An Introduction to the Theory of

Critical Phenomena

and the

Renormalization Group

by

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Introductory Remarks

The aim of the present lectures is to give on an introductory level an outline of the theory of critical phenomena as developed during the last decade. Some of the important theoretical ideas that have contributed to our present understanding of critical phenomena are:

1873 van der Waals:	Theory of liquid-gas transition
1907 Weiss:	Molecular field theory of the ferromagnet
1937 Landau:	Mean field approach to critical phenomena
1944 Onsager:	Exact solution of the Ising model in two dimensions
1965 Widom, Domb, Hunter:	Scaling
1966 Kadanoff:	Scaling and block spin picture
1969 Polyakov, Migdal, Gribov:	Renormalized field theory approach to critical phenomena
1969 Di Castro, Jona-Lasinio:	Renormalization group in the renormalized field theory approach (Stückelberg, Petermann, Gell-Mann, Low 1954)
1970 Griffith, Kadanoff:	Universality
1970 Fisher:	Dependence of critical exponents on the dimension and symmetries
1971 Wilson:	Renormalization group for Ising and Landau-Ginzburg model
1972 Wilson, Fisher:	ε-expansion, critical exponents in $4-\varepsilon$ dimensions
1972 Wegner:	Fixed point structure, Corrections
1973 Niemeyer, van Leeuwen:	Renormalization group for the two dimensional triangular Ising model
1974 Kadanoff, Houghton, Wilson:	Renormalization group calculations for square (d=2) and cubic (d=3) lattice Ising models

These are not all the important contributions. (The choice depends on my personal taste and ignorance.)

Not listed here are the many important experimental results (the basis for that which theory has to explain) (see e.g. Kadanoff et al. (1967), Levelt-Sengers (1973) and de Jongh-Miedema (1974)) and the numerical calculations on models (see e.g. Wortis (1973)) which were necessary to support and stimulate theoretical developments.

Also there are many structural results concerning the connection between statistical mechanics of phase transitions quantum field theory and probability theory which are not mentioned.

I will mainly concentrate on developing the main ideas needed to understand some of the main aspects of the physics of phase transitions using the example of the ferromagnetic Ising model with short range interactions in d-dimensional space. With suitable modifications the considerations presented will apply to a large number of other systems. Numerous calculations have been performed for a large class of models (by means of doubtful approximation schemes).

The main emphasis of my lectures will be to create an intuitive understanding of the physics behind the theory describing the characteristic features of critical phenomena and to show how to calculate critical exponents in simple approximations. This will not be an easy task. A respectable amount of labor has to be done to understand "how and why it works".

Part I. Critical Behavior and the Renormalization Group

1. Introduction

The transition from one phase to another like melting or boiling changes the properties of a system discontinuously. Such discontinuous transitions are called first order phase transitions. One ore more derivatives of the free energy

$$F = F(T, \zeta)$$

with respect to the temperature T or an appropriate ordering field ζ are dinscontinuous. A standard example is the liquid gas transition of a single component system with typical phase diagram depicted in Fig. 1. The ordering field is the pressure p in this case.

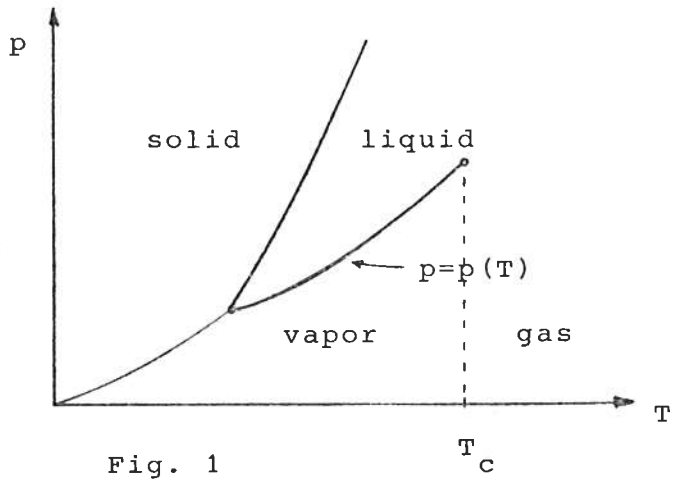


Fig. 1

On the phase boundary (coexistence curve) $\zeta = \zeta(T)$ below some temperature T_c two distinct phases can coexist. The two phases are characterized by two different values of the order parameter ψ, the basic variable being thermodynamically conjugate to ζ (i.e. below T_c the order parameter takes different values under identical physical (external) conditions). In the (T, ζ) phase diagram the phase boundary may come to an end. Such an end point (T_c, ζ_c) is the critical point. At this point the two coexisting phases become more and more similar e.g. the liquid takes on a cloudy appearance (critical opalescence). Above the critical point all differences between the two phases disappear. The order parameter shows a finite jump in the course of a first order phase transition and this jump describes the amount of difference between

the coexisting phases. As the temperature is increased along the coexistence curve the jump goes to zero at the critical point (second order phase transition).

There are many examples of critical points observed in nature belonging to very different systems (Tab. 1).

Transition	Order parameter	Ordering field (symmetry breaking parameter)
Liquid-gas	Density $\rho-\rho_c$	Chem. Potential μ
Ferromagnetic	Magnetization M	Magnetic Field H
Antiferromagnetic	Sublattice Magnetization	Alternating Magnetic Field (non physical)
Ferroelectric	Polarization P	Electric Field E
Antiferroelectric	Sublattice Polarization	Alternating Electric Field (non physical)
Phase Separation (Mixtures, Alloys)	Concentration $x-x_c$	Chemical Potential $\mu-\mu_c$
Superfluids	Expectation of Condensate Wave Function $\langle\psi\rangle$	Staggerd Field (non physical)
Superconductors	Gap Parameter	non physical
Ising Model	Magnetization $\langle\sigma\rangle = M$	Magnetic Field h

Tab. 1

Phase diagrams in general may have a more complicated structure e.g. for superfluid Helium there is a line of critical points (λ-line) and a tricritical point at which three distinct phases become identical.

Surprisingly enough it has been observed that many quite differently looking systems (e.g. different substances, different types of transitions like liquid-gas, ferromagnetic and antiferromagnetic) not only show qualitatively similar phenomena but quantitatively identical behavior in the critical region (universal

properties); only the names of variables change. It seems that there are not too many distinct classes (universality classes) of critical systems characterized by only a few criteria like dimensionality and symmetry of the ordered state.

Apparently many details of a system undergoing a phase transition are irrelevant. This observation is the basis for a unified description of second (or higher) order phase transitions. We shall limit ourselves considering a restricted class of systems (models) explaining the main characteristic features of critical phenomena and neglecting a number of peculiar features of a given system. So we will ignore quantum effects. Since critical phenomena become apparent on a macroscopic scale it seems to be justified to treat the problem in terms of continuous classical variables. Long range interactions will also be neglected, e.g. the dipolar interactions due to lattice distortions etc..

The idea of universality goes back to Landau who proposed that all critical behavior might be described by a mean field theory. It needed the result of Onsager on the two-dimensional Ising model to convince theoretical physicists that Landau's suggestion could not be generally true; but early experimental results (dating back to the first decade of this century) if taken seriously would also have lead to this conclusion. Recent developments towards a theory of critical phenomena give (partial) answers to questions like:

What are the characteristic properties of critical systems and how do we calculate these? How does universality come about? Why and when does the mean field approximation fail? How do we classify critical systems? What are the relevant and what are the irrelevant features of the characteristics of critical behavior? etc..

The general method for the study of critical behavior will be illustrated in detail below, using the example of the ferromagnetic Ising model. At the same time this model serves as a good description of many physical systems near the critical region.

2. Thermodynamical Quantities in Terms of Local Variables

The Lenz-Ising model we are going to define now may be considered as a microscopic description of a certain physical system.

In the d-dimensional configuration space a regular lattice G_a with lattice spacing a is given. To each lattice point $x \in G_a$ there is associated a classical discrete spin σ_x taking the values ±1 (i.e. site spin distribution $\delta(\sigma_x^2-1)$). Each spin interacts with its nearest neighbor (n.n.) only. Parallel spins are attractive with energy -K, a spin parallel to an external field H has energy -H; for antiparallel spins the energy is K and H. Accordingly the Lenz-Ising Hamiltonian reads

$$\mathcal{H}(\sigma) = -K \sum_{n.n.} \sigma_x\sigma_y - H\sum\sigma_x \quad . \tag{2.1}$$

It associates to each spin configuration $\{\sigma\}$ a real number $\mathcal{H}(\sigma)$. For a finite system with N spins we obtain the thermodynamical quantities from the partition function

$$Z_N = \sum_{\substack{conf.\\ \{\sigma\}}} \exp - H_N \quad ; \quad H = \beta\mathcal{H} \; ; \beta = \frac{1}{k_B T} \; . \tag{2.2}$$

In particular we define the free energy density

$$f(k,h) = \frac{F_N}{N} \quad ; \; F_N = - \ln Z_N \; ; \; (k=\beta K, \; h=\beta H) \tag{2.3}$$

the spin correlation functions

$$\begin{aligned} \langle\sigma_x\rangle &= Z_N^{-1} \sum_{conf.} \sigma_x \, e^{-H_N} \\ \langle\sigma_x\sigma_o\rangle &= Z_N^{-1} \sum_{conf.} \sigma_x\sigma_o \, e^{-H_N} \quad \text{etc.} \end{aligned} \tag{2.4}$$

and the energy correlation functions

$$\begin{aligned} \langle E_x\rangle &= Z_N^{-1} \sum_{conf.} E_x \, e^{-H_N} \\ \langle E_x E_o\rangle &= Z_N^{-1} \sum_{conf.} E_x E_o \, e^{-H_N} \quad \text{etc.} \end{aligned} \tag{2.5}$$

with $E_x \doteq \sum_{\substack{y \\ |x-y|=a}} \sigma_x \sigma_y$ (energy density fluctuation).

We always assume that a suitable thermodynamic limit $N \to \infty$ exists for these quantities (unique for $\beta < \beta_c$). From the free energy density we define the thermodynamical quantities

$$M \doteq -\frac{\partial f}{\partial h} = \langle\sigma_x\rangle = \langle\sigma_o\rangle \qquad \text{magnetization density}$$

$$E \doteq -\frac{\partial f}{\partial k} = \langle E_x\rangle = \langle E_o\rangle \qquad \text{energy density}$$

$$\chi \doteq -\frac{\partial^2 f}{\partial_h{}^2} = \sum_x \{\langle\sigma_x\sigma_o\rangle - \langle\sigma_x\rangle\langle\sigma_o\rangle\} \qquad \text{susceptibility} \tag{2.6}$$

$$= \sum_x \langle\sigma_x\sigma_o\rangle^{conn}$$

$$C \doteq -\frac{\partial^2 f}{\partial k^2} = \sum_x \{\langle E_x E_o\rangle - \langle E_x\rangle\langle E_o\rangle\} \qquad \text{specific heat}$$

$$= \sum_x \langle E_x E_o\rangle^{conn} \quad .$$

The second equality in each case gives the important expression of the thermodynamic quantity in terms of the local fluctuation variables σ_x and E_x. Above we have assumed (appropriate for the thermodynamical limit) translation invariance.

We will now enter into qualitative discussion of how to understand critical behavior of the Ising ferromagnet in terms of spin fluctuations.

Away from the critical point the spin correlation shows the characteristic Ornstein-Zernike fall of

$$\langle\sigma_x\sigma_o\rangle^{conn} \simeq \frac{e^{-|x|/\xi}}{|x|^{d-2}} \quad ; \quad |x| \gg a \tag{2.7}$$

for large distances.

This relation defines the <u>correlation length ξ</u> which is the fundamental parameter in the study of second order transitions. The phase

diagram for a ferromagnetic system is depicted in Fig. 2.

