawa couplings to the fermions. The leading terms are those where each vertex carries a factor y_t. For fixed vacuum expectation value v heavy top means strong Yukawa coupling and the effects are large due to strong coupling. At one-loop we get terms proportional to y_t^2 at two-loop to y_t^4 etc. Reducible and 1pi two-loop diagrams give contributions of comparable size and actually, as discussed above, there is a large cancellation between the reducible (the ones one gets by resummation) and the 1pi ones which one only can get by an actual calculation. Thus this case is very different from the running coupling scenario.

5. Applications

Once Δr is given the W mass can be predicted by using the values of α, G_μ and M_Z from LEP1. According to Eqs. (34,35) we obtain

$$M_W^2 = \frac{M_Z^2}{2}\,(1+\sqrt{1-\frac{4A_0^2}{M_Z^2}\frac{1}{1-\Delta r}}\,) \tag{212}$$

and, equivalently,

$$\sin^2\Theta_W = \frac{1}{2}\,(1-\sqrt{1-\frac{4A_0^2}{M_Z^2}\frac{1}{1-\Delta r}}\,). \tag{213}$$

with A_0 given in Eq. (152). Explicit expressions for the various quantities which have been mentioned in this section can be found in Ref. [48,41], for example. Numerical results are given in Tab. 2. In Fig. 13 the m_t-dependence of Δr is shown for various Higgs masses. The W mass measurement is equivalent to a determination of

$$\Delta r = 1 - \frac{\pi\alpha}{\sqrt{2}G_\mu}\frac{1}{M_Z^2\frac{M_W^2}{M_Z^2}\,(1-\frac{M_W^2}{M_Z^2})}. \tag{214}$$

Table 2. Prediction of M_W and related parameters ($M_Z = 91.176$ GeV, $\alpha_s = 0.117$). Masses in GeV. $\sin^2\Theta_e$, $\sin^2\Theta_b$ and $\sin^2\bar{\Theta}$ will be considered below.

m_t	m_H	M_W	Δr	$\sin^2\Theta_W$	$\sin^2\Theta_e$	$\sin^2\Theta_b$	$\sin^2\bar{\Theta}$
90	*100*	*79.928*	*0.06032*	*0.2315*	*0.2334*	*0.2335*	*0.2326*
110	*100*	*80.037*	*0.05430*	*0.2294*	*0.2329*	*0.2333*	*0.2322*
130	*50*	*81.182*	*0.04607*	*0.2266*	*0.2321*	*0.2328*	*0.2313*
130	*100*	*80.151*	*0.04786*	*0.2272*	*0.2324*	*0.2330*	*0.2316*
130	*1000*	*80.002*	*0.05623*	*0.2301*	*0.2334*	*0.2341*	*0.2327*
150	*100*	*80.275*	*0.04068*	*0.2248*	*0.2318*	*0.2328*	*0.2310*
200	*100*	*80.642*	*0.01840*	*0.2177*	*0.2299*	*0.2321*	*0.2292*
230	*100*	*80.905*	*0.00133*	*0.2126*	*0.2286*	*0.2315*	*0.2278*

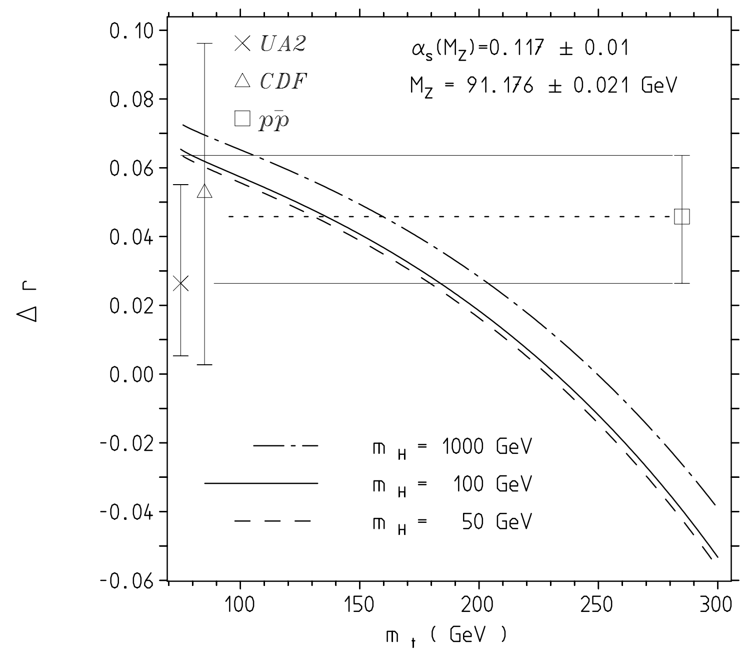


Figure 13: Δr as a function of the top mass for various m_H

Using the experimental values Eqs. (37,38) for M_Z and M_W, Δr is determined fairly well and since Δr is strongly dependent on the top-quark mass we can use the results *to find a* <u>direct</u> *constraint on the top-quark mass. Within one standard deviation we read off from Fig. 13 (the second uncertainty in m_t comes from the change of m_H)*

$$\Delta r = 0.046^{+0.018}_{-0.019} \quad \Leftrightarrow \quad m_t = 136^{+47+21}_{-57-5} \quad GeV \tag{215}$$

assuming $m_H \leq 1$ TeV. We notice that the direct lower limit $m_t > 89$ GeV is stronger than the indirect one obtained here.

In future one expects to be able to achieve a precision of $\delta M_W = 70$ MeV at LEP2. An accuracy $\delta M_W = 100$ MeV possibly may be achieved by combining the hadron collider results from CDF and D0 by the end of 1995 with an integrated luminosity of about $70 pb^{-1}$ [79]. This corresponds to an error in Δr of $\delta\Delta r = 0.0056$, and using $\frac{\delta m_t}{m_t} = -\frac{1}{2}\frac{\delta\Delta r}{\Delta r}$ this would determine m_t to an accuracy better than $\delta m_t = 10$ GeV. Of course we are waiting for the direct discovery of the top which is within reach in the next years at the Tevatron.

Appendix: Hadronic contributions to coupling shifts (update of Ref. [69]).

The Crystal Ball (CB) Collaboration has carefully reanalyzed their old e^+e^--annihilation data and now obtain $R(s)$ values substantially lower than the Mark I data [80] and in agreement with other experiments (LENA). The results now are in much better agreement with perturbative QCD. The change of the data is mainly due to an up to date treatment of the QED radiative corrections and τ subtraction. If we include the new results from CB and discard the Mark I data, which systematically lie 28% higher, in average, we obtain updated values for the hadronic contributions to the photon vacuum polarization. The results for $\Delta\alpha/e^2 = \Delta\pi_\gamma(M_Z)$ are collected in Tab. 3.

Table 3a: Contributions to $\Delta\pi_\gamma(M_Z) \times 10^3$

(final state)	(energy range)	(contribution) (stat) (syst)
ρ	(0.28, 1.20)	37.36 (0.15) (1.12)
ω	(0.42, 2.00)	3.74 (0.38) (0.11)
ϕ	(0.42, 2.00)	5.75 (0.26) (0.17)
J/ψ		11.08 (1.46) (1.66)
Υ		1.27 (0.04) (0.08)
hadrons	(0.84, 3.10)	38.59 (0.99) (7.72)
hadrons	(3.10, 3.60)	6.52 (0.34) (1.25)
hadrons	(3.60, 5.20)	19.04 (0.19) (1.27)
hadrons	(5.20, 9.46)	35.78 (0.52) (2.16)
hadrons	(9.46,40.00)	102.07 (1.36) (3.18)
perturbative	(40.0,∞)	46.53 (0.32) (0.64)

Table 3b: "Distribution" of errors

	$\Delta\pi_\gamma \times 10^3$	relat. error
Resonances:	59.20 (2.53)	4.3 %
($\rho\omega\phi$:	46.85 (1.24)	2.6 %)
Background:		
$E < M_{J/\psi}$	38.59 (7.78)	20.2 %
$M_{J/\psi} <$	6.52 (1.30)	19.9 %
E	19.04 (1.28)	6.7 %
$> M_\Upsilon$	35.78 (2.22)	6.2 %
$40GeV > E > M_\Upsilon$	102.07 (3.46)	3.4 %
$E <$ 40 GeV data	261.12 (9.34)	3.6 %
$E >$ 40 GeV QCD	46.53 (0.72)	1.5 %
total	307.65 (9.36)	3.0 %
($\star$)	(6.62)	(2.1 %)

The last line ($\star$) shows the error one would get if the experimental error on $R(s)$ would be reduced to 5% in the regions with larger errors.