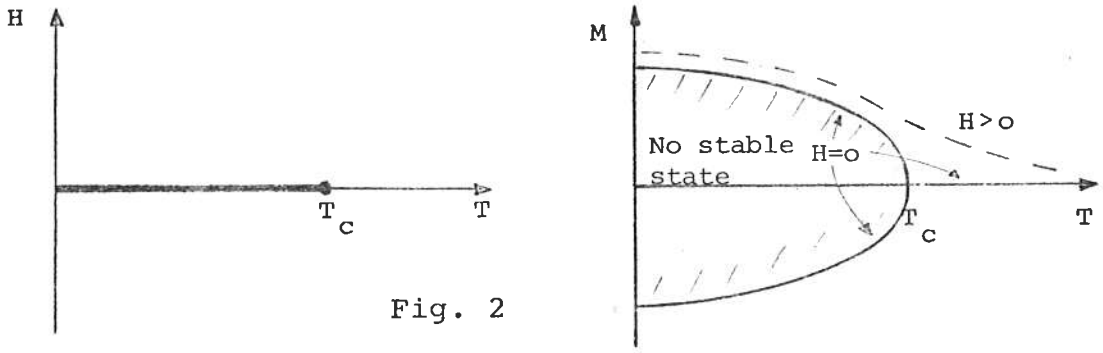


Fig. 2

Below we will talk about the Ising spin configurations as if they were snapshots of a dynamically fluctuating system at different times. The typical configurations considered are those appearing with "large" Gibbs factor exp-H. For $H \neq o, T$ arbitrary and for $H = o, T > T_c$ all thermodynamic quantities are <u>analytic functions</u> of T and H due to the finite correlation length ξ which means that the physics takes place in a finite box of size $L \gtrsim \xi$. By increasing the size of the box nothing new can be learned about the system; the densities remain unchanged (independent of L). For $H = o, T > T_c$ the correlation length ξ can be interpreted as the maximal size of islands of spins pointing in the same direction with fluctuations (subislands) down to microscopic scale (up-down symmetry) (Fig. 3). The coupling between spins is weak (for $T \to \infty$: $k \to o$) and no long range order is present.

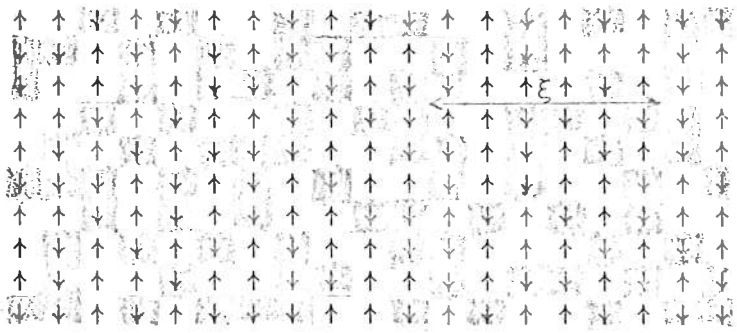


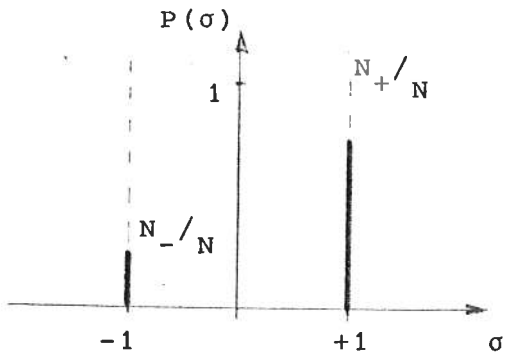
Fig. 3

For $H = o$, $T < T_c$ the coupling between spins is strong enough (for $T \to o$: $k \to \infty$) to line up the spins i.e. a spontaneous magnetization

$$\langle\sigma\rangle = \pm M(T) \; ; \; |M| > o$$

occurs. The state is no longer uniquely defined as a function of T. It is an arbitrary mixing of two extremal (pure) states

located on the two branches ($H = \pm o$) of the coexistence curve. For a pure state the probability $P(\sigma)$ for a spin of pointing up or down is unique. The other pure state then has $P'(\sigma) = P(-\sigma)$. At $T = o$, in a pure state all spins point either up or down.



A first order transition takes place as H changes sign and the magnetization of the system jumps from M to -M. This jump disappears as T increases to T_c, the point of second order transition. This point is necessarily a point of <u>non analyticity</u> as $M(H=o;T) \equiv o$ for $T > T_c$ but $\neq o$ for $T < T_c$. For $H = o$, $T < T_c$ the meaning of the correlation length has changed. Now there is a net magnetization say in the z direction and ξ has the meaning of maximal size of clusters of spins pointing in the "wrong" direction again with subclusters down to microscopic scale (no up-down symmetry!) (Fig. 4).

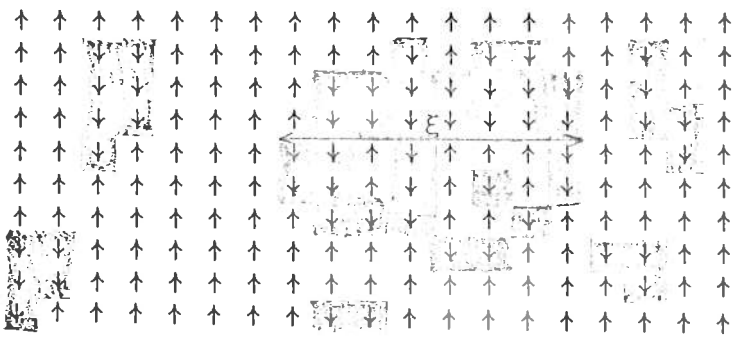


Fig. 4

The formation of clusters of "wrong" phase requires an extra of free energy to be added to the system proportional to the surface of the cluster

$$\text{free energy cost} \propto \text{area} \times \left(\frac{\text{energy}}{\text{unit area}}\right).$$

The energy per unit volume remains the same in the two phases. The formation of a cluster due to fluctuations is very unlikely if the free energy cost is much greater than $k_B T$. Thus

$$\begin{array}{l}\text{surface area}\\ \text{of cluster}\end{array} \leq \frac{k_B T}{\left(\frac{\text{energy}}{\text{unit area}}\right)}.$$

The critical point is characterized by a divergent correlation length $\xi \to \infty$ as $T \to T_c$, $H = o$. If T_c is approached from below ($H = o$) the difference in magnetization of the "right" and the "wrong" phase approaches zero and together with it the energy per unit area that determines the cost in free energy for cluster formation. Now infinite clusters are formed at no energy cost. This is the region of large scale weak fluctuations in magnetization (Fig. 5).

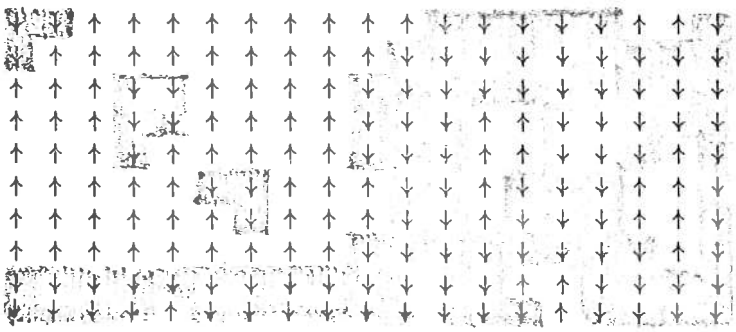


Fig. 5

The physics is no longer determined by what is happening in a finite box. We are faced with a system with infinitely many degrees of freedom. In a strict sense cooperative phenomena only appear in infinite systems (i.e. thermodynamical limit must always be performed) (see Fig. 7).

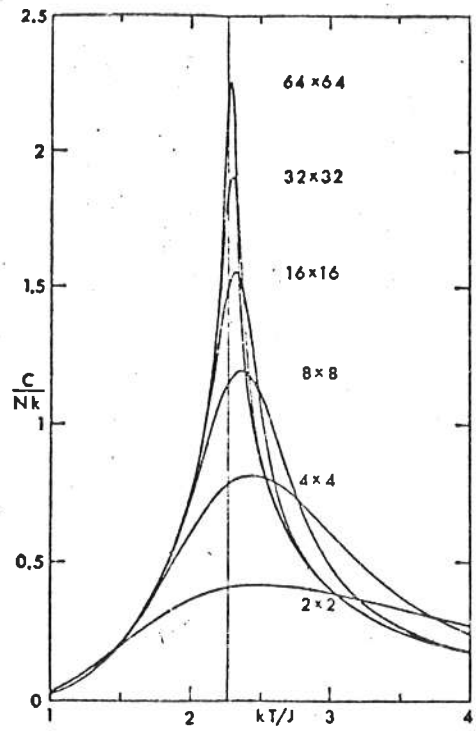
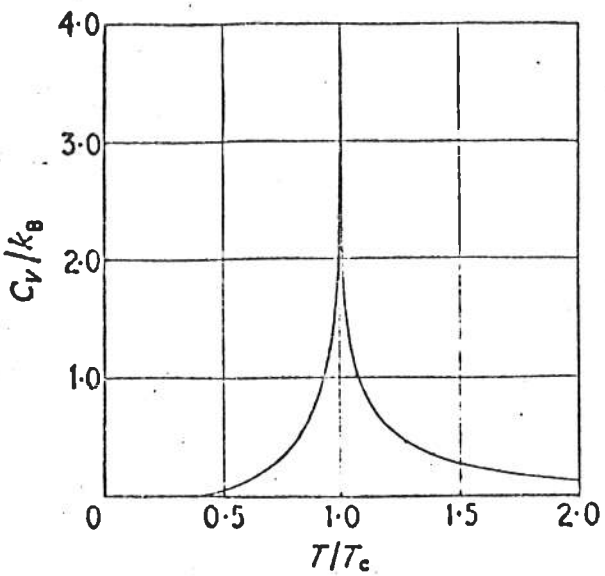


FIGURE 7. Specific heat of finite Ising models: the plots are for n^2 lattices with periodic boundary conditions (Ferdinand and Fisher, 1969).



The limiting specific heat per spin for an infinite plane square Ising model Onsager (1944).

The Ornstein-Zernike behavior of the spin correlation (2.7) changes discontinuously at the critical point. Roughly speaking the r.h.s. of equation (2.7) looses its scale ξ and must hence take a homogeneous (scaling) form

$$\begin{aligned} \langle \sigma_x \sigma_o \rangle &\simeq \frac{\text{const.}}{|x|^{2d\sigma}} \\ \langle E_x E_o \rangle &\simeq \frac{\text{const.}}{|x|^{2d_E}} \end{aligned} \quad ; \; |x| \to \infty \quad . \tag{2.8}$$

The weak fall off of correlations over very large distances in space is the microscopic cause for the non analyticities and divergences in thermodynamical quantities , e.g. the divergences of χ and C turn out by virtue of (2.6) as divergences of the infinite volume sums over weakly decreasing densities.

The singularities may be parametrized in general by power laws (in degenerate cases logarithms appear instead). This has been confirmed by model calculations and experiments.

The fundamental divergence of ξ at the critical point is described by an exponent ν:

$$\xi \propto \begin{cases} t^{-\nu} \\ (-t)^{-\nu'} \end{cases} \quad ; \; t \to \pm o \; ; \; H = o. \tag{2.9}$$

Similarly the other standard exponents are defined (see Fig. 6 and Tab. 2):

$$M\Big|_{H=o} \propto \begin{cases} o \\ (-t)^{\beta} \end{cases} ;\ t \to \pm o \qquad M\Big|_{t=o} \propto \pm|H|^{1/\delta} ;\ H \to \pm o$$

$$\chi\Big|_{H=o} \propto \begin{cases} t^{-\gamma} \\ (-t)^{-\gamma'} \end{cases} ;\ t \to \pm o \qquad C\Big|_{H=o} \propto \begin{cases} t^{-\alpha} \\ (-t)^{-\alpha'} \end{cases} ;\ t \to \pm o \tag{2.10}$$

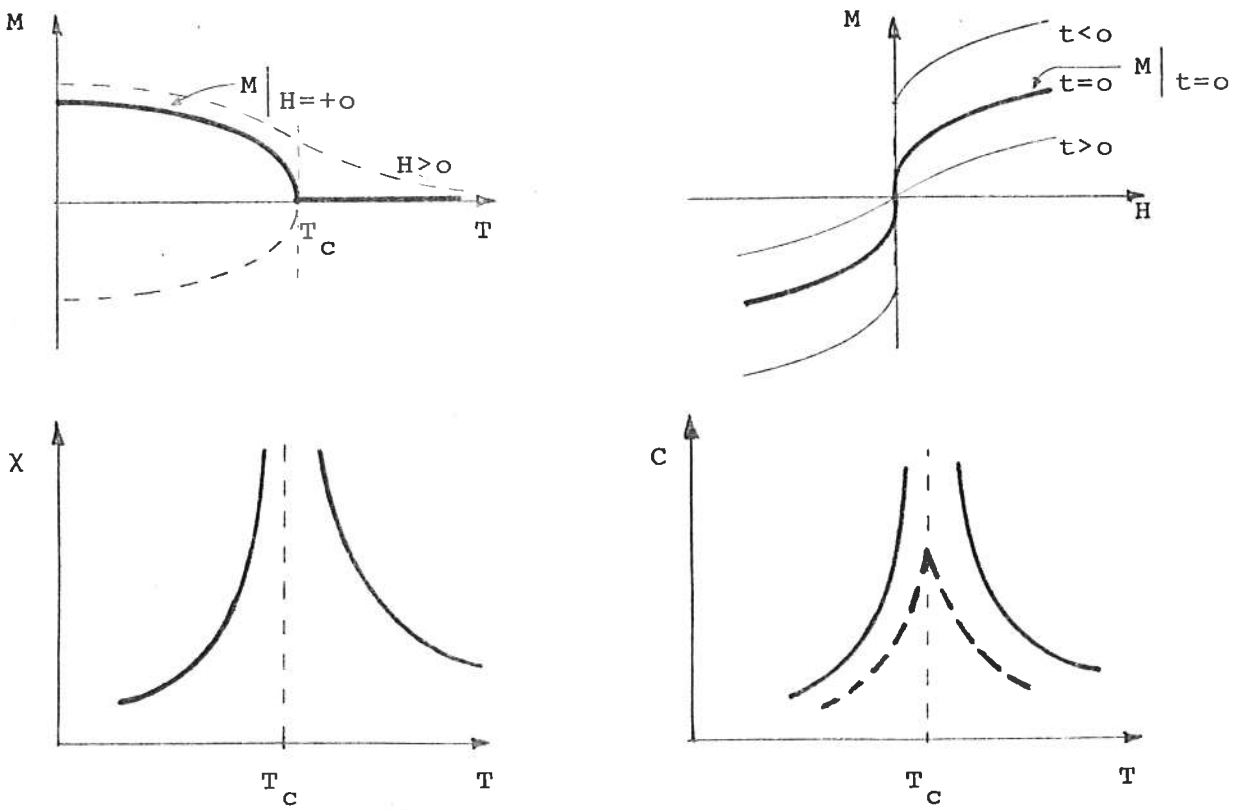
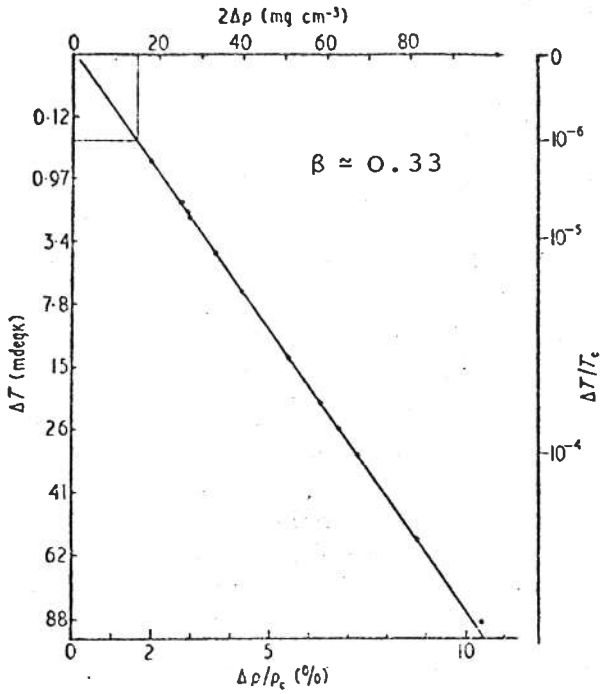


Fig. 6.a

Model system	α	α'	β	γ	γ'	δ	η	ν	ν'
MF	0 (discont.)	0 (discont.)	$\frac{1}{2}$	1	1	3	0	$\frac{1}{2}$	$\frac{1}{2}$
Spherical ($d=3$)	-1	—	$\frac{1}{2}$	2	—	5	0	1	—
Ising ($d=3$)	$\simeq\frac{1}{8}$	$\frac{1}{8}-\frac{1}{16}$	0·312	1·25	1·25–1·31	$\simeq 5$	0·03–0·05	$\simeq 0{\cdot}63$	—
XY ($d=3$)	$\simeq 0$ (logar.)	—	—	$\simeq 1{\cdot}33$	—	—	—	—	—
Heisenberg ($d=3$)	$\simeq -0{\cdot}1$	—	$\simeq 0{\cdot}36$	$\simeq 1{\cdot}40$	—	$\simeq 5$	0·03–0·04	$\simeq 0{\cdot}71$	—
Ising ($d=2$)	0 (logar.)	0 (logar.)	$\frac{1}{8}$	1·75	1·75	15	0·25	1	1

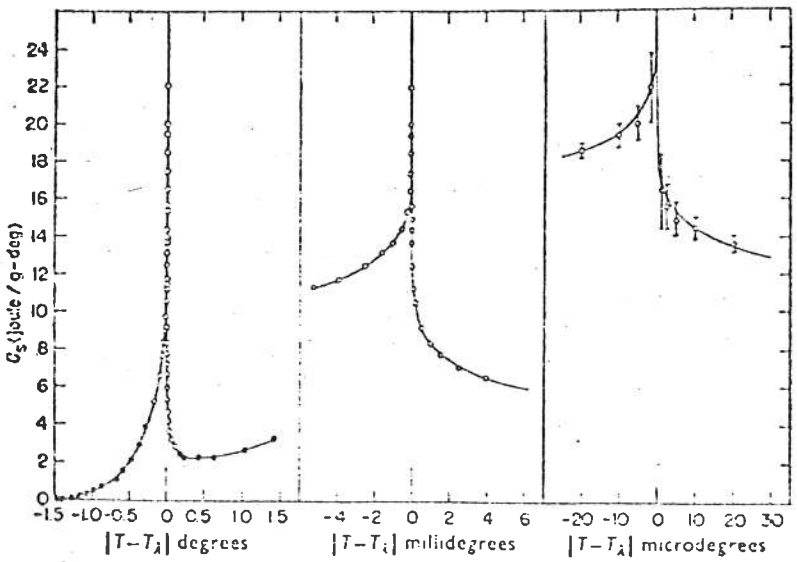
Tab. 2. Critical exponents predicted from various models

Experimental plots:



Plot of the cube root of $\Delta T = T_c - T$ against $\Delta\rho = \rho_L - \rho_G$ for CO_2 demonstrating the validity of the 'one-third' law to high accuracy over three decades in temperature. (After Lorentzen 1965.)

Range of a transition region



Specific heat of liquid helium vs $T - T_\lambda$ in °K.

The specific heat of liquid helium (Fairbank *et al.* 1957).

Sharpness of a transition

Fig. 6.b

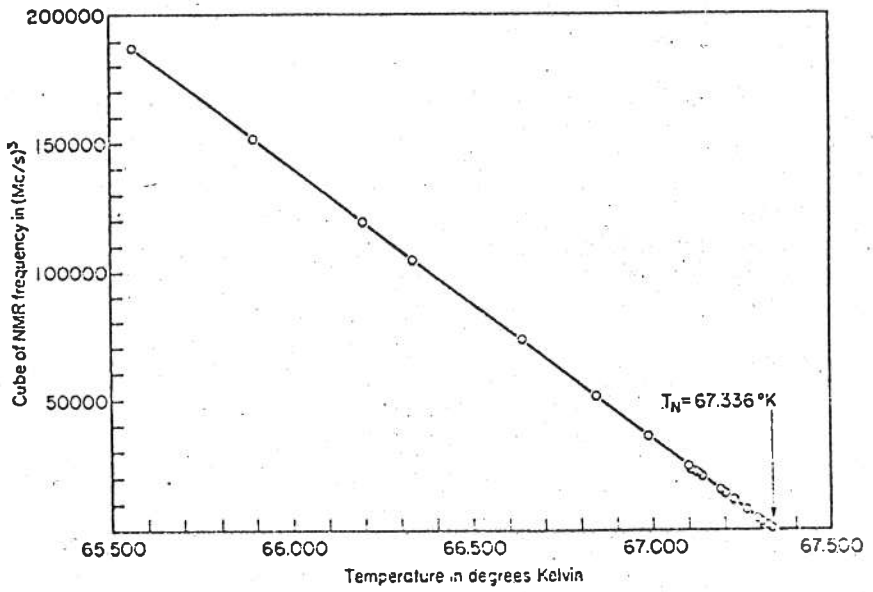


FIGURE 4. Cube of the sub-lattice magnetization of MnF_2 versus T demonstrating that $\beta \simeq 0.33$ (Heller and Benedek, 1962).

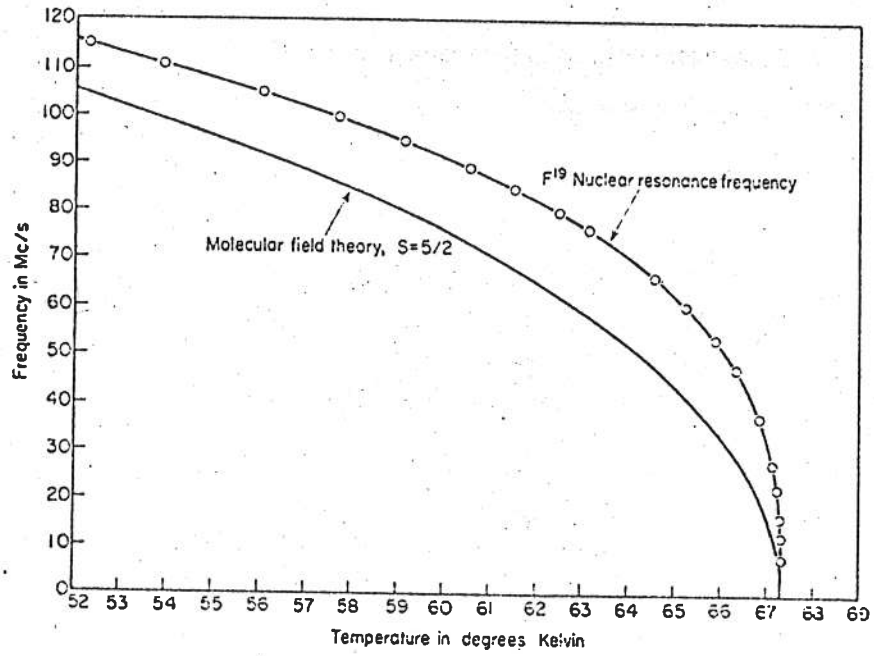


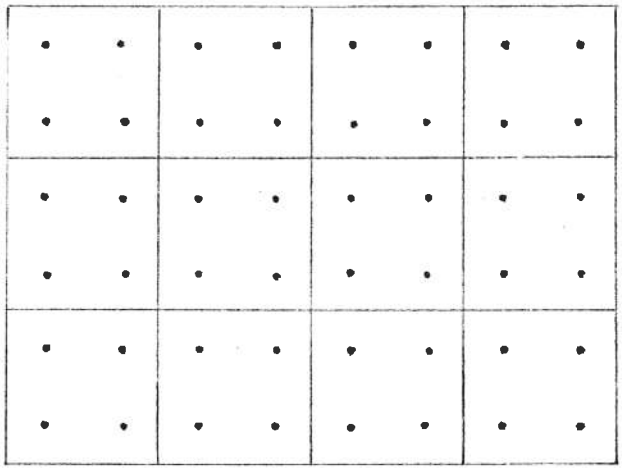
FIGURE 3. The sub-lattice magnetization of MnF_2 as measured by nuclear magnetic resonance (Heller and Benedek, 1962).