$\Delta\alpha^{(5)}_{2\,had}$ may be determined using a partial separation of flavors, as explained in Ref. [69]. The following results are obtained:

Partial flavor separation of $\Delta\pi_\gamma(M_Z)$			
	uds	c	b
$E < M(J/\psi)$	85.44		
$M(J/\psi) < E < M(\Upsilon)$	43.94	28.48	
$M(\Upsilon) < E$	55.68	37.12	10.55

Using the approximate relation

$$\Delta\pi_{3\gamma} = \frac{1}{2}\Delta\pi_\gamma^{uds} + \frac{3}{8}\Delta\pi_\gamma^{c} + \frac{3}{4}\Delta\pi_\gamma^{b} \ ,$$

which derives from assuming $SU(3)_{flavor}$ for (u, d, s) and the OZI-rule for the heavy flavors c, b and t, the hadronic contributions to the shift of the $SU(2)$ coupling α_2 is the given by

$$\Delta\alpha^{(5)}_{2\,had} = g^2\ \Delta\pi_{3\gamma}(M_Z)\ ,\quad g^2 = e^2/\sin^2\Theta_W \ .$$

For $\sin^2\Theta_W = 0.23$ we obtain

$$\begin{aligned}\Delta\alpha^{(5)}_{had} &= 0.0282 \pm 0.0009(6)\\ \Delta\alpha^{(5)}_{2\,had} &= 0.0587 \pm 0.0018(12)\end{aligned}$$

where the error in brackets is the ($\star$) value mentioned above. Since the errors of $\Delta\alpha^{(5)}_{had}$ and $\Delta\alpha^{(5)}_{2\,had}$ are correlated the error in the renormalization of the weak mixing *angle from neutrino scattering*

$$\sin^2 \bar{\Theta} = \left(\frac{1 - \Delta\alpha_2}{1 - \Delta\alpha} + \cdots \right) \sin^2 \Theta_{\nu_\mu N(e)}$$

remains quite small. We get

$$\delta \left(\frac{1 - \Delta\alpha_2}{1 - \Delta\alpha} \right) \simeq 0.0009$$

or

$$\delta \sin^2 \Theta_{\nu_\mu N(e)} \simeq 0.00021$$

which is negligibly small relative to the experimental error 0.006 shown in Tab. 1.

V. LEP/SLC PHYSICS

Radiative corrections play a crucial role in the interpretation of electroweak precision measurements. In this last section, we will concentrate on discussing radiative corrections for LEP1/SLC physics near the Z peak.

The basic processes investigated at LEP1/SLC are fermion pair production $e^+e^- \to f\bar{f}(f \neq e)$ *and Bhabha scattering* $e^+e^- \to e^+e^-$. *At LEP2 W-pair production* $e^+e^- \to W^+W^-$ *will be the main process.*

The large cross-section at the Z-peak, $\sigma^{f\bar{f}}_{peak} \simeq 1.45\,(1.95)$ *nb for* $f = e, \mu, \tau$ *and* $30.08\,(40.65)$ *nb for hadrons, (in brackets, the value without QED corrections) gives easily a production of 1 million Z's per year at LEP1. The cross-section is enhanced relative to the pure QED process by a factor* $(M_Z/\Gamma_Z)^2 \simeq 10^3$ *or about 150 for leptons and 750 for hadrons.*

For precision physics the most important aims are

- *the detailed investigation of* $e^+e^- \to f\bar{f}$ *around the Z resonance which should allow to observe small calculable deviations of the partial and total cross-sections* $\sigma_f = \sigma(e^+e^- \to f\bar{f})$ *and* $\sigma_{tot} = \sum_f \sigma_f$ *and the partial and total widths* $\Gamma_f = \Gamma(Z \to f\bar{f})$ *and* $\Gamma_Z = \sum_f \Gamma_f$ *from their lowest order predictions*

$$\Gamma_{Zf\bar{f}} = \frac{\sqrt{2}G_\mu M_Z^3}{12\pi}(v_f^2 + a_f^2)N_{cf}\ ;\ \ \sigma^{f\bar{f}}_{peak} \ \simeq \ \frac{12\pi}{M_Z^2}\frac{\Gamma_e\Gamma_f}{\Gamma_Z^2} \tag{216}$$

  where $v_f = T_{3f} - 2Q_f \sin^2\Theta_W$ *and* $a_f = T_{3f}$ *are, respectively, the vector and axial-vector neutral current (NC) couplings for fermions with flavor f.* N_{cf} *is the color factor which is 1 for leptons and 3 for quarks.*

- *Additional information will be obtained from the on-resonance asymmetries, the forward-backward asymmetries* $A^{f\bar{f}}_{FB}$ *and the* τ *polarization-asymmetry* A^τ_{pol}. *If longitudinally polarized beams would be realized, the measurement of the left-right asymmetry* A_{LR} *and the polarized forward-backward asymmetries* $A^{f\bar{f}}_{FB,pol}$ *would allow to substantially improve the results. All the asymmetries are functions of the specific ratios*

$$A_f = \frac{2v_f a_f}{v_f^2 + a_f^2} \tag{217}$$

  of the NC couplings, and thus provide accurate determinations of the weak mixing angle $\sin^2\Theta_W$. *At the tree level the on-resonance asymmetries are given by*

$$A^{f\bar{f}}_{FB} = \frac{3}{4}A_e A_f, \quad A_{LR} = A^\tau_{pol} = A_e, \quad A^{f\bar{f}}_{FB,pol} = \frac{3}{4}A_f\ . \tag{218}$$

The "weak" (non-QED) radiative corrections reveal the asymmetries to be very interesting quantities, mainly because the different asymmetries exhibit different sensitivities to various interesting effects. The measurement of many independent quantities, *which depend in their own way on unknown physics, is important in order to be able to disentangle the origin of possible deviations from lowest order predictions.*

Since higher order predictions depend on the unknown mass of the Higgs boson, the remnant of the spontaneous symmetry breaking, and the mass of the unknown top quark, the missing member of the 3rd fermion family and other possible unknown physics, as a first step, data mainly constrain the unknown parameters of the SM. At the same time bounds on possible extensions of the SM gradually improve.

While the higher order predictions of physical quantities depend substantially on the unknown top mass the dependence on the unknown Higgs mass is much weaker. The first important goal thus is to restrict the range for the top mass.

1. Effective Couplings at the Z Resonance

Radiative corrections for the NC process $e^+e^- \to f\bar{f}$ *have been calculated by may groups [81]. The diagrams for the "weak" (=non-photonic) one-loop corrections are depicted in the Fig. 14. Diagrams involving ghost particles are not shown.*

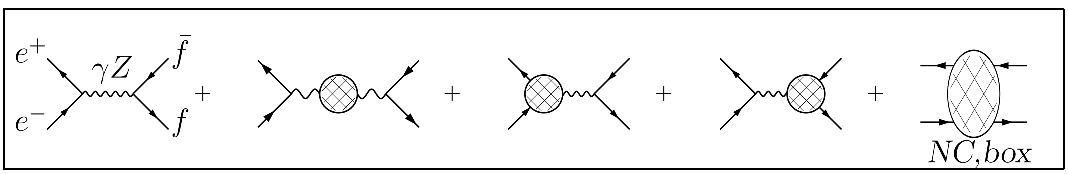


Figure 14a: Radiative corrections to $e^+e^- \to f\bar{f}$

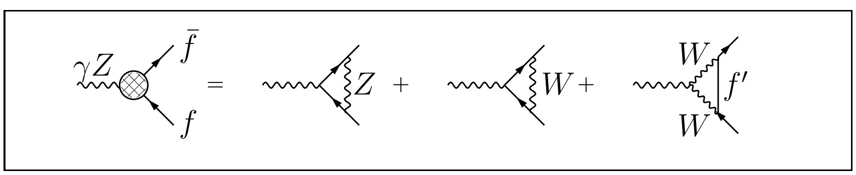


Figure 14b: NC vertex diagrams

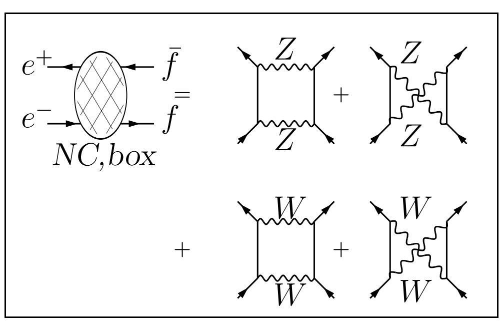
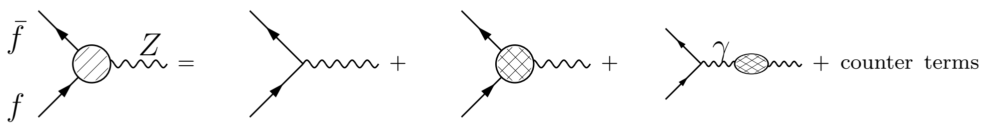