3. The Block Spin RG-Transformation

3.1. Kadanoff's Block Spin Picture of Critical Behavior

The main difficulty in the quantitative understanding of critical behavior are the infinitely many degrees of freedom relevant to this cooperative phenomenon. Fluctuation over an infinite hierachy of length scales $a < |x| < \xi = \infty$ must be taken into account. If one restricts oneself to a discussion of only the typical features of a critical system like long range correlations, critical singularities etc. one might expect to be able to simplify the problem by disregarding a lot of details. The main observation is that close to the critical point we expect the spins to be "strongly" correlated over large distances, i.e. because of the extension to infinity of up and down clusters of spins the probability of finding, for a given spin, spins pointing in the same direction is large over large distances. For a description of the characteristic critical long range behavior then it suffices to consider the critical system as consisting of "large" blocks of spins which (as site spins are strongly correlated within blocks) essentially behave as one big spin. The interaction among the block spins is expected (at criticality) to be again of the nearest neighbor Ising form. By forming block spins one "simplifies" the system as one eliminates a large number of degrees of freedom expected to be irrelevant at the critical point. This is the basic idea suggesting to study critical behavior by investigation of renormalization groups (RG). The block spin RG expresses the effective block spin Hamiltonian in terms of the original site spin Hamiltonian. Before we give a precise definition of the RG let us briefly discuss some of the essential steps involved:

We subdivide the "microscopic" system of site spins with cell size a^d into cells of size L^d with $L=2^n a$ (n=positive integer)



and form block spins

$$\hat{\sigma}_{\hat{x}} = \sum_{x \in \mathrm{cell}_{\hat{x}}} \sigma_x \quad .$$

For exactly lined up spins

$$\hat{\sigma}_{\hat{x}} = \pm 2^{nd} \quad .$$

If we form larger and larger blocks the block spins do not stay bounded

$$\hat{\sigma}_{\hat{x}} \to \pm \infty \qquad (n \to \infty) .$$

In order to get bounded block spin variables one must <u>renormalize</u> the $\hat{\sigma}$

$$\sigma'_{\hat{x}} = Z_n \, \hat{\sigma}_{\hat{x}}$$

such that

$$\sigma'_{\hat{x}} = Z_n \, \hat{\sigma}_{\hat{x}} \to \pm 1 \qquad (n \to \infty) \tag{3.0}$$

For exactly aligned spins

$$Z_n = 2^{-nd} = (L/a)^{-d}$$

would be the appropriate renormalization. However, (and this is the crucial point) there are fluctuations within the cells such that the spins are <u>not</u> exactly lined up (this would be likely only for small T and then not with up and down symmetry). Thus Z_n must be somewhat larger. We set

$$Z_n = 2^{-nx} = (L/a)^{-x}; \quad o < x < d$$

where x is an unknown parameter (critical exponent) to be determined from the condition (3.0).

The renormalized block spins are spins on a regular lattice with spacing $a' = 2^n a$ and for larger and larger blocks the lattice spacing does not stay finite. In order to have the block spins on a finite lattice (with spacing a) also as $n \to \infty$ one has to <u>dilate</u> (shrink) the system (scale transformation)

$$x' = \frac{\hat{x}}{2^n} \quad .$$

Thus to the site spin we have related a block spin system on the same lattice (Fig. 8).

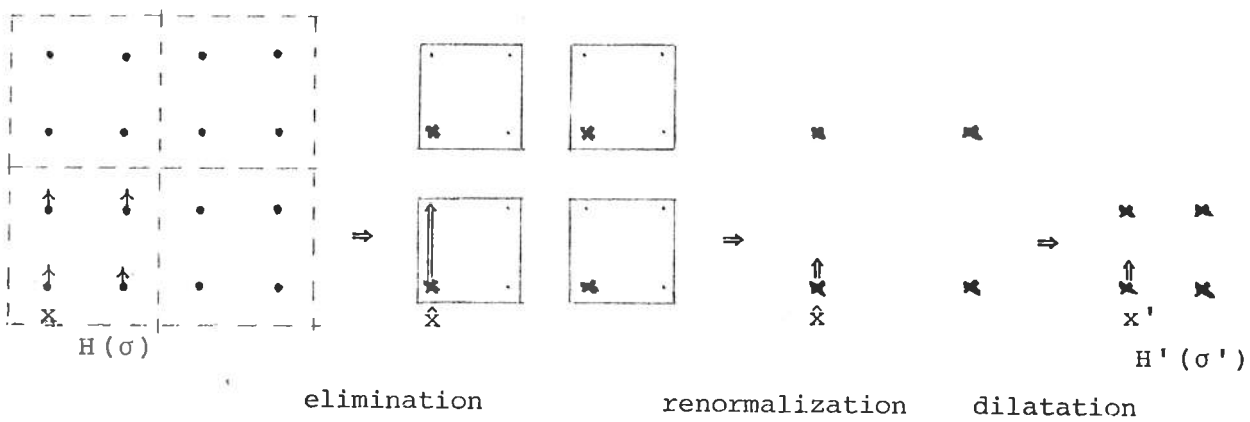


Fig. 8

The relation between the Hamiltonians of the two systems is given

by the renormalization group transformation

$$H'(\sigma') = R_n \, H(\sigma) \; .$$

Note L fixed
$x = 2^n x' \to \infty \; (n \to \infty)$

In the region $|x| > L$ for $\xi >> L > a$ we expect the two Hamiltonians to give an equivalent description of the system i.e.

$$H'(\sigma') \simeq H(\sigma) \; .$$

In particular at the critical point $\xi = \infty$ correlations over distances

$$|x| > L >> a$$

should be independent of the cell size L i.e.

$$H^{crit.}(\sigma) \simeq H'^{crit.}(\sigma') \simeq H^*$$

and thus the long range asymptote* may be calculated from an L-independent fixed point Hamiltonian.

The transformation R_n thus must have a fixed point

$$H^* = R_n \, H^* \quad ; \quad n \text{ arbitrary.}$$

The characteristic features of 2nd order phase transitions are thus described by a fixed point Hamiltonian. The renormalization group is the essential tool to construct H^* e.g. by iteration of R_1:

$$H_c \xrightarrow{R_1} H_1 \xrightarrow{R_1} H_2 \to \ldots\ldots \to H^* \; .$$

By using this iterative scheme it becomes possible to run through the whole infinite hierarchy of fluctuations. At each step only a finite number of degrees of freedom must be taken into account.

* i.e. we can about the properties of the original system $2^n L >> \xi >> |x| >> a$ i.e in the limit

1) thermodyn. Limit $2^n L \to \infty$
2) criticality $\xi \to \infty$
3) Long distance $|x| >> a$

(long distance properties of an infinite critical system)

3.2. Definition of the Block Spin RG

The precise definition of the RG transformation includes the block spin transformation, renormalization of spins, dilatation of the system and elimination of the fluctuations within cells. For a finite system with N site spins we always enlarge the system first to $N' = 2^{nd} N$ site spins in order to end up with N block spins. This step is trivial as we will always be interested in the thermodynamical limit $N \to \infty$ only.

Definition of R_n:

A. Enlargement of the system

$$N \to N' = 2^{nd} N$$

B. Block spin transformation

$$\hat{\sigma}_{\hat{x}} = \sum_{x \in (\hat{x},n)} \sigma_x \tag{3.1}$$

where we subdivided the system into cells

$$(\hat{x},n) = \{x \; ; \; \hat{x}_i \leq x_i < \hat{x}_i + 2^n a \quad i = 1,\ldots,d\}$$

$$x \in G_a \; ; \; \hat{x} \in G_{2^n a}$$

C. Renormalization

$$\sigma'_{\hat{x}} = 2^{-nx} \hat{\sigma}_{\hat{x}} \tag{3.2}$$

D. Elimination of irrelevant degrees of freedom:

The partition function

$$Z_{N'} = \sum_{\{\sigma\}} \exp - H_{N'}(\sigma)$$

$$= \sum_{\{\sigma'\}} \sum_{\{\sigma\}'} \exp - H_{N'}(\sigma) \tag{3.3}$$

$$= \sum_{\{\sigma'\}} \exp - H'_N(\sigma')$$

may be calculated as indicated by first sum over all configurations $\{\sigma\}'$ with fixed block spin configuration $\{\sigma'\}$ then sum over the block spin configurations $\{\sigma'\}$. What is summed over in the second step must be the $\exp - H_N'(\sigma')$ where $H_N'(\sigma')$ is the effective Hamiltonian of the N block spins. Thus

$$\exp - H_N'(\sigma') \doteq \sum_{\{\sigma\}'} \exp - H_{N'}(\sigma) \tag{3.4}$$

defines the block spin Hamiltonian $H_N'(\sigma')$ in terms of the site spin Hamiltonian $H_N(\sigma)$ by averaging over the cell fluctuations $\{\sigma\}'$.

E. Dilatation of the system: $\hat{x} = 2^n x'$ i.e.

$$\sigma'_{x'} = 2^{-nx} \sum_{x \in (2^n x', n)} \sigma_x \, . \tag{3.5}$$

The transformations A to E define the block spin renormalization group transformation

$$H_N'(\sigma') = R_n \, H_N(\sigma) = - \ln \sum_{\{\sigma\}'} \exp - H_{N'}(\sigma) . \tag{3.6}$$

The RG transformation defines a one parameter semi-group (group composition law without inverses)

$$\begin{aligned} R_{n_1} \cdot R_{n_2} &= R_{n_1+n_2} \; ; \; n_1, \, n_2 \in Z^+ \\ R_n &= (R_1)^n . \end{aligned} \tag{3.7}$$

Below we therefore only have to study R_1. R_n follows by iteration.

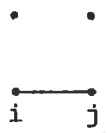
In quite general systems the Hamiltonian is characterized by a set of parameters

$$H(\sigma) = -\sum_{A \subset G_a} K_\alpha \sigma_A \tag{3.8}$$

with A a subset of lattice points including the empty set $\emptyset$ (constant term in H). σ_A denotes a product of spins at different lattice points of a set A i.e.

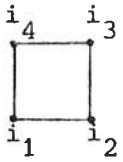
$$\sigma_A = \prod_{i \in A} \sigma_i \quad .$$

Examples:



$\sigma_i \sigma_j$ n.n. interaction

$\sigma_i \sigma_j$ n.n.n. interaction



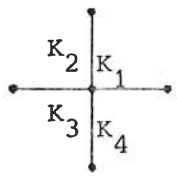
$\sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_4}$ four spin interaction

etc..

The index α denotes the type of coupling i.e. α is characterized by the set of subsets A having the same coupling

$$\alpha = \{A' \; ; \; K_{A'} = K_\alpha\} \quad .$$

Example: isotropic n.n. interaction



$$K_1 = K_2 = K_3 = K_4 = K_{n.n.} \quad .$$

In general the transformed Hamiltonian $H'(\sigma')$ can be written in the same way as $H(\sigma)$, i.e.

$$H'(\sigma') = -\sum_{A \subset G_a} K'_\alpha \; \sigma_A \tag{3.9}$$

so that the RG transformation can be written as a matrix-transformation in the parameter space characterizing the system:

$$K'_\alpha = R_{\alpha\beta}(K)\ K_\beta \quad . \tag{3.10}$$

This is in general a tremendously complicated non linear transformation. This is the prize we have to pay. By elimination of irrelevant degrees of freedom we have obtained a system with less degrees of freedom but with a much more complicated Hamiltonian e.g. if $H(\sigma)$ is a n.n. ferromagnetic Ising Hamiltonian $H'(\sigma')$ will no longer be of the n.n. Ising form. The RG transformation generates all sorts of couplings that are compatible with the symmetries of the system. However it will turn out that near the critical point $H'(\sigma')$ can be "controlled", i.e. $H'(\sigma')$ will be essentially of the n.n. Ising form. This fact makes the RG a useful concept for the study of critical behavior.

The explicit discussion of the RG in concrete examples is complicated. Before we go into these model calculations we will discuss in general the implications of the RG transformation for the critical behavior.