Figure 14c: NC box diagrams

Here we discuss the non-photonic corrections for the observables Eqs. (179,181), measured in resonant production and decay of Z's in $e^+e^- \to Z \to f\bar{f}$*. Because of the factorization of the "weak" corrections at the resonance, we restrict ourselves to con-*

sider the Z vertex corrections



They can be cast into an overall renormalization of the $Zf\bar{f}$ vertex

$$(\sqrt{2}G_\mu)^{1/2} M_Z \gamma^\mu (-2Q_f \sin^2\Theta_W + (1-\gamma_5)T_{3f})$$

by $\rho_f^{1/2}$ and a renormalization of $\sin^2\Theta_W$ in the NC vector-coupling [83]:

$$G_\mu \to \rho_f G_\mu \quad , \qquad \sin^2\Theta_W \to \kappa_f \sin^2\Theta_W \quad , \tag{219}$$

where $\rho_f = 1 + \Delta\rho_{se} + \Delta\rho_{f,vertex}$ and $\kappa_f = 1 + \Delta\kappa_{se} + \Delta\kappa_{f,vertex}$. In terms of the corrections δv_f and δa_f of the vector and axial-vector couplings we have

$$\Delta\rho = 2\frac{\delta a_f}{a_f}, \quad \Delta\kappa = \frac{a_f \delta v_f - v_f \delta a_f}{-2Q_f a_f \sin^2\Theta_W} \ .$$

Using the counter terms defined in Eqs. (115-117) and (123,147) we find

$$\begin{aligned}
\delta v_f &= A_v^{Zff} + \frac{v_f}{2}\left(\delta Z_Z + \frac{\delta M_Z^2}{M_Z^2} + \frac{\delta G_\mu}{G_\mu}\right) - 2Q_f \sin^2\Theta_W \left(\frac{\delta \sin^2\Theta_W}{\sin^2\Theta_W} + \frac{c_W}{s_W}\frac{\Pi_{\gamma Z}(M_Z^2)}{M_Z^2}\right) \\
\delta a_f &= A_a^{Zff} + \frac{a_f}{2}\left(\delta Z_Z + \frac{\delta M_Z^2}{M_Z^2} + \frac{\delta G_\mu}{G_\mu}\right)
\end{aligned}$$

where the lepton wave function terms (124) have been added to the bare vertex corrections $A_v \to A_v + vz_v - az_a$, $A_a \to A_a + az_v - vz_a$. Inserting the explicit expressions for the counter terms we may write $\Delta\rho$ and $\Delta\kappa$ in terms of the bare self-energies plus vertex corrections. The potentially large self-energy contributions (se) are universal. The analogues of Eq. (157) for $\Delta\rho$ and $\Delta\kappa$ read

$$\begin{aligned}
\Delta\rho_{se} &= \Delta\bar{\rho} = \Delta\rho + \Delta\rho_{se,rem} \qquad (220)\\
\Delta\kappa_{se} &= \Delta\bar{\kappa} = \frac{c_W^2}{s_W^2}\Delta\rho + \Delta\kappa_{se,rem}
\end{aligned}$$

with $\Delta\rho$ defined in Eq. (158). The self-energy terms are given by

$$\begin{aligned}
\Delta\rho_{se,rem} &= \Delta_Z = \frac{\Pi_Z(M_Z^2)}{M_Z^2} - \frac{\Pi_Z(0)}{M_Z^2} - \left(\frac{d\Pi_Z}{dq^2}\right)(M_Z^2) \\
\Delta\kappa_{se} &= \frac{c_W^2}{s_W^2}\Delta\hat{\rho} \qquad (221)
\end{aligned}$$

where $\Delta\hat{\rho}$ is given in Eq. (173). The vertex contributions are (if $f \neq b$) relatively small (but not negligible) and flavor dependent [30] *. We may define effective $\sin^2\Theta$'s by*

$$\sin^2\Theta_f = \kappa_f \sin^2\Theta_W = \tilde{\kappa}_f \tilde{s}^2 \tag{223}$$

where

$$\tilde{s}^2 = \sin^2\tilde{\Theta} = \frac{1}{2}\,(1 - \sqrt{1 - 4A_0^2/M_Z^2}\,) = 0.2122(1) \tag{224}$$

is the lowest order $\sin^2\Theta$ in terms of α, G_μ and M_Z. We have

$$\tilde{\kappa}_f = \kappa_f + \frac{\tilde{c}^2}{\tilde{c}^2 - \tilde{s}^2}\Delta r = \frac{\tilde{c}^2}{\tilde{c}^2 - \tilde{s}^2}\Delta r_f \tag{225}$$

and, generalizing Eq. (176),

$$\sqrt{2}G_\mu M_Z^2 \cos^2\Theta_f \sin^2\Theta_f = \frac{\pi\alpha}{(1-\Delta r_f)} \;\; ; \;\; \Delta r_f \;\; = \;\; \Delta r + \frac{\tilde{c}^2 - \tilde{s}^2}{\tilde{c}^2}\Delta\kappa_f \;\; . \tag{226}$$

Using Eqs. (157) and (183) we obtain

$$\Delta r_f = \Delta\alpha - \Delta\rho + \Delta r_{f,rem} \;\; . \tag{227}$$

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[30] The explicit expressions for the light fermion vertex corrections are [59, 82]

$$\begin{aligned}
\Delta\rho_{f,vertex} &= \frac{\sqrt{2}G_\mu M_Z^2}{16\pi^2}\left\{2(3v_f^2 + a_f^2)\Lambda_2(s, M_Z)\right. \\
&\left. -4c_W^2(1 - 2(1-|Q_f|)s_W^2)\Lambda_2(s, M_W) + 24c_W^4\Lambda_3(s, M_W)\right\} - \Delta r_{vertex+box} \\
\Delta\kappa_{f,vertex} &= \frac{\sqrt{2}G_\mu M_Z^2}{16\pi^2}\left\{-(1 - 4|Q_f|s_W^2)(1 - 2|Q_f|s_W^2)\Lambda_2(s, M_Z)\right. \\
&\left. +2c_W^2(1 - 2(1-|Q_f|)s_W^2)\Lambda_2(s, M_W) - 12c_W^4\Lambda_3(s, M_W)\right\}
\end{aligned} \tag{222}$$

where $\Delta r_{vertex+box}$ is given by Eq. (155) and comes in through the $\alpha \to G_\mu$ replacement used here. The functions $\Lambda_i(s, M)$ are given ($y = M^2/s$ with $M = M_Z$ or M_W, $s > 0$)

$$\begin{aligned}
\Lambda_2(s, M) &= -\frac{7}{2} - 2y - (2y+3)\ln(y) \\
&\quad +2(1+y)^2\left[\ln(y)\ln(\frac{1+y}{y}) - Sp(-\frac{1}{y})\right] \\
&\quad - i\pi\left[3 + 2y - 2(y+1)^2\ln\left(\frac{1+y}{y}\right)\right] \\
\Lambda_3(s, M) &= \frac{5}{6} - \frac{2y}{3} + \frac{2}{3}(2y+1)\sqrt{4y-1}\,\arctan\frac{1}{\sqrt{4y-1}} \\
&\quad -\frac{8}{3}y(y+2)\left(\arctan\frac{1}{\sqrt{4y-1}}\right)^2 .
\end{aligned}$$

where the formula for Λ_3 is valid for $s < 4M^2$ only. The Spence function is defined by $Sp(x) = -\int_0^1 \frac{dt}{t}\ln(1-xt)$. For f=b the expressions are more complicated and may be found in Ref. [84].

and we may calculate

$$\sin^2\Theta_f \quad = \quad \kappa_f \sin^2\Theta_W = \frac{1}{2}(1-\sqrt{1-\frac{4A_0^2}{M_Z^2}\frac{1}{1-\Delta r_f}}\,) \tag{228}$$

which compares to Eq. (175). Figs. 15 and 16 exhibit the different behavior as a function of m_t.

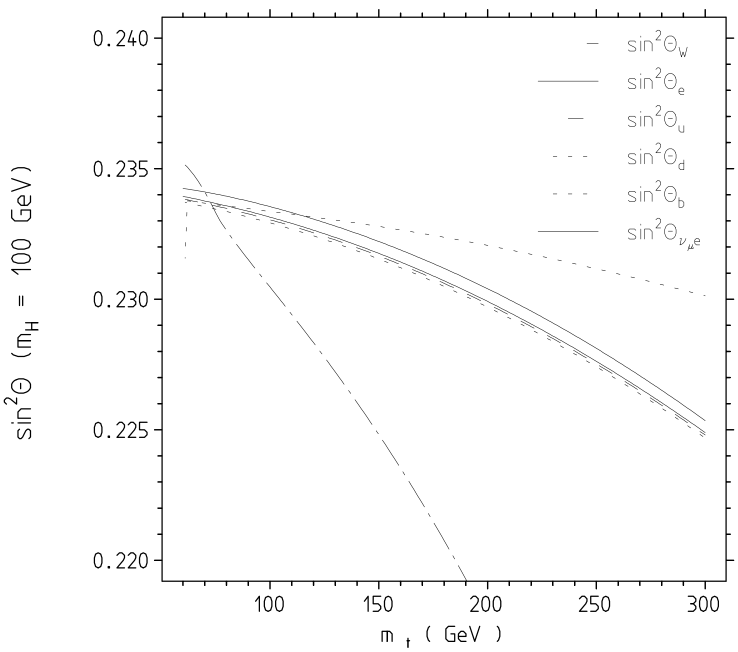


Figure 15: Flavor dependence of effective $\sin^2\Theta$'s.

Comparing (190) with (157), we notice that the LEP1 versions Δr_f and $\sin^2\Theta_f$ of Δr and $\sin^2\Theta_W$ (obtained from the W-mass measurement) are by a factor $c_W^2/s_W^2 \simeq 3.3$ less sensitive to heavy particle effects (see Fig. 15 below). But in both cases it is the same quantity , namely $\Delta\rho$, which is measured. Also, one finds that the sensitivity to a heavy Higgs is lower by a factor $(1+9s_W^2)/(11c_W^2) \simeq 2.8$. This does not mean that LEP1 experiments are less suitable to get important information on heavy physics, however. Thanks to the higher statistics of LEP1 experiments, LEP1 observables are measured with higher precision. Furthermore, the relative sensitivity to the Higgs is higher at LEP1, a welcome fact, since the Higgs remains "the big unknown" in the Standard Model.

From the measured effective $\sin^2\Theta_i$'s we may evaluate

$$\Delta r_i^{exp} = 1 - \frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2}\frac{1}{\sin^2\Theta_i^{exp}\cos^2\Theta_i^{exp}}. \tag{229}$$

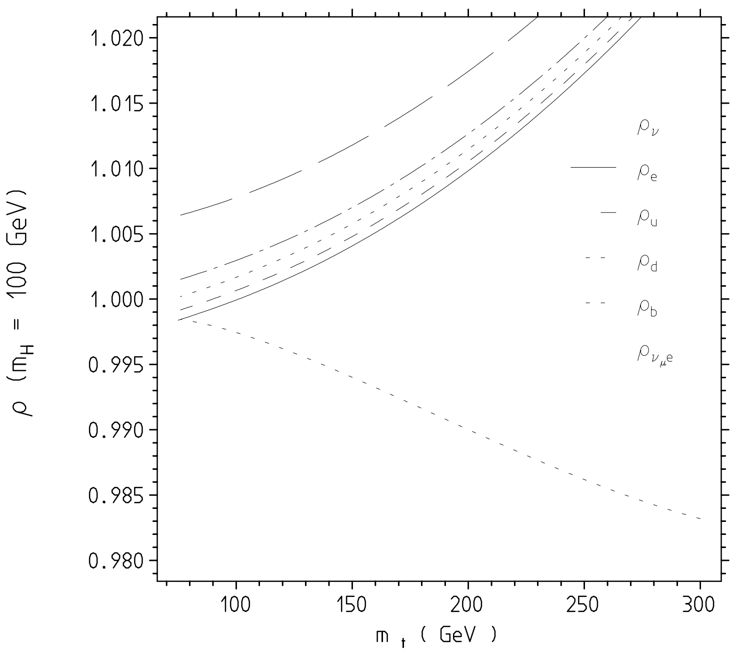


Figure 15: Flavor dependence of effective ρ's.

The values for $\sin^2\Theta_f^{exp}$ *can be obtained, using the tree level formulae, from the on-resonance asymmetries which have been corrected for QED effects, experimental cuts and detector efficiencies. For example, from the experimental left-right asymmetry we get*

$$\sin^2\Theta_e^{exp} = \sin^2\Theta_{LR} = \frac{A_{LR} - 1 + \sqrt{1 - A_{LR}^2}}{4A_{LR}}, \qquad (230)$$

which confronts with the theoretical prediction (191). The last equation may also be used to determine $\sin^2\Theta_e^{exp}$ *from the forward-backward asymmetry* $A_{FB}^{\mu^+\mu^-}$ *if we identify*

$$A_{LR} = \sqrt{\frac{4}{3}A_{FB}^{\mu^+\mu^-}}\ .$$

The weak mixing parameter most precisely measured at LEP is

$$\sin^2\Theta_e(M_Z^2) = 0.2302 \pm 0.0025 \quad \Leftrightarrow \quad m_t = 196^{+54+24}_{-76-16}\ \ GeV\ . \qquad (231)$$

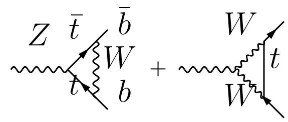
We see that the m_t*-bound is weaker than the one obtained from the hadron collider results. The smaller error cannot yet compensate for the weaker* m_t*-dependence of* $\sin^2\Theta_e$ *in comparison to* $\sin^2\Theta_W$*. While this measurement does not improve the upper limit, it does improve the lower limit to* $m_t > 104$ *GeV. LEP has dramatically improved the precision of the leptonic Z couplings*

Particle Data 90 [14]	LEP 90 [18]
g_V^e = -0.045 ± 0.022	-0.037 ± 0.005
g_A^e = -0.513 ± 0.025	-0.501 ± 0.003

Since $g_A^e = -\rho_e/2$ *and* $g_V^e/g_A^e = 1 - 4\,(1 + \Delta\tilde{\kappa}_e)\,\tilde{s}^2 = 1 - 4\sin^2\Theta_e$ *we obtain*

$$\Delta\rho_e = 0.002 \pm 0.006 \;\;,\;\; \Delta\tilde{\kappa}_e = 0.126 \pm 0.048 \;\;,\;\; \sin^2\Theta_e = 0.2315 \pm 0.0027 \;\;.$$

Due to virtual b-t transitions in the $Zb\bar{b}$ *vertex*



one finds large vertex corrections from a heavy top quark, given by [83, 84]

$$\begin{aligned} \Delta\kappa_{b,vertex} &= \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\left\{2\frac{m_t^2}{M_W^2} + \frac{1}{3}(16 + \frac{1}{c_W^2})\ln\frac{m_t^2}{M_W^2} + \cdots\right\} \\ \Delta\rho_{b,vertex} &= -2\Delta\kappa_{b,vertex}. \end{aligned} \tag{232}$$

These corrections lead to a much weaker top mass dependence of quantities (partial width, asymmetries) associated with $b\bar{b}$ *final states. Thus, in comparison with other channels the production of* $b\bar{b}$ *is particularly interesting since*

$$\begin{aligned} \sin^2\Theta_b - \sin^2\Theta_e &= \tilde{s}^2(\Delta\kappa_{b,vertex} - \Delta\kappa_{e,vertex}) \\ g_A^b/g_A^e &= 1 + (\Delta\rho_{b,vertex} - \Delta\rho_{e,vertex}) \end{aligned}$$

measure the large top contribution of the Zbb-vertex. They are completely independent of Higgs and other heavy particle effects and hence they are ideal heavy top meters. As an example, for $m_t = 200$ *GeV we obtain* $\sin^2\Theta_b - \sin^2\Theta_e$*= 0.0020 and* g_A^b/g_A^e*= 0.9821. For* $\sin^2\Theta_b$ *an experimental accuracy of 0.0009 is supposed to be achievable.*

We may define a flavor independent effective $\sin^2\Theta$ *by*

$$\sin^2\bar{\Theta} \;=\; (1 + \Delta\kappa_{se})\sin^2\Theta_W \tag{233}$$

and include the small vertex corrections in a second step

$$\sin^2\Theta_f \;=\; (1 + \Delta\kappa_{f,vertex})\sin^2\bar{\Theta} \tag{234}$$

up to negligible higher order terms.

The flavor independent auxiliary quantity $\sin^2\bar{\Theta}$ *is used in Ref. [74, 82] and is very similar to* s_*^2 *introduced in Ref. [62]. The "barred"(or "starred")-quantities are obtained by ignoring (small) corrections different from the vector boson self-energies.*

The leading heavy top and heavy Higgs dependence is given by

$$\begin{aligned} \Delta\bar{r}^{top} &= \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\left\{-3\frac{m_t^2}{M_W^2} + \frac{2}{3c_W^2}\ln\frac{m_t^2}{M_W^2} + \cdots\right\} \\ \Delta r_b^{top} &= \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\left\{-\frac{1+s_W^2}{c_W^2}\frac{m_t^2}{M_W^2} + \frac{16c_W^2(c_W^2 - s_W^2) - 1}{3c_W^4}\ln\frac{m_t^2}{M_W^2} + \cdots\right\} \end{aligned} \tag{235}$$

and

$$\Delta\bar{r}^{Higgs} \simeq \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\left\{\frac{1+9s_W^2}{3c_W^2}(\ln\frac{m_H^2}{M_W^2}-\frac{5}{6})\right\} \tag{236}$$

respectively. Except from extra top contributions in the case $f=b$, all heavy particle effects are universal i.e. $\Delta r_{f\neq b}^{top}=\Delta\bar{r}^{top}$ and $\Delta r_f^{Higgs}=\Delta\bar{r}^{Higgs}$.