3.3. How to use the RG

By our definition of the RG the partition function of the 2nd N site spins may be calculated from the N block spins:

$$Z_{N2^{nd}}(K) = Z'_N(K') \quad . \tag{3.11}$$

The free energy densities

$$f_{N2^{nd}}(K) = -\frac{1}{N2^{nd}} \ln Z_{N2^{nd}}(K) \qquad \text{(site spin system)}$$

$$f'_N(K') = -\frac{1}{N} \ln Z'_N(K') \qquad \text{(block spin system)}$$

are related by

$$f_{N2^{nd}}(K) = 2^{-nd} f'_N(K')$$

or in the infinite volume limit

$$f(K) = 2^{-nd} f'(K') \quad . \tag{3.12}$$

For the correlation functions

$$\langle \sigma_{x_1} \ldots \sigma_{x_m} \rangle = \sum_{\{\sigma\}} \sigma_{x_1} \ldots \sigma_{x_m} \; e^{-H(\sigma)} \qquad \text{(site spin system)}$$

$$\langle \sigma'_{x'_1} \ldots \sigma'_{x'_m} \rangle = \sum_{\{\sigma'\}} \sigma'_{x'_1} \ldots \sigma'_{x'_m} \; e^{-H'(\sigma')} \qquad \text{(block spin system)}$$

we have a somewhat complicated relation. $\boxtimes$ indicates division by Z. Using the relation between site spin and block spin variables

$$\sigma'_{x'} = 2^{-nx} \sum_{x \in (2^n x', n)} \sigma_x \tag{3.13}$$

we may write

$$\sum_{x_1 \in (2^n x_1', n)} \cdots\cdots\cdots \sum_{x_m \in (2^n x_m', n)} \langle \sigma_{x_1} \cdots\cdots \sigma_{x_m} \rangle_K$$

$$= \sum_{\substack{\{\sigma\} \\ \boxtimes}} \sum_{x_1} \cdots\cdots \sum_{x_m} \sigma_{x_1} \cdots\cdots \sigma_{x_m} \, e^{-H(\sigma)}$$

$$= 2^{nmx} \sum_{\substack{\{\sigma'\} \\ \boxtimes}} \sigma'_{x_1'} \cdots\cdots \sigma'_{x_m'} \, e^{-H'(\sigma')} \tag{3.14}$$

$$= 2^{nmx} \langle \sigma'_{x_1'} \cdots\cdots \sigma'_{x_m'} \rangle_{K'} \; .$$

At the critical point we expect $\sum_{x \in (2^n x', n)} \sigma_x \simeq 2^{nd} \sigma_{2^n x'}$

i.e.

$$\sigma'_{x'} \simeq 2^{n(d-x)} \, \sigma_{2^n x'}$$

and hence (3.14) takes the simpler form

$$\langle \sigma_{2^n x_1'} \cdots\cdots \sigma_{2^n x_m'} \rangle_{K_c} \simeq (2^n)^{m(x-d)} \langle \sigma'_{x_1'} \cdots\cdots \sigma'_{x_m'} \rangle_{K'_c} \; .$$

4. Behavior at the Critical Point; Scaling

What are the characteristic properties of the critical correlation functions? We consider the correlation functions in the infinite volume limit $N \to \infty$. For a system with large correlation length we expect the block spin system to be essentially independent of the size of blocks L provided

$$a \ll L \ll \xi$$

i.e. the blocks are so large that the memory on the lattice (short distance behavior) is lost but they are still small compared to the correlation length. Then

$$H^{(n+m)} = (R_1)^m H^{(n)} \simeq (R_1)^n H = H^{(n)}$$

if $a \ll 2^n a < 2^{n+m} a \ll \xi$. Similarly if $m \ll n$ we expect

$$\sigma_{x'}^{(n+m)} = 2^{-mx} \sum_{x \in (2^m x', m)} \sigma_x^{(n)} \simeq 2^{-mx}\, 2^{md}\, \sigma_{2^m x'}^{(n)}$$

because the large block spins $\sigma^{(n)}$ may be considered to be exactly lined up within <u>small</u> blocks (factor 2^{md}). Notice that the renormalization (factor 2^{-mx}) must be chosen independent of the block size in case of very large blocks. Fluctuations in general make $x < d$. Hence we expect for

$$a \ll 2^n a < 2^{n+m} a \ll \xi \; ; \; m \ll n$$

$$\begin{aligned} &1.\; H^{(n+m)} \simeq H^{(n)} \\ &2.\; \sigma_x^{(n+m)} \simeq 2^{m(d-x)} \sigma_{2^m x}^{(n)} \end{aligned} \tag{4.1}$$

and if

$$2^{n+m} a < |\Delta x| \ll \xi \quad ; \quad \Delta x_{ij} = x_i - x_j$$

$$\langle \sigma_{x_1}^{(n+m)} \ldots \sigma_{x_p}^{(n+m)} \rangle_{H^{(n+m)}} \simeq 2^{m(d-x)p} \langle \sigma_{2^m x_1}^{(n)} \ldots \sigma_{2^m x_p}^{(n)} \rangle_{H^{(n)}} \tag{4.2}$$

At the critical point $\xi = \infty$ i.e. $K = K_c$ n may be chosen under the conditions above as large as we want i.e.

$$\lim_n H_c^{(n+m)} = \lim_n (R_1)^m H_c^{(n)} = \lim_n H_c^{(n)} \quad .$$

This means

$$\begin{aligned} &1. \quad R_1 \text{ has a fixed point: } R_1 H^* = H^* \\ &2. \quad H_c^{(n)} = (R_1)^{(n)} H_c \to H^* \ . \end{aligned} \tag{4.3}$$

The existence of a 2nd order phase transition is equivalent to the existence of a fixed point Hamiltonian H^* for the RG-transformation. The fixed point Hamiltonian is obtained from the critical Hamiltonian by iteration of the RG-transformation.

In parameter space (4.3) reads

$$\begin{aligned} &1. \quad R_1(K^*)\ K^* = K^* \\ &2. \quad K_c^{(n)} = R_1^n(K_c) \to K^* . \end{aligned} \tag{4.4}$$

If R_1 is given we can (in principle) determine the fixed points K^* and the critical values of K for which the sequence $K_c^{(n)}$ converges to K^*.

By the condition $|\Delta x| > 2^n a$ we can determine from H^* the <u>long range asymptote</u> of the correlation functions. So we can indeed determine the characteristic critical long range behavior from a (simple) fixed point Hamiltonian.

The parameter x defined in (3.5) has to be determined such that the two point correlation is nonsingular.

$$o < \lim_{n\to\infty} \langle \sigma_x^{(n)}\ \sigma_o^{(n)} \rangle \ < \infty. \tag{4.5}$$

We then expect the limits

$$\lim_{n\to\infty} \langle \sigma_{x_1}^{(n)} \ldots . \sigma_{x_p}^{(n)} \rangle_c = \langle \sigma_{x_1}^{*} \ldots . \sigma_{x_p}^{*} \rangle_* \tag{4.6}$$

to exist. From equation (4.2) it follows that setting $2^{-m} = \kappa$ for $n \to \infty$:

$$<\sigma^*(\frac{x_1}{\kappa}) \ldots\ldots \sigma^*(\frac{x_p}{\kappa})>_{K^*} = \kappa^{(d-x)p} <\sigma^*(x_1) \ldots\ldots \sigma^*(x_p)>_{K^*}. \quad (4.7)$$

This is the <u>scaling</u> formula for the long range tails of the correlation functions. These fixed point correlation functions are homogeneous under dilatation of the system.

Thus at the critical point the long range asymptotes of correlation functions coincide with the fixed point scaling functions i.e. for $x_i \neq x_j$ $(i \neq j)$

$$<\sigma(\frac{x_1}{\kappa}) \ldots\ldots \sigma(\frac{x_p}{\kappa})>_{K_c} \overset{\kappa \to o}{\to} \kappa^{(d-x)p} <\sigma^*(x_1) \ldots\ldots \sigma^*(x_p)>_{K^*}. \quad (4.8)$$

By comparing our result for the two point function with (2.8)

$$<\sigma(x)\ \sigma(o)>_c \simeq \frac{cst.}{|x|^{2d_\sigma}} \qquad ; \ |x| \to \infty$$

$$<\sigma(x)\ \sigma(o)>_c \simeq \kappa^{2(d-x)}\ <\sigma^*(\kappa x)\ \sigma^*(o)>_*$$

choosing κ such that $\kappa|x| = 1$ we have

$$<\sigma(x)\ \sigma(o)>_c \simeq \frac{<\sigma^*(1)\sigma^*(o)>_*}{|x|^{2(d-x)}} \quad ; \ |x| \to \infty \qquad (4.9)$$

$$\text{thus } d_\sigma = d-x = \frac{d-2}{2} + \frac{\eta}{2}\ . \qquad (4.10)$$

The part $\frac{d-2}{2}$ is called canonical part (Gaussian, mean field) of the dimension of σ. $\frac{\eta}{2}$ is the anomalous (nontrivial) part of the dimension. d_σ is called dynamical dimension of σ. So we have found the relation between the conventional critical exponent η and our renormalization exponent x

$$x = \frac{d+2-\eta}{2} \ ; \ \eta = d + 2 - 2x. \qquad (4.11)$$

5. Behavior in a Neighborhood of the Critical Point

We will now briefly consider the implications of fixed point properties for the approach to criticality. We will make some plausible assumptions that we will have to prove in model calculations. We consider a system with a critical point and assume the limit $(R_1^n)H_c \to H^*$ to exist.

i) We assume there exists a complete set of operators $\mathcal{O}_\alpha$ so that at least in a neighborhood of the critical point the Hamiltonian can be written in the form

$$H = H^* + \sum_\alpha h_\alpha \mathcal{O}_\alpha + O(h^2)$$
$$h_\alpha = f_\alpha(K-K^*) = K_\alpha - K_\alpha^* + O((K - K^*)^2) \quad . \tag{5.1}$$

ii) We assume that the RG-transformation can be linearized about H^*, i.e.

$$R_1 H = H^* + \sum_\alpha h_\alpha T_1 \mathcal{O}_\alpha + O(h^2) \tag{5.2}$$

where T_1 is a linear operator.

iii) We assume T_1 can be diagonalized i.e. there exist eigenoperators $\mathcal{O}_\alpha^*$ and eigenvalues λ_α such that

$$T_1 \mathcal{O}_\alpha^* = \lambda_\alpha \mathcal{O}_\alpha^* \quad . \tag{5.3}$$

Then

$$R_1 H = H^* + \sum_\alpha \bar{h}_\alpha \lambda_\alpha \mathcal{O}_\alpha^* + O(h^2)$$
$$= H^* + \sum_\alpha \bar{h}'_\alpha \mathcal{O}_\alpha^* + O(h^2) \quad . \tag{5.4}$$

In general the eigenvalues will turn out to be real and positive. Often they are degenerate but we will not consider this case here.

By the semi-group property

$$\lambda_\alpha^{(n_1+n_2)} = \lambda_\alpha^{(n_1)} \cdot \lambda_\alpha^{(n_2)} \tag{5.5a}$$

this implies that $\lambda_\alpha^{(n)}$ must have the form

$$\lambda_\alpha^{(n)} = 2^{ny_\alpha} . \tag{5.5b}$$

Now we are able to get a clearer picture of what means "universality" or "the effective Hamiltonian becomes simple at criticality" etc..

By using (5.4) the parameters $\bar{h}_\alpha$ transfor under the RG-transformation near the fixed point as

$$\bar{h}_\alpha^{(n)} = \bar{h}_\alpha \cdot 2^{ny_\alpha} \to \begin{cases} \infty & \text{if } y_\alpha > o \\ ? & \text{if } y_\alpha = o \\ o & \text{if } y_\alpha < o \end{cases} \tag{5.6}$$

(if y_α complex use $\mathrm{Re}\, y_\alpha$ for classification).

Accordingly the operators $\mathcal{O}_\alpha^*$ and their conjugate parameters $\bar{h}_\alpha$ are classified as

$y_\alpha > o$ relevant

$y_\alpha = o$ marginal

$y_\alpha < o$ irrelevant.

The marginal case in general deserves refined investigations. The system is <u>critical</u>, if the relevant parameters are zero ($\bar{h}_\alpha = o$ if $y_\alpha > o$). These are the parameters that we have to adjust in experiments in order to observe the phase transition. A reasonable physical system exhibiting critical behavior should have only <u>few relevant variables.</u> If in a system marginal variables are absent

$$H_c \to H^*$$

<u>has a unique fixed point</u> and there are no parameters left in the system. In particular all irrelevant variables (in general ∞-many) do not affect the critical behavior! We say the system exhibits <u>universality</u> (Kadanoff). In case there is a persistent marginal

variable the fixed point is not unique (e.g. one parameter family of fixed points) and some of the critical exponents y_α are not either. Critical behavior in this case depends on the initial system H_c and universality in its strict sense is violated although again the limit does not depend on ∞-many irrelevant details.

The transformation of the free energy density and the correlation functions in the neighborhood of the critical point then takes the form: (transformed system: length $x\kappa$, parameters $\bar{h}_\alpha \kappa^{-y_\alpha}$, field $\sigma\ \kappa^{(d-x)}$

$$f\{\bar{h}_\alpha\} \simeq \kappa^{+d}\ f\{\bar{h}_\alpha \kappa^{-y_\alpha}\} \tag{5.7}$$

$$\langle\sigma(x_1)\ldots\sigma(x_p)\rangle\{\bar{h}_\alpha\} \simeq \kappa^{p(d-x)}\ \langle\sigma(x_1\kappa)\ldots\sigma(x_p\kappa)\rangle\{\bar{h}_\alpha\kappa^{-y_\alpha}\} \tag{5.8}$$

Such a transformation law we call <u>Kadanoff scaling law</u>. It is a homogeneous substitution law, i.e. it expresses the invariance under simultaneous substitutions:

$$\begin{aligned} x &\to \kappa x \\ f &\to \kappa^d f \\ \sigma &\to \kappa^{(d-x)}\sigma \\ \bar{h}_\alpha &\to \kappa^{-y_\alpha}\bar{h}_\alpha\ . \end{aligned} \tag{5.9}$$

By (5.7) the singularities of thermodynamical quantities are associated to the presence or vanishing of the relevant variables. All critical exponents defined in (2.10) can be read off from (5.7) and are given by the $y_\alpha > o$. Corrections to the laws (5.7) and (5.8) will be discussed later on.