What is the proper resummation of the large higher terms in case $\Delta\rho$ is large? Using Eqs. (183), (170) and (172) we have

$$\begin{aligned}\sin^2\Theta_f &= \left(1+\frac{\cos^2\Theta_W}{\sin^2\Theta_W}\Delta\rho+\cdots\right)\sin^2\Theta_W \\ &= 1-\frac{M_W^2}{\rho M_Z^2}+\cdots=\frac{1}{2}(1-\sqrt{1-\frac{4A_0^2}{\rho M_Z^2}(\frac{1}{1-\Delta\alpha}+\cdots)}\ )+\cdots\end{aligned}$$

where the ellipses stand for the small remainder terms. As a result we obtain

$$\frac{1}{1-\Delta r_f}=\frac{1}{1-\Delta\alpha}\left(1-(\Delta\rho)_{irr}\right)+\Delta r_{f,rem} \tag{237}$$

for the proper resummation of the large terms in Eqs. (189) and (191). This leads to the important relation

$$\sqrt{2}G_\mu\bar{\rho}M_Z^2\cos^2\Theta_f\sin^2\Theta_f=\pi\bar{\alpha}(1+\Delta r_{f,vertex}) \tag{238}$$

where

$$\bar{\rho}=\frac{1}{1-\Delta\bar{\rho}}\simeq\frac{1}{1-\Delta\rho}\quad,\quad\bar{\alpha}=\frac{\alpha}{1-\Delta e}\simeq\frac{\alpha}{1-\Delta\alpha} \tag{239}$$

with $\Delta\bar{\rho}$ and Δe given in Eqs. (183) and (173), respectively. Ignoring vertex corrections we obtain the universal relation

$$\sqrt{2}G_\mu\bar{\rho}M_Z^2\cos^2\bar{\Theta}\sin^2\bar{\Theta}=\pi\bar{\alpha}\ . \tag{240}$$

For completeness we mention that $\sin^2\Theta_e$ measured at the Z peak is the high energy analogue of $\sin^2\Theta_{\nu_\mu e}$ measured in low momentum transfer $\nu_\mu e-$scattering. In fact, the two versions of $\sin^2\Theta$ are related in a way which is practically independent of unknown effects (they differ by γZ mixing and ν_μ charge radius contributions only, which, by accident, largely cancel each other numerically). Formally we have

$$\sin^2\Theta_e=(1+\Delta_{se}+\Delta_{\nu_\mu e,vertex+box}+\Delta\kappa_{e,vertex})\sin^2\Theta_{\nu_\mu e} \tag{241}$$

where

$$\begin{aligned}\Delta_{se} &= \frac{\cos\Theta_W}{\sin\Theta_W}\left\{\Pi'_{\gamma Z}(M_Z^2)-\frac{d\Pi_{\gamma Z}}{dq^2}(0)\right\} \qquad (242)\\ &= \Delta\alpha-\Delta\alpha_2 \\ \Delta_{\nu_\mu e,vertex+box} &= \frac{\alpha}{4\pi s_W^2}\left\{\frac{2}{3}\left(\ln\frac{M_W^2}{m_\mu^2}+1\right)+\frac{24c_W^4-14c_W^2+9}{4c_W^2}\right\}\end{aligned}$$

and $\Delta\kappa_{e,vertex}$ is the same as in Eq. (182) (see (185)). The shift $\Delta\alpha_2$ in the $SU(2)_L$ coupling $\alpha_2 = \frac{g^2}{4\pi}$ is analogous to $\Delta\alpha$ Eq. (158)

$$\begin{aligned}\Delta\alpha_2 &= \Pi'_{3\gamma}(0) - \Pi'_{3\gamma}(M_Z^2) \\ &= \frac{\alpha_2}{12\pi}\Sigma_l|Q_l|(\ln\frac{M_Z^2}{m_l^2} - \frac{5}{3}) + \Delta\alpha^{(5)}_{2,hadrons}\end{aligned} \tag{243}$$

where the sum extends over the light leptons and [69] (see Appendix Sec. IV)

$$\begin{aligned}\Delta\alpha^{(5)}_{2,hadrons}(s) &= 0.0587 \pm 0.0018 \\ &+0.006184 \cdot \{\ln(s/s_0) + 0.005513 \cdot (s_0/s - 1)\}\end{aligned} \tag{244}$$

is the hadronic contribution of the 5 known light quarks u,d,s,c,b ($\sqrt{s_0} = 91.176$ GeV).

The proper summation of the higher order effects in this case reads

$$\sin^2\Theta_e = \left\{\frac{1-\Delta\alpha_2}{1-\Delta\alpha} + \Delta_{\nu_\mu e,vertex+box} + \Delta\kappa_{e,vertex}\right\}\sin^2\Theta_{\nu_\mu e} \tag{245}$$

The ratio $\sin^2\Theta_{\nu_\mu e}/\sin^2\Theta_e$ is shown in Fig. 6 as a function of m_t. The value of this ratio is close to 1.002. This relation provides a short of "model independent" constraint for the Standard Model . The CHARM II value for 0.240 ± 0.012 [85] is in agreement with the SM. The precise definition of the low energy ρ-parameter is (to linear order)

$$\rho_{\nu_\mu e} = \frac{G_{NC}(0)}{G_{CC}(0)} = 1 + \Delta\rho + \Delta\rho_{vertex+box} \tag{246}$$

with $\Delta\rho$ given in Eq. (158) and

$$\Delta\rho_{vertex+box} = \frac{\sqrt{2}G_\mu M_Z^2}{16\pi^2}\left\{24c_W^4 - 44c_W^2 + 15 - 2\frac{c_W^2}{s_W^2}(4c_W^2+3)\ln c_W^2\right\} .$$

Similar to the asymmetries, the corrected partial widths $\Gamma_{Zf\bar{f}} = \frac{\sqrt{2}G_\mu M_Z^3}{3\pi}(v_f^2 + a_f^2)N_{cf}K_{QCD}\,(1+\delta_{QED})$ and the peak cross-sections $\sigma^{f\bar{f}}_{peak} \simeq \frac{12\pi}{M_Z^2}\frac{\Gamma_e\Gamma_f}{\Gamma_Z^2}$ are given by the Born formulae using the effective parameters Eq. (182). The uncertainty in α_s implies an uncertainty of 12 MeV in $\Gamma_{Z,tot}$. The QED-correction including real photon emission is given by $\delta_{QED} = \frac{3\alpha}{4\pi}Q_f^2$. In Tab. 4 some values are given for the widths and peak cross-sections. Full QCD corrections are taken into account [86, 87]. In contrast to other authors we use a running $\overline{MS}$ top mass. QCD corrections for the heavy top are small in this case, i.e. the results are close to the results which do not include QCD corrections for the heavy top.

Table 4. Z widths and peak cross-sections for M_Z = 91.176 GeV and α_s = 0.117. Masses are given in GeV, widths in MeV and cross sections in nb.

m_t	m_H	Γ_Z	Γ_h	Γ_ℓ	Γ_{inv}	Γ_c	Γ_b	R_{had}	σ_μ^{peak}	σ_{had}^{peak}
90	100	2482	1733	83.4	499	296	378	20.787	1.9927	41.423
110	100	2485	1735	83.5	499	296	378	20.782	1.9937	41.432
130	50	2490	1739	83.7	500	297	378	20.780	1.9944	41.443
130	100	2489	1738	83.7	500	297	377	20.775	1.9949	41.444
130	1000	2481	1732	83.5	499	296	376	20.755	1.9971	41.449
150	100	2494	1741	83.9	501	298	377	20.767	1.9963	41.456
200	100	2508	1751	84.4	504	301	375	20.745	2.0002	41.494
230	100	2519	1759	84.9	506	303	375	20.731	2.0028	41.521

2. Results from LEP at the Z Resonance

The results from LEP based on 600,000 Z decays (presented at the Aspen Conference January 1991) are collected in Tab. 5.

The central values are given for $m_t = 136$ *GeV and* $m_H = 100$ *GeV. The uncertainties for the SM predictions include variations of the parameters within the one standard deviation bounds* 89 *GeV* $< m_t < 204$ *GeV, from the UA2 and CDF data, and* 50 *GeV* $< m_H < 1$ *TeV. More precisely, the allowed range for* m_t *depends on* m_H*. Since, in the range of interest, all quantities are monotonic functions of* m_H *and* m_t *we may inspect the extremal cases simply: For* $m_H = 50$ *GeV the* 1σ *range for* m_t *is (74,180) GeV or (89,180) GeV if we take into account the direct bound (26). For* $m_H = 1$ *TeV we get (104,204) GeV. The bounds given in Tab. 5 are then the maximum or minimum values from the two extremal cases. Taking an upper bound 1 TeV for the Higgs mass is of course a theoretical prejudice.*

The mass and the total width of the Z are determined from the line-shape. The separate analysis of the visible channels $e^+e^- \to hadrons$ *and* $e^+e^- \to \ell^+\ell^-$ *allows to determine* Γ_{had} *and* Γ_ℓ *(*$\ell = e, \mu, \tau$*), respectively. Using that the total Z-width is given by*

$$\Gamma_Z = \Gamma_{had} + 3\,\Gamma_\ell + \Gamma_{invisible} \;\; ; \;\; \Gamma_{invisible} = N_\nu (\Gamma_\nu)_{SM} \tag{247}$$

in terms of the hadronic, leptonic and neutrinic contributions, $\Gamma_{invisible}$ *is determined.* N_ν *is the effective number of SM neutrinos. The most important result established by the LEP experiments is that* $N_\nu = 2.95 \pm 0.05$ *and hence no additional light (*$m_\nu \lesssim 45 GeV$*) neutrino (sneutrino, Majoron etc.) exists [32]. This rules out the existence of further family replicas of the known type with (within experimental limits) massless neutrinos.*

Table 5. LEP results on the Z peak

	ALEPH	*DELPHI*	*L3*	*OPAL*	*LEP*	*SM*	$\sin^2\bar{\Theta}$
Z decays	*195,000*	*130,000*	*125,000*	*156,000*	*600,000*		
M_Z (GeV)	91.182 ±0.009 ±0.020	91.175 ±0.010 ±0.020	91.180 ±0.010 ±0.020	91.160 ±0.009 ±0.020	91.176 ±0.005 ±0.020		0.2315 $^{+.0018}_{-.0019}$
Γ_Z (MeV)	2488 ±17	2454 ±21	2500 ±17	2497 ±17	2485 ±10	2490 ±22	0.2322 $^{+.0017}_{-.0024}$
σ_{had}^{peak} (nb)	41.76 ±0.39	41.98 ±0.63	40.92 ±0.47	41.23 ±0.47	41.45 ±0.21	41.45 ±0.12	0.2313
Γ_{had} (MeV)	1756 ±15	1718 ±22	1739 ±19	1747 ±19	1744 ±10	1739 ±18	0.2314 ±.0022
Γ_ℓ (MeV)	83.6 ±0.7	83.4 ±1.0	83.3 ±0.8	83.4 ±0.7	83.4 ±0.4	83.7 ±0.5	0.2326 ±.0021
R_{had}	21.07 ±0.19	21.61 ±0.33	20.88 ±0.28	20.94 ±0.24	20.92 ±0.13	20.77 ±0.12	
Γ_{inv} (MeV)	487 ±14	486 ±21	511 ±18	499 ±17	496 ±9	500 ±3	
N_ν	2.90 ±.08	2.93 ±.13	3.08 ±.10	3.00 ±.10	2.95 ±.05	3	
$(v_e/a_e)^2$	0.0081 ±.0028	0.0028 ±.0056	0.0081 ±.0051	0.0024 ±.0028	0.0056 ±.0016	0.0051 ±.0013	0.2315 ±.0027
A_{FB}^b	0.141 ±.044		0.130 ±.043	0.080 ±?	0.117 ±.027	0.0962 $^{+.012}_{-.006}$	0.2241 ±.0077
$A_{FB}^{\mu^+\mu^-}$	0.0239 ±.0082	0.0084 ±.0168	0.0239 ±.0150	0.0072 ±.0084	0.0166 ±.0047	0.0151 ±.004	0.2313 ±.0027

Of particular interest is the observable $R_{had} = \Gamma_{had}/\Gamma_\ell$ *which is almost independent of* m_t*, due to an accidental cancellation of the* m_t*-dependence between the Zbb-vertex and the self-energies. A deviation from the SM would be a direct signal for non-standard physics. The experimental value* 20.92 ± 0.13 *is slightly higher than the SM prediction* 20.77 ± 0.12*. Also the hadronic peak cross-section* σ_{had}^{peak} *is weakly dependent on* m_t

only. The experimental value is in perfect agreement with the prediction. Before more stringent tests are possible one has to pin down further the allowed mass ranges for the top and the Higgs. We do not expect that the errors on M_Z and α_s can be substantially improved further.

Some major results obtained in the first year of LEP ($\sim$ 600 000 Z's) are shown together with theoretical predictions in Figs. 17 and 18. All Figures show the data together with the theoretical prediction as a function of the top mass for m_H= 50,100 and 1000 GeV. An uncertainty $\delta\alpha_s = \pm 0.01$ in the strong interaction coupling constant is shown as a inner error band whereas the outer error band shows the uncertainty in the prediction due to the experimental error $\delta M_Z = \pm 0.021$ in the Z-mass. The agreement between the experimental numbers and the theoretical predictions is impressive.

ACKNOWLEDGEMENTS

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Appendix A: Leading Higgs and top-quark contributions

The asymtotic contributions to various corrections factors are summarized in this Appendix. Except from the shifts in the $Z\bar{b}b$ effective parameters $\Delta\rho_b$ and $\Delta\kappa_b$ which recieve large contributions from the virtual heavy top–quark of the $Z\bar{b}b$–vertex, all Higgs and top corrections come in through gauge boson self–energy functions. The basic combinations are ($\Pi(k^2) \equiv \Pi(0) + k^2\, \Pi'(k^2)$, $\quad s = \sin\Theta_W$, $\quad c = \cos\Theta_W$)

$$
\begin{aligned}
\Delta\alpha &= \Pi'_\gamma(0) - Re\,\Pi'_\gamma(M_Z^2) \\
\Delta e &= \Pi'_\gamma(0) - \Pi'_W(M_W^2) + \frac{c}{s}\,\Pi'_{\gamma Z}(M_Z^2) \\
\Delta\rho &= \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} + 2\frac{s}{c}\frac{\Pi_{\gamma Z}(0)}{M_Z^2} \\
\Delta\hat{\rho} &= \frac{\Pi_Z(M_Z^2)}{M_Z^2} - \frac{\Pi_W(M_W^2)}{M_W^2} + \frac{s}{c}\frac{\Pi_{\gamma Z}(M_Z^2) + \Pi_{\gamma Z}(0)}{M_Z^2} \\
\Delta_V &= \frac{\Pi_V(M_V^2)}{M_V^2} - \frac{\Pi_V(0)}{M_V^2} - \left(\frac{d\Pi_V}{dq^2}\right)(M_V^2)\ .
\end{aligned}
$$

In terms of these basic combinations the "pseudo observables" of interest read:

$$
\Delta r = \Delta e - \Delta\bar{\kappa}\ \ , \quad \Delta\bar{r} = \Delta e - \Delta\hat{\rho}\ \ , \quad \Delta\bar{\kappa} = \frac{c^2}{s^2}\Delta\hat{\rho}\ \ , \quad \Delta\bar{\rho} = \Delta\rho + \Delta_Z\ .
$$

The $Z\bar{f}f$ vertex corrections are encoded by

$$
\begin{aligned}
\Delta r_{f\neq b} &= \Delta\bar{r}\ \ , \quad \Delta r_b = \Delta\bar{r} + (1 - \frac{s^2}{c^2})\,\Delta\kappa_{b,vertex} \\
\Delta\kappa_{f\neq b} &= \Delta\bar{\kappa}\ \ , \quad \Delta\kappa_b = \Delta\bar{\kappa} + \Delta\kappa_{b,vertex} \\
\Delta\rho_{f\neq b} &= \Delta\bar{\rho}\ \ , \quad \Delta\rho_b = \Delta\bar{\rho} - 2\Delta\kappa_{b,vertex}\ \ ,
\end{aligned}
$$

$$
K = \frac{\sqrt{2}G_\mu M_W^2}{16\pi^2}\ \ , \quad X_H = \ln\frac{m_H^2}{M_W^2} - \frac{5}{6}\ \ , \quad h_V = \frac{m_H^2}{M_V^2}\ \ , \quad t_V = \frac{m_t^2}{M_V^2} \quad V = W, Z\ .
$$