Wegner has introduced <u>global scaling fields</u>

$$\bar{h}_\alpha = \sum_\beta C_{\alpha\beta}(K-K^*)_\beta + \frac{1}{2}\sum_{\beta\gamma} C_{\alpha\beta\gamma}(K-K^*)_\beta(K-K^*)_\gamma + \ldots \tag{5.10}$$

which satisfy

$$\bar{h}_\alpha^{(n)} = \bar{h}_\alpha\ 2^{n y_\alpha}$$

in some global region in parameter space. The asymptotic forms (5.7) and (5.8) then become equalities that hold in the region of convergence of (5.10).

6. Critical Behavior of the Ising Model

The general structure discussed above will be illustrated now for the Ising Hamiltonian:

$$H = -k \sum_{n.n} \sigma_i\sigma_j - h\sum\sigma_i - k_1 \sum_{n.n.n} \sigma_i\sigma_j - k_2 \sum_{n.n} \sigma_i\sigma_j\sigma_k\sigma_m + \dots$$

$$= -k_c \sum_{n.n} \sigma_i\sigma_j - (k-k_c) \sum_{n.n} \sigma_i\sigma_j - h\sum\sigma_i - k_1 \sum_{n.n.n} \sigma_i\sigma_j - \dots \qquad (6.1)$$

$$= H_c - (k-k_c)\,\mathcal{O}_E - h\,\mathcal{O}_\sigma \,.$$

There are two relevant operators (fluctuation variables)

$$\mathcal{O}_E = \sum_{n.n} \sigma_i\sigma_j \qquad \text{energy}$$

$$\mathcal{O}_\sigma = \sum \sigma_i \qquad \text{magnetization} \qquad (6.2)$$

and correspondingly two relevant exponents y_E and y_σ. There is no marginal operator, i.e. there is a unique H^* so that the critical Hamiltonian is

$$H_c = H^* + \text{irrelevant terms.}$$

Accordingly critical behavior of the Ising system is universal i.e. it does not depend particularly on the next nearest neighbor (n.n.n), the four spin, and more complicated interactions. (Remark: A model showing up a persistent marginal variable is the Baxter model in field theory the Thirring model.) In Fig. 9 we depict some trajectories of Ising Hamiltonians under the action of the RG in the subspace $h = o$, $k_i = o$; $i \geq 2$.

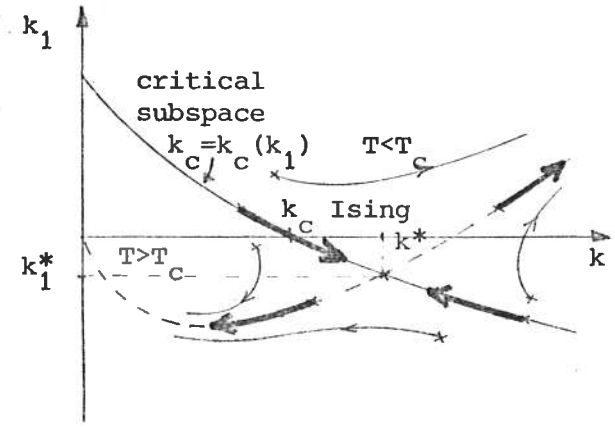


Fig. 9

In the following we ignore the irrelevant variables.

The singular part of the free energy density transforms as follows:

$$f(t,h) = \kappa^d f(t^{-y_E}, h\kappa^{-y_\sigma}) \; ; \; t = \frac{T-T_c}{T_c} . \tag{6.3}$$

Thus the <u>equation of state</u> reads

$$M = \frac{\partial f}{\partial h} = \kappa^{d-y_\sigma} \frac{\partial f}{\partial \tilde{h}} (t\kappa^{-y_E}, \tilde{h}) . \tag{6.4}$$

If we choose κ such that

$$|t| \; \kappa^{-y_E} = 1 \quad (\text{e.g. } \kappa = |t|^{1/y_E})$$

we have on the <u>coexistence curve</u> $h = o$:

$$M(t,h = o) = |t|^{\frac{d-y}{y_E}} \; M(\pm 1, o)$$

$$= \begin{cases} o & \\ (-t)^\beta \text{ const.} & t \to \pm o. \end{cases} \tag{6.5}$$

Equation (6.3) actually shows that Kadanoff's idea that critical behavior might be calculated from noncritical properties has a precise meaning. In the expression

$$M(t,h=o) = \left(\frac{t}{t_o}\right)^{\beta} \tilde{M}(t_o,o) \; ; \; t_o \text{ as reference temperature,}$$

we see that the critical singularity is isolated in the singular factor $(t/t_o)^{\beta}$ whereas $\tilde{M}(t_o,o)$ is the magnetization in a noncritical system, which is <u>analytic</u> in the parameter space (t,h) referring to a system with a finite number of degrees of freedom. This fact represents the crutial point making scaling laws like (6.5) useful.

On the <u>critical isotherm</u> $t = o$ we have, with κ chosen according to

$$|h|\kappa^{-y_\sigma} = 1 \; ,$$

$$M(o,h) = |h|^{\frac{d-y_\sigma}{y_\sigma}} M(o,\pm 1) = \text{sign}(h) \; |h|^{1/\delta} \text{ const.} \; . \qquad (6.6)$$

$$(h \to \pm o)$$

Thus

$$\beta = \frac{d-y_\sigma}{y_E} \; , \quad \delta^{-1} = \frac{d-y_\sigma}{y_\sigma} \; . \qquad (6.7)$$

On the other hand (5.8)

$$\langle\sigma\rangle(t,h) = \kappa^{d-x}\langle\sigma\rangle(t\kappa^{-y_E}, \; h\kappa^{-y_\sigma})$$

compared with (6.4) implies

$$x = y_\sigma \; . \qquad (6.8)$$

The <u>susceptibility</u>

$$\chi = \chi(t,h) = \sum_x \langle\sigma \; \sigma\rangle^c \; (|x|,t,h) \qquad (6.9)$$

$$= \kappa^{2d-y_\sigma}\kappa^{-d} \sum_x \langle\sigma \; \sigma\rangle^c(|x|,t\kappa^{-y_E},h\kappa^{-y_\sigma})$$

behaves as

$$\chi(t,o) = |t|^{\frac{d-2y_\sigma}{y_E}} \qquad \chi(\pm 1,o) = \begin{cases} t^{-\gamma}\text{const.} \\ \\ (-t)^{-\gamma'}\text{const.} \end{cases} \qquad t \to \pm o \qquad (6.10)$$

thus

$$\gamma = \gamma' = \frac{2y_\sigma - d}{y_E} \quad .$$

The <u>specific heat</u>

$$C = C(t,h) = \sum_x \langle EE\rangle^c \; (|x|,t,h) \qquad (6.11)$$

$$= \kappa^{2d-y_E} \; \kappa^{-d} \sum_x \langle EE\rangle^c \; (|x|, t\kappa^{-y_E}, h\kappa^{-y_E})$$

behaves as

$$C(t,o) = |t|^{\frac{d-2y_E}{y_E}} \qquad C(\pm 1,o) = \begin{cases} t^{-\alpha} \text{ const.} \\ \\ (-t)^{-\alpha'} \text{ const.} \end{cases} \; ; \quad t \to \pm o \quad (6.12)$$

and

$$\alpha = \alpha' = \frac{2y_E - d}{y_E} \quad .$$

The <u>correlation length</u> ξ is defined by

$$\langle\sigma\sigma\rangle^c \simeq \frac{e^{-|x|/\xi}}{x^{d-2}} \quad \text{const., for } |x| >> a. \qquad (6.13a)$$

An alternative definition that for large ξ coincides with the one above is

$$\xi^2 \doteq \sum_x x^2 \langle\sigma\sigma\rangle^c / \sum_x \langle\sigma\sigma\rangle^c \; . \qquad (6.13b)$$

From

$$<\sigma\sigma>^c (|x|,t,h) = \kappa^{2(d-y_\sigma)} <\sigma\sigma>^c (|x|\kappa, t\kappa^{-y_E}, h\kappa^{-y_\sigma}) \quad (6.14)$$

we have for $t > o$, $h = o$

$$\xi^2 = t^{-2/y_E} \sum_x x^2 <\sigma\sigma>^c (|x|,1,o,) \Big/ \sum_x <\sigma\sigma>^c (|x|,1,o) \quad (6.15)$$

$$\simeq t^{-2\nu} \text{const.} \quad (t \to o)$$

so the exponent ν is

$$\nu = 1/y_E \ . \quad (6.16)$$

As a result we make the following observations (to the extent we are able to show that the Ising Hamiltonian has a unique fixed point and there are only two (nontrivial) relevant operators):

- All critical indices α, β, γ, δ, ν, η are determined by y_E and y_σ i.e. they are related by scaling sum rules

  $$\alpha = \alpha', \quad \gamma = \gamma', \quad \nu = \nu'$$

  $$\beta(\delta-1) = 2\beta + \gamma = 2-\alpha$$

- Near criticality the singular part of the free energy density scales
- At criticality the system is described by a set of homogeneous (dilatation invariant) correlation functions and two exponents y_E and y_σ
- The approach to criticality is completely governed by the critical system (i.e. by two exponents y_E , y_σ).

7. Summary of Important Formulae:

1. Ising Hamiltonian

$$H = H_c - (k-k_c)\mathcal{O}_E - h\,\mathcal{O}_\sigma \;;\; k-k_c \propto t = \frac{T-T_c}{T_c}$$

Relevant operators

$$\mathcal{O}_E = \sum_{n.n} \sigma_i\sigma_j \qquad \text{"energy" (even)}$$

$$\mathcal{O}_\sigma = \sum \sigma_i \qquad \text{"magnetization" (odd)} \tag{7.1}$$

2. Critical Hamiltonian

$$H_c = -k_c\,\mathcal{O}_E + \text{irrelevant terms}$$

$$R_n\,H_c \underset{n\to\infty}{\to} H^* \tag{7.2}$$

3. Action of R_n near the critical point

$$\begin{aligned} x &\to \kappa x \\ f &\to \kappa^d f \\ \sigma &\to \kappa^{(d-x)}\sigma \\ E &\to \kappa^{(d-y)}E \\ t &\to \kappa^{-y}t \\ h &\to \kappa^{-x}h \\ M &\to \kappa^{(d-x)}M \end{aligned} \tag{7.3}$$

(with $y = y_E$; $x = y_\sigma$)

4. Scaling laws:

a) free energy density (singular part)

$$f(t,h) = \kappa^d f(t\kappa^{-y}, h\kappa^{-x}) \tag{7.4}$$

b) equation of state: $h = h(t,M)$

$$h(t,M) = \kappa^{-x}\,h(t\kappa^{-y}\,,\,M\kappa^{(d-x)}) \tag{7.5}$$

c) spin correlation

$$\langle\sigma\sigma\rangle\ (|x|,t,h) = \kappa^{2(d-x)}\langle\sigma\sigma\rangle(\kappa|x|,t\kappa^{-y},h\kappa^{-x}) \tag{7.6}$$

d) energy correlation

$$\langle EE\rangle\ (|x|,t,h) = \kappa^{2(d-y)}\langle EE\rangle(\kappa|x|,t\kappa^{-y},h\kappa^{-x}) \tag{7.7}$$

5. Critical exponents

$$\begin{aligned} M(t,o) &\propto \begin{cases} o \\ (-t)^{\beta} \end{cases} && t \to \pm o \\ M(o,h) &\propto \pm h^{1/\delta} \ ; && h \to \pm o \\ \chi(t,o) &\propto \begin{cases} t^{-\gamma} \\ (-t)^{-\gamma'} \end{cases} \ ; && t \to \pm o \\ C(t,o) &\propto \begin{cases} t^{-\alpha} \\ (-t)^{-\alpha'} \end{cases} \end{aligned} \tag{7.8}$$

$$\begin{aligned} \beta &= \frac{d-x}{y} \ ; \quad \delta^{-1} = \frac{d-x}{y} \\ \gamma = \gamma' &= \frac{2x-d}{y} \ ; \quad \alpha = \alpha' = \frac{2y-d}{y} \end{aligned} \tag{7.9}$$

$$\xi \propto \begin{cases} t^{-\nu} \\ (-t)^{-\nu'} \end{cases} \ ; \quad t \to \pm o \ ; \ h = o \tag{7.10}$$

$$\nu = \nu' = \frac{1}{y} \quad (y = \nu^{-1} \ ; \ x = \nu^{-1}\beta\delta) \tag{7.11}$$

6. Critical correlations

$$\begin{aligned} \langle\sigma\sigma\rangle(|x|,o,o) &\simeq \frac{\text{const.}}{|x|^{2d_\sigma}} \\ \langle EE\rangle(|x|,o,o) &\simeq \frac{\text{const.}}{|x|^{2d_E}} \end{aligned} \ ; \ |x| \to \infty \tag{7.12}$$

$$d_\sigma = \frac{d-2}{2} + \frac{\eta_\sigma}{2} = d-x \quad ; \quad \eta_\sigma = d+2-2x$$
$$d_E \doteq 2\frac{d-2}{2} + \frac{\eta_E}{2} = d-y \quad ; \quad \eta_E = 4-2y \tag{7.13}$$

7. Scaling sum rules:

$$\alpha = \alpha' \qquad \gamma = \gamma' \qquad \nu = \nu'$$
$$\beta(\delta-1) = 2\beta + \gamma = 2-\alpha$$
$$2-\alpha = d\nu \tag{7.14}$$
$$d_\sigma = \frac{\beta}{\nu} \quad ; \quad d_E = \frac{1-\alpha}{\nu} \quad .$$

Equivalent forms of scaling formulae are easily derived e.g.