Correction	$m_H \gg M_W$	$m_H \ll M_W$
$\Delta\alpha^{Higgs}$	0	0
Δe^{Higgs}	$\frac{1}{3}\, X_H$	$-\frac{79}{18}$
$\Delta\rho^{Higgs}$	$-3\frac{s^2}{c^2}\, X_H$	$3\,\frac{\ln c^2}{c^2} + \frac{5}{2}\frac{s^2}{c^2}$
$\Delta\hat{\rho}^{Higgs}$	$-\frac{10}{3}\,\frac{s^2}{c^2}\, X_H$	$\frac{10}{3}\,\frac{\ln c^2}{c^2} + \frac{62}{9}\frac{s^2}{c^2}$
Δ_Z^{Higgs}	0	$\frac{1}{c^2}\,\left(2\ln h_V + \frac{47}{6}\right)$
$\Delta\kappa_{b,vertex}^{Higgs}$	0	0
Δr^{Higgs}	$\frac{11}{3}\, X_H$	$-\frac{10}{3}\,\frac{\ln c^2}{s^2} - \frac{203}{18}$
$\Delta\bar{r}^{Higgs}$	$\frac{1+9s^2}{3c^2}\, X_H$	$-\frac{10}{3}\,\frac{\ln c^2}{c^2} - \frac{62}{9c^2} + \frac{5}{2}$
Δr_b^{Higgs}	$\frac{1+9s^2}{3c^2}\, X_H$	$-\frac{10}{3}\,\frac{\ln c^2}{c^2} - \frac{62}{9c^2} + \frac{5}{2}$
$\Delta\bar{\kappa}^{Higgs}$	$-\frac{10}{3}\, X_H$	$\frac{10}{3}\,\frac{\ln c^2}{s^2} + \frac{62}{9}$
$\Delta\kappa_b^{Higgs}$	$-\frac{10}{3}\, X_H$	$\frac{10}{3}\,\frac{\ln c^2}{s^2} + \frac{62}{9}$
$\Delta\bar{\rho}^{Higgs}$	$-3\frac{s^2}{c^2}\, X_H$	$\frac{1}{c^2}\,\left(2\ln h_Z + \frac{47}{6} + 3\,\ln c^2 + \frac{5}{2}s^2\right)$
$\Delta\rho_b^{Higgs}$	$-3\frac{s^2}{c^2}\, X_H$	$\frac{1}{c^2}\,\left(2\ln h_Z + \frac{47}{6} + 3\,\ln c^2 + \frac{5}{2}s^2\right)$

Correction	$m_t \gg M_W$	$m_t \ll M_W$
$\Delta\alpha^{top}$	0	$-\frac{64}{9}\,s^2\,(\ln t_Z + \frac{5}{3})$
Δe^{top}	$\frac{4}{3}\,\ln t_W - \frac{2}{3}\,\ln c^2 + \frac{8}{9}$	$\Delta\alpha^{top} + 2\,\ln c^2$
$\Delta\rho^{top}$	$3\,t_W$	0
$\Delta\hat{\rho}^{top}$	$3\,t_W + \hat{c}_1\ln t_W + \hat{c}_2$	$\hat{c}_3$
Δ_W^{top}	-2	2
Δ_Z^{top}	0	$\frac{2}{c^2}\,(1 - \frac{8}{3}s^2 + \frac{32}{9}s^4)$
$\Delta\kappa_{b,vertex}^{top}$	$2\,t_W + \frac{1}{3}\,(16 + \frac{1}{c^2})\,\ln t_W + \cdots$	$\cdots$
Δr^{top}	$-\frac{c^2}{s^2}\,3\,t_W + c_1\ln t_W + c_2$	$\Delta\alpha^{top} + c_3$
$\Delta\bar{r}^{top}$	$-3\,t_W + \bar{c}_1\ln t_W + \bar{c}_2$	$\Delta\alpha^{top} + \bar{c}_3$
Δr_b^{top}	$-\frac{1+s^2}{c^2}\,t_W + c_{b,1}\ln t_W + \cdots$	$\Delta\alpha^{top} + \cdots$
$\Delta\bar{\kappa}^{top}$	$\frac{c^2}{s^2}\,(3\,t_W + \hat{c}_1\ln t_W + \hat{c}_2)$	$\frac{c^2}{s^2}\,\hat{c}_3$
$\Delta\kappa_b^{top}$	$\frac{2+c^2}{s^2}\,t_W + \tilde{c}_{b,1}\ln t_W + \cdots$	$\cdots$
$\Delta\bar{\rho}^{top}$	$3\,t_W$	Δ_Z^{top}
$\Delta\rho_b^{top}$	$-\,t_W - \frac{2}{3}\,(16 + \frac{1}{c^2})\,\ln t_W + \cdots$	$\cdots$

$\hat{c}_1 = 2\,(1 + \frac{1}{3}\frac{s^2}{c^2})$	$\hat{c}_2 = (\frac{2}{3}\,\frac{s^2}{c^2}\ln c^2 + 1 + \frac{1}{9}\frac{s^2}{c^2})$	$\hat{c}_3 = 2\,\ln c^2$
$c_1 = -2\,(\frac{c^2}{s^2} - \frac{1}{3})$	$c_2 = -(\frac{4}{3}\,\ln c^2 + \frac{c^2}{s^2} - \frac{7}{9})$	$c_3 = -2\,(\frac{c^2}{s^2} - 1)\,\ln c^2$
$\bar{c}_1 = -\frac{2}{3}\frac{1}{c^2}$	$\bar{c}_2 = -(\frac{2}{3}\ln c^2 + \frac{1}{9})\,\frac{1}{c^2}$	$\bar{c}_3 = 0$
$c_{b,1} = \frac{16c^2\,(c^2-s^2)-1}{3c^2}$	$c_{b,2} =?$	$c_{b,3} =?$
$\tilde{c}_{b,1} = 6 + \frac{2c^2}{s^2} + \frac{1}{3c^2}$	$\tilde{c}_{b,2} =?$	$\tilde{c}_{b,2} =?$

In the SM the Higgs serves to provide a physical cut–off to the massive vector boson theory and taking the Higgs heavy is like taking a large cut–off. By Veltman's

screening theorem at $O(\alpha)$ a weak logarithmic Higgs mass dependence shows up only. All physical observables exhibit some logarithmic Higgs mass dependence in the large Higgs Mass limit (via Δe, $\Delta\rho$, $\Delta\hat{\rho}$). There is one interesting case which shows a logarithmic Higgs mass dependence for small Higgs masses (via Δ_Z). It originates from the derivative of the gauge boson self energy on the mass shell, which corresponds to the gauge boson wave functon renormalization. The latter enters the (partial) width of the gauge boson Γ_Z^f. For the Higgs dependence via Δ_Z, we find $\Delta_Z^{Higgs} = 0$ for $m_H \gg M_Z$ and $\Delta_Z^{Higgs} = \frac{2K}{c^2}\,(\ln\frac{m_H^2}{M_Z^2} + \frac{47}{12})$ for $m_H \ll M_Z$. However, the potentially interesting infrared log for a light Higgs disappears if Higgs Bremsstrahlung off the final state Z's is taken into account. Actually, one finds $\delta\Gamma_{Z\to Hf\bar{f}}/\Gamma_{Z\to f\bar{f}} = -\Delta_Z^{Higgs}$ such that $\Delta_{Z,virtual+soft\ Higgs} = 0$ for $m_H \ll M_Z$.
Higgs contribution to inclusive Z–width [90].
The $Z \to Hf\bar{f}$ decay rate is given by

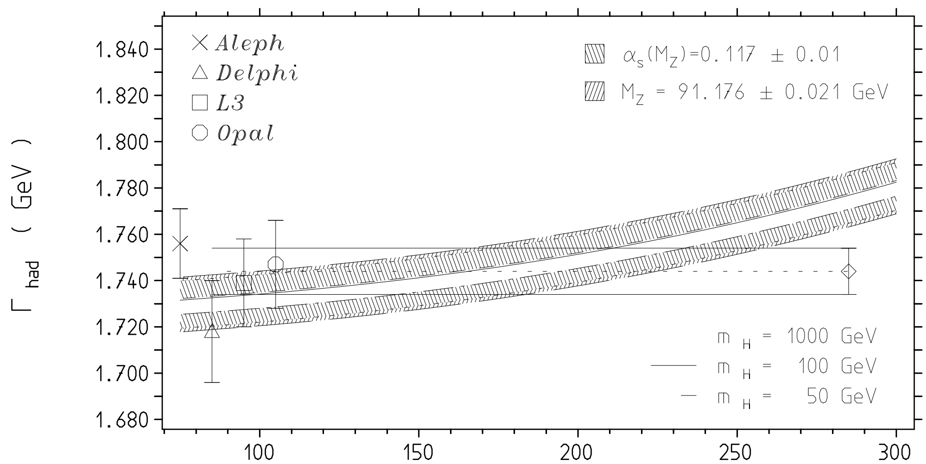
$$\frac{1}{\Gamma \to f\bar{f}}\frac{d\Gamma}{dx} = \frac{\alpha}{4\pi\sin^2\Theta_W\cos^2\Theta_W}\frac{\left[1 - x + \frac{x^2}{12} + \frac{2}{3}\,a^2\right]\sqrt{(x^2 - 4a^2)}}{(x - a^2)^2}$$

with $a = \frac{m_H}{M_Z}$ and $x = \frac{2E_{\mathrm{Higgs}}}{M_Z}$. and the kinematical limits

$$2a \le x \le 1 + a^2$$

$$\frac{\alpha}{4\pi\sin^2\Theta_W\cos^2\Theta_W} = \frac{\sqrt{2}G_\mu M_Z^2}{4\pi^2} = \frac{\sqrt{2}G_\mu M_W^2}{4\pi^2 c^2} = K\frac{4}{c^2}$$

Similarly, for the Higgs dependence, we find $\Delta C^{WH} = 0$ for $m_H \gg M_W$ and $\Delta C^{WH} = 2K(\ln\frac{m_H^2}{M_W^2} + \frac{47}{12})$ for $m_H \ll M_W$. The potentially interesting infrared log for a light Higgs disappears if Higgs Bremsstrahlung off the final state Z's is taken into account. Actually, one finds $\delta\Gamma_{Z\to Hf\bar{f}}/\Gamma_{Z\to f\bar{f}} = -\Delta_Z^{Higgs}$ such that $\Delta_{Z,virtual+soft\ Higgs} = 0$ for $m_H \ll M_Z$.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Delphi
L3
Opal
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M_Z = 91.176 ± 0.021 GeV
m_H = 1000 GeV
m_H = 100 GeV
m_H = 50 GeV
Γ_{had} (GeV)
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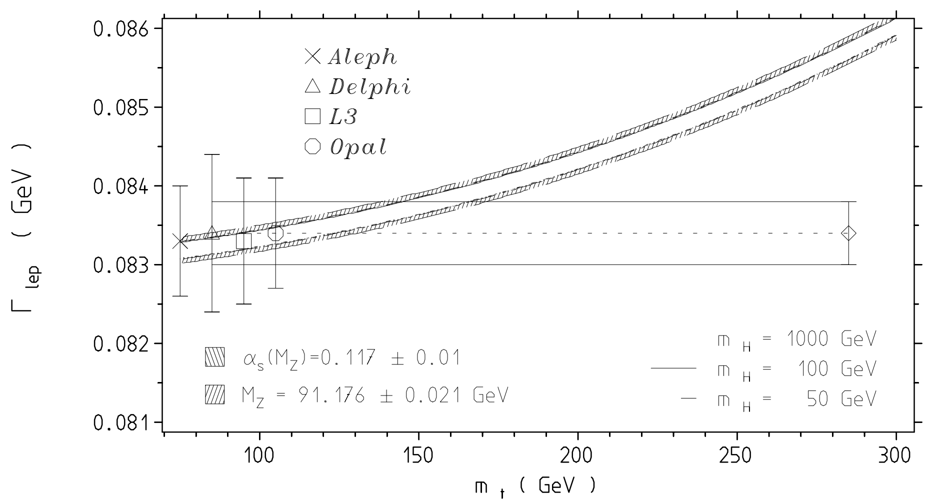
1.800
1.780
1.760
1.740
1.720
1.700
1.680
100
150
200
250
300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m_H = 1000 GeV
m_H = 100 GeV
m_H = 50 GeV
Γ_{lep} (GeV)
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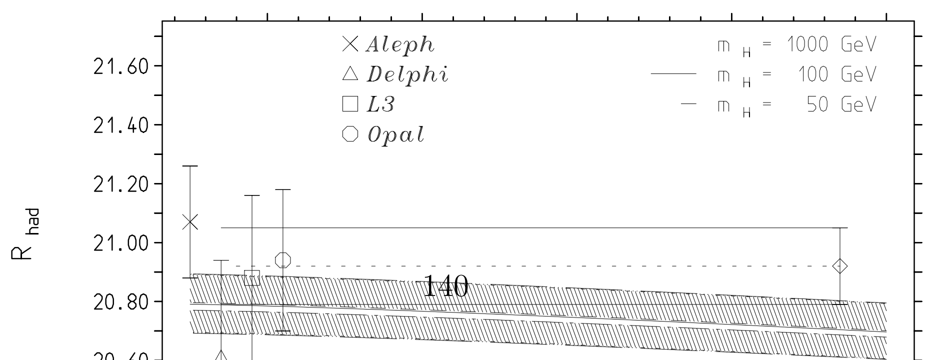
0.082
0.081
100
150
200
250
300
m_t (GeV)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Delphi
L3
Opal
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m_H = 100 GeV
m_H = 50 GeV
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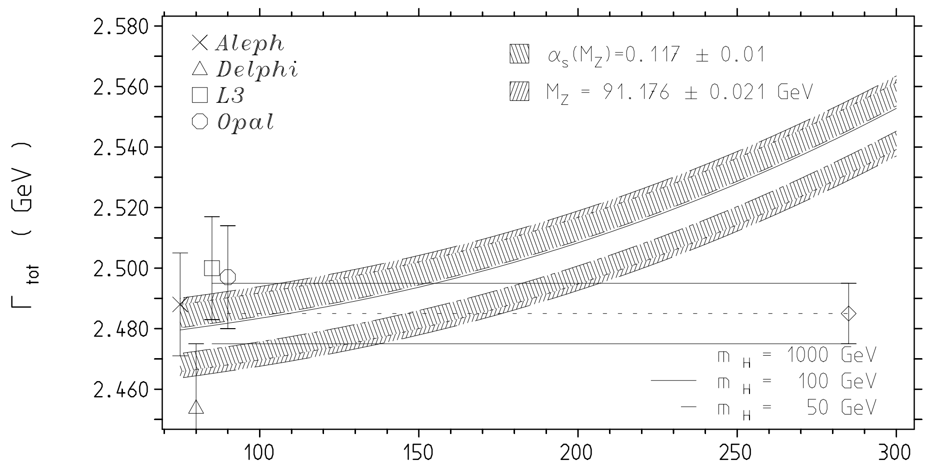
21.40
21.20
21.00
20.80


Aleph
Delphi
L3
Opal
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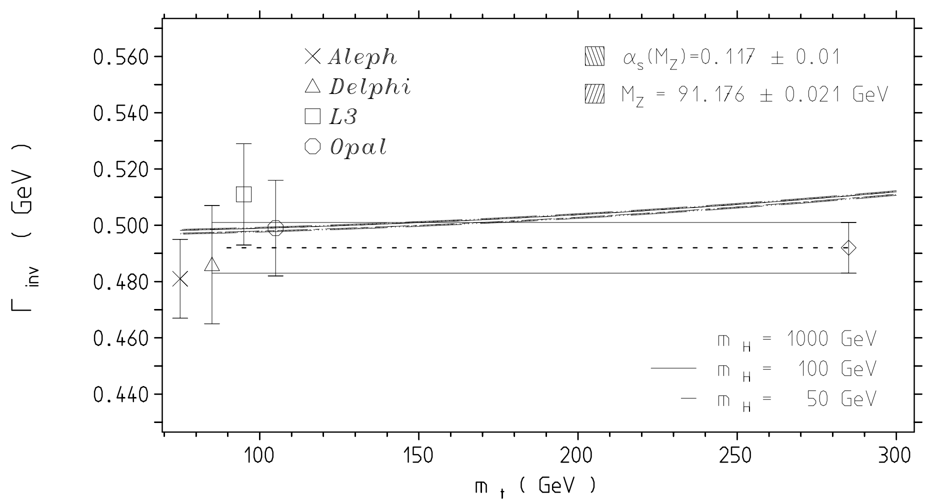
M_Z = 91.176 ± 0.021 GeV
m_H = 1000 GeV
m_H = 100 GeV
m_H = 50 GeV
Γ_{tot} (GeV)
2.580
2.560
2.540
2.520
2.500
2.480
2.460
100
150
200
250
300


Aleph
Delphi
L3
Opal
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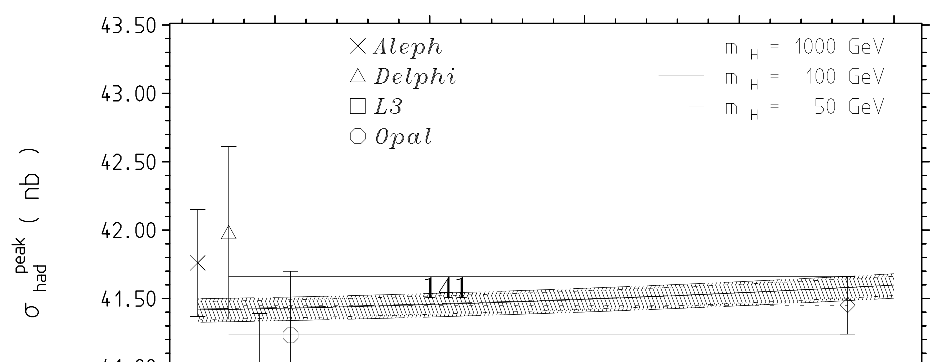
M_Z = 91.176 ± 0.021 GeV
m_H = 1000 GeV
m_H = 100 GeV
m_H = 50 GeV
Γ_{inv} (GeV)
0.560
0.540
0.520
0.500
0.480
0.460
0.440
100
150
200
250
300
m_t (GeV)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Delphi
L3
Opal
m_H = 1000 GeV
m_H = 100 GeV
m_H = 50 GeV
σ_{had}^{peak} (nb)
43.50
43.00
42.50
42.00
41.50

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