4. a) free energy density (singular part)

$$f(t,h) = |t|^{2-\alpha} f(\pm 1, \frac{h}{|t|^\Delta}) \quad ; \tag{7.15}$$

$\Delta = \beta + \gamma$ gap exponent

4. b) equation of state

$$\frac{h(t,M)}{M^\delta} = h(\frac{t}{M^{1/\beta}}, 1) \tag{7.16}$$

A remark concerning scaling:
Scaling formulae have been tested by experiments e.g. we may write the equation of state as

$$\frac{h}{M|t|^\gamma} = w_\pm(\frac{M}{|t|^\beta}) \quad .$$

Scaling means that the function $w_\pm$ is a function of $M|t|^{-\beta}$ only! The data for $CrBr_3$ are shown in Fig. 10. The magnetization M measured as a function of temperature and magnetic field is plotted in the variables $M/|t|^\beta$ and $h/M|t|^\gamma$. If scaling (homogeneity) did not hold the data points would be scattered in the whole plot.

Since the data follow the homogeneity assumption, they lie on two lines corresponding to the behavior above and below T_c.

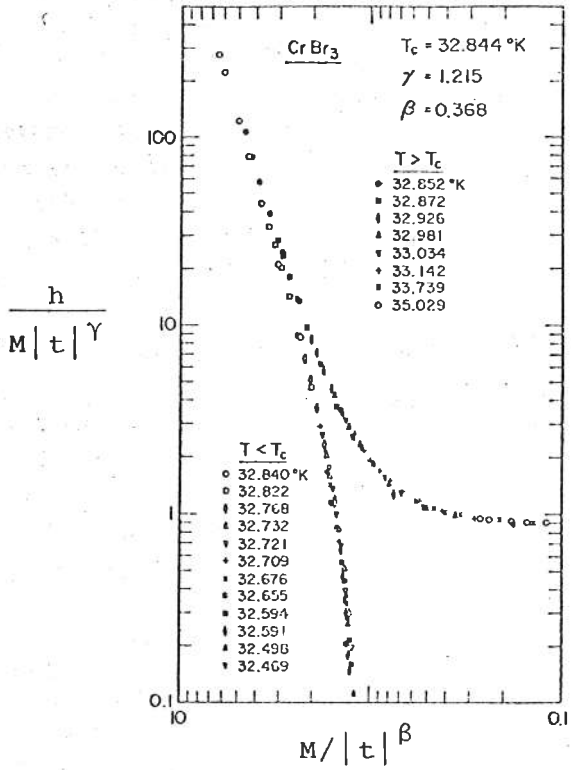


Fig.10 This plot of $h/(M|t|^{\gamma})$ against $M/|t|^{\beta}$ confirms the scaling hypothesis for $CrBr_3$. The two branches are for $T>T_c$ and $T<T_c$. After J.T. Ho and J.D. Litster, J. Appl. Phys. 40, 1270 (1969)

The validity of scaling sum rules for various models can be checked looking at the numerical values for critical exponents given in Tab. 2.

A remark concerning universality:

The Ising exponents are numbers depending only on the dimensions of the system. There are only two independent exponents and they do not depend on any details of a ferromagnetic Ising system provided it has short range interaction (there is no proof of these statements; however, as far as model calculations have been done this picture is confirmed).

Physical universality says that different physical systems (like ferromagnetic and liquid-gas systems) have identical critical behavior (critical exponents) only the names of variables change

(see Table 1). The fact that one indeed finds different physical systems with the same critical exponents can be seen from Table 3.

System	α	β	γ	δ	ν	η
CO_2		0.352±0.008	1.22±0.01	4.47±0.12		
X_e	0.04-0.06	0.35±0.07	1.232±0.02	4.6±0.1		
H_e^4	0.04-0.11	0.355±0.009	1.22±0.002	4.44±0.16		
N_i		0.374±0.016	1.35±0.02	4.45±0.1		
Gd		0.370±0.01	1.33	4.39±0.1		
$CrBr_3$		0.368	1.215	4.3		
MnF_2		0.335±0.01	1.27		0.63	0.05
fluids		0.35±0.01	1.21±0.05			
ferrom.		0.37±0.01				
binary fluids		0.342±0.016				
Ising (d=3)	0.13	0.312	1.25	≈5	≈0.63	0.03-0.05
Heisenberg (d=3)	-0.1	≈0.36	≈1.38	≈5	≈0.71	0.03-0.04

Tab. 3

The physical universality can be understood to some extent (at least intuitively) from our general discussion and from model calculations.

For theoretical physicists it is an exciting observation that within one universality class of critical theories one can find very differently looking physical model theories. It is either expected or "proved" that e.g. within the classes of quantum spin systems ferromagnetic and antiferromagnetic classical spin systems (and equivalent models), Euclidean field theory models (with or without cut-off) relavistic field theories (bare or renormalized) conformal invariant field theories and probabilistic models there are corresponding subclasses having identical critical long range behavior. This "fact" is frequently used in the study of critical behavior in that one does not investigate the model at hand but some equivalent model which is simpler or for which more advanced techniques are available.

Part II: Renormalization Groups for Lattice Systems

8. The RG for the Two Dimensional Triangular Ising Model

The study of the block spin RG in two dimensional Ising like systems provides an important test of the ideas developed in part I of these lectures.

The planar triangular Ising model is defined as follows: the plane is covered by regular triangles of side a. To every point of the triangular lattice there is assigned a site spin $s_i = \pm 1$. To every configuration $\{s\}$ of site spins there is associated an energy $H(s)$. The Hamiltonian may be of n.n. type or more generally have any isotropic ferromagnetic short ranged form.

The RG for this model was studied by Niemeyer-van Leeuwen. We closely follow their treatment. The peculiar feature of this model is that one can perform a block spin transformation with an odd (and small) number of spins. As we will see this makes life much easier as compared to the square lattice system. We now define the RG for this system. Block spins are formed as depicted in Fig. 11.

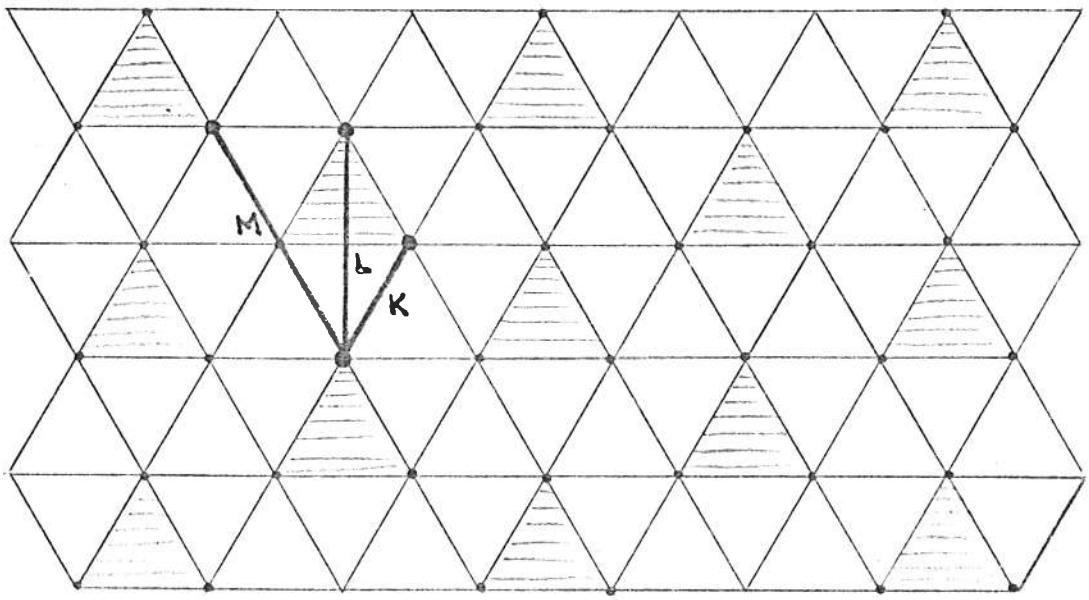


Fig. 11

The site spin index is i the cell spin index i'. s_i is the site spin and $s'_{i'}$ denotes the cell spin. Inside a cell the site spins are labeled as $s^1_{i'}$, $s^2_{i'}$, $s^3_{i'}$. The renormalized cell spin can be defined by

$$s'_{i'} = \text{sign}\,(s^1_{i'} + s^2_{i'} + s^3_{i'}) \; . \tag{8.1}$$

This transformation is unique because of the odd number of site spins within a cell. For a given $s'_{i'}$ the site spins $s^1_{i'}$, $s^2_{i'}$, $s^3_{i'}$ can assume four different configurations. They may be labeled by a fluctuation variable $\sigma_{i'}$ as follows:

$\sigma_{i'}$	$s^1_{i'}$	$s^2_{i'}$	$s^3_{i'}$
o	$s'_{i'}$	$s'_{i'}$	$s'_{i'}$
1	$-s'_{i'}$	$s'_{i'}$	$s'_{i'}$
2	$s'_{i'}$	$-s'_{i'}$	$s'_{i'}$
3	$s'_{i'}$	$s'_{i'}$	$-s'_{i'}$

(8.2)

The site spin configurations $\{s_i\}$ are thus alternatively specified by $\{s'_{i'}, \sigma_{i'}\}$.

The RG-transformation is then defined by

$$\exp - H'(s') = \sum_{\{\sigma\}} \exp - H(s',\sigma) . \tag{8.3}$$

We write the site spin Hamiltonian in the form

$$H(s) = - \sum_A K_\alpha \, S_A \quad ; \quad S_A = \prod_{i \in A} s_i \tag{8.4}$$

Analogously the cell spin Hamiltonian may be decomposed as

$$H'(s') = - \sum_{A'} K'_\alpha \, S'_A \quad . \tag{8.5}$$

Here $A \subset G_a$ and $A' \subset G_{a\sqrt{3}}$ are subsets on the lattices. α denotes the type of interaction as: constant, nearest neighbor, next nearest neighbor, threespin, etc.. We assume the interactions to be isotropic, ferromagnetic and short ranged.

We now define a simple perturbation expansion for the calculation of the RG-transformation

$$K'_\alpha = K'_\alpha(K) \quad . \tag{8.6}$$

(An exact evaluation of this transformation is more complicated than the solution of the Ising model due to the appearance of arbitrary types of interactions.)

The perturbation expansion is based on a separation of H into a part H_o which can be handled exactly and a remainder V:

$$H = H_o + V \quad . \tag{8.7}$$

We then define averages $< >_o$ with respect to the Hamiltonian H_o by:

$$<A>_o = \sum_{\{\sigma\}} A(s',\sigma)\, e^{-H_o(s',\sigma)} \Big/ \sum_{\{\sigma\}} e^{-H_o(s',\sigma)} \quad . \tag{8.8}$$

The RG-transformation (8.3) can now be written in the form

$$\begin{aligned} e^{-H'(s')} &= \sum_{\{\sigma\}} e^{-H_o(s',\sigma)} \cdot \sum_{\{\sigma\}} e^{-V(s',\sigma)}\, e^{-H_o(s',\sigma)} \Big/ \sum_{\{\sigma\}} e^{-H_o(s',\sigma)} \\ &= \Big(\sum_{\{\sigma\}} e^{-H_o}\Big) <e^{-V}>_o \quad . \end{aligned} \tag{8.9}$$

We will use the cumulant expansion

$$\ln <e^{-V}>_o = \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} <V^n>_o^c \tag{8.10}$$

with

$$\langle V\rangle_o^c = \langle V\rangle_o$$
$$\langle V^2\rangle_o^c = \langle V^2\rangle_o - \langle V_o\rangle^2 \qquad (8.10a)$$

etc..

The perturbation expansion for H'(s') takes the form

$$-H'(s') = \ln\left(\sum e^{-H_o}\right) + \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \langle V^n\rangle_o^c \quad . \qquad (8.11)$$

The main problem is a manageable and physically reasonable choice for H_o and V. A natural choice is

H_o contains all intra cell interactions

V contains all inter cell interactions.

We then easily evaluate

$$-H_o = -\sum_{i'} H_{oi'} = K \sum_{i'} (s_{i'}^1 s_{i'}^2 + s_{i'}^2 s_{i'}^3 + s_{i'}^3 s_{i'}^1)$$
$$= K \sum_{i'} (4\delta_{\sigma_{i'},o} - 1) \qquad (8.12)$$

with K the n.n. interaction in H(s). Notice that H_o only depends on σ. Thus the first factor in (8.9) is

$$\sum_{\{\sigma\}} e^{-H_o(\sigma)} = \prod_{i'} \sum_{\sigma_{i'}} e^{-H_{oi'}(\sigma)} = (z_o)^N \qquad (8.13)$$

where

$$z_o = e^{3K} + 3\, e^{-K} \qquad (8.14)$$

and N the number of blocks of the finite system.

1. The 1st order approximation

a) Zero external field

We calculate the first order term in V in the expansion (8.11). We will assume first that H(s) has n.n. interactions only. A typical interaction between two adjacent cells i' and j' (Fig. 12) is given by

$$V_{i'j'} = -K(s^2_{i'} + s^3_{i'})s^1_{j'} \quad . \tag{8.15}$$

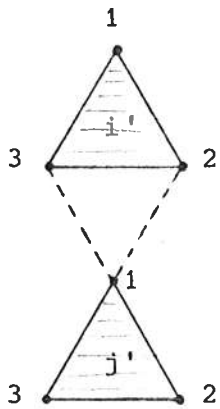


Fig. 12

For the calculation of expectation values of $V_{i'j'}$ we need the averages

$$f_1 = s'_{i'}\langle s^\alpha_{i'}\rangle_o \qquad = z_o^{-1}\, s'_{i'} \sum_{\{\sigma\}} s^\alpha_{i'}\, e^{K(4\delta_{\sigma,o}-1)}$$

$$= z_o^{-1}\, s'_{i'}\, s'_{i'}(e^{3K}+e^{-K}) = z_o^{-1}(e^{3K}+e^{-K}) \qquad \alpha=1,2,3$$

$$f_2 = \langle s^\alpha_{i'}\, s^\beta_{i'}\rangle_o \qquad = z_o^{-1} \sum_{\{\sigma\}} s^\alpha_{i'}\, s^\beta_{i'}\, e^{K(4\delta_{\sigma,o}-1)} \tag{8.16}$$

$$= z_o^{-1}\,(e^{3K} - e^{-K}) \qquad \alpha\neq\beta=1,2,3$$

$$f_3 = s'_{i'}\langle s^1_{i'}s^2_{i'}s^3_{i'}\rangle_o \qquad = z_o^{-1}\, s'_{i'} \sum_{\{\sigma\}} s^1_{i'}s^2_{i'}s^3_{i'}\, e^{K(4\delta_{\sigma,o}-1)}$$

$$= z_o^{-1}\, s'_{i'}s'_{i'}(e^{3K}-3e^{-K}) = z_o^{-1}\,(e^{3K} - 3e^{-K}) \quad .$$

With these expressions we find for the n.n. cells

$$\langle V_{i'j'}\rangle_o = -K \quad \langle s^2_{i'} + s^3_{i'}\rangle_o \langle s'_{j'}\rangle_o = -2f_1^2 \, K \, s'_{i'} \, s'_{j'} \tag{8.17}$$

and hence

$$\langle V\rangle_o = \sum_{\langle i'j'\rangle} \langle V_{i'j'}\rangle_o = -2f_1^2 \, K \sum_{\langle i'j'\rangle} s'_{i'} \, s'_{j'} \quad . \tag{8.17a}$$

To first order the cell spin Hamiltonian (8.11) then reads:

$$H'(s') = -f'_o - \sum_{\langle i'j'\rangle} K' \, s'_{i'} \, s'_{j'} = - N \ln z_o - 2f_1^2 \, K \sum_{\langle i'j'\rangle} s'_{i'} \, s'_{j'} \tag{8.18}$$

and the RG-transformation in this approximation is

$$\begin{aligned} f'_o &= N \ln z_o \\ K' &= 2f_1^2 \, K \quad . \end{aligned} \tag{8.19}$$

Note that in this first approximation the block spin Hamiltonian is again of n.n. form.

b) <u>Non zero external field</u>

In the presence of an external magnetic field we have to consider the replacement

$$V \to V_e + h \, \mathcal{O}_s \; ; \quad \mathcal{O}_s = \sum_i s_i \tag{8.20}$$

with V_e the even n.n. potential considered above. In the cumulant expansion

$$\begin{aligned} \langle V\rangle_o^c &= \langle V_e\rangle_o + h\langle \mathcal{O}_s\rangle \\ \langle V^2\rangle_o^c &= \langle (V_e + h\mathcal{O}_s)^2\rangle_o - \langle V_e + h\mathcal{O}_s\rangle_o^2 \\ &= (\langle V_e^2\rangle_o - \langle V_e\rangle_o^2) + 2h(\langle V_e \mathcal{O}_s\rangle_o - \langle V_e\rangle_o \langle \mathcal{O}_s\rangle_o) \\ &\quad + h^2(\langle \mathcal{O}_s^2\rangle_o - \langle \mathcal{O}_s\rangle_o^2) \quad . \end{aligned} \tag{8.21}$$

To first order in h thus

$$-\langle V\rangle_o^c + \frac{1}{2}\langle V^2\rangle_o^c + \ldots = -\langle V_e\rangle_o - h[\langle \mathcal{O}_s\rangle_o - (\langle V_e \mathcal{O}_s\rangle_o - \langle V_e\rangle_o\langle \mathcal{O}_s\rangle_o)] + .$$

$$= -\langle V_e\rangle_o - h\langle \mathcal{O}_s\rangle_o + h\langle V_e \mathcal{O}_s\rangle_o^c + \ldots\ldots \tag{8.22}$$

We evaluate:

$$\langle \mathcal{O}_s\rangle_o = \sum_{i'} \langle s^1_{i'} + s^2_{i'} + s^3_{i'}\rangle_o = 3\, f_1 \sum_{i'} s'_i$$

$$\langle V_e \mathcal{O}_s\rangle_o^c = \sum_{l'} \sum_{\langle i'j'\rangle} \{\langle V_{i'j'}(s^1_{l'} + s^2_{l'} s^3_{l'})\rangle_o - \langle V_{i'j'}\rangle_o \langle s^1_{l'} + s^2_{l'} + s^3_{l'}\rangle_o\}$$

$$= -K \sum_{l'} \sum_{\langle i'j'\rangle} \{\langle (s^2_{i'} + s^3_{i'}) s^1_{j'}\ (s^1_{l'} + s^2_{l'} + s^3_{l'})\rangle_o - \langle (s^2_{i'} + s^3_{i'}) s^1_{j'}\rangle_o \langle \cdot\rangle_o\}$$

$$= 3\, f_1 4Ka \sum_{i'} s'_{i'} \tag{8.23}$$

with $a = 1 + 2f_2 - 3f_1^2$.

The only contributions come from l' = i' or j' i.e.

$$\sum_{\langle i'j'\rangle} \{\langle (s^2_{i'} + s^3_{i'})(s^1_{i'} + s^2_{i'} + s^3_{i'})\rangle_o \langle s^1_{j'}\rangle_o + \langle s^2_{i'} + s^3_{i'}\rangle_o \langle s^1_{j'}(s^1_{j'} + s^2_{j'} + s^3_{j'})\rangle_o$$

$$-\langle s^2_{i'} + s^3_{i'}\rangle_o \langle s^1_{i'} + s^2_{i'} + s^3_{i'}\rangle_o \langle s^1_{j'}\rangle_o - \langle s^2_{i'} + s^3_{i'}\rangle_o \langle s^1_{j'}\rangle_o \langle s^1_{j'} + s^2_{j'} + s^3_{j'}\rangle_o\}$$

$$= \sum_{\langle i'j'\rangle} \{\langle s^2_{i'} s^1_{i'}\rangle_o \langle s^1_{j'}\rangle_o + \langle s^2_{i'}\ s^2_{i'}\rangle_o \langle s^1_{j'}\rangle_o + \ldots$$

$$+ \langle s^2_{i'}\rangle_o \langle s^1_{i'}\ s^1_{j'}\rangle_o + \langle s^2_{i'}\rangle_o \langle s^1_{j'}\ s^2_{j'}\rangle_o + \ldots$$

$$-\langle s^2_{i'}\rangle_o \langle s^1_{i'}\rangle_o \langle s^1_{j'}\rangle_o - \langle s^2_{i'}\rangle_o \langle s^2_{i'}\rangle_o \langle s^1_{j'}\rangle_o - \ldots$$

$$-\langle s^2_{i'}\rangle_o \langle s^1_{j'}\rangle_o \langle s^1_{j'}\rangle_o - \langle s^2_{i'}\rangle_o \langle s^1_{j'}\rangle_o \langle s^2_{j'}\rangle_o - \ldots\}$$

$$= \sum_{<i'j'>} \{f_1 f_2\, s'_{j'} + f_1\, s'_{j'} + \ldots$$

$$+ f_1\, s'_{i'} \quad + f_1 f_2\, s'_{i'} + \ldots$$

$$- f_1^3\, s'_{j'} \quad - f_1^3\, s'_{j'} - \ldots$$

$$- f_1^3\, s'_{i'} \quad - f_1^3\, s'_{j'} - \ldots \qquad \}$$

$$= 3 \sum_{i'} s'_{i'} \{f_1 f_2 + f_1 + \ldots + f_1 + f_1 f_2 + \ldots \quad - f_1^3 - f_1^3 - \ldots - f_1^3 - f_1^3 \ldots\}$$

$$= 3 \cdot 2 (2f_1 + 4f_1 f_2 - 6f_1^3) \sum_{i'} s'_{i'}$$

by symmetry in i' and j' and as each cell has 3 n.n. cells in a half plane.

Thus the additional term $\Delta H'(s')$ is again linear (single site → single cell) and according to (8.11)

$$\Delta H'(s') = - h' \sum_{i'} s'_{i'} = - 3f_1(1+4Ka)h \sum_{i'} s'_{i'} \tag{8.24}$$

or

$$h' = 3f_1(1+4Ka)h \ . \tag{8.25}$$

2. <u>The 2nd order approximation</u>

We consider only the case with zero external field. We have to evaluate

$$<V^2>_o^c = <V^2>_o - <V>_o^2 \ .$$

The <u>n.n. contribution</u> is given by

$$\sum_{<i',j'>} \sum_{<k',l'>} <(V_{i'j'} - <V_{i'j'}>_o)\ (V_{k'l'} - <V_{k'l'}>_o)>_o$$

with $V_{i'j'}$ from (8.15). Terms contribute only when <i'j'> and

$< k',l'>$ have cells in common since otherwise

$$<V_{i'j'}\ V_{k'l'}>_o = <V_{i'j'}>_o\ <V_{k'l'}>_o\ .$$

A constant contribution arises from the case $<i',j'> = <k',l'>$:

$$\sum_{<i'j'>} <V^2_{i'j'}>^c_o = \sum_{<i'j'>} \{K^2<(s^2_{i'}+s^3_{i'})^2>_o<(s^1_{j'})^2>_o - <V_{i'j'}>^2_o\} \tag{8.26}$$

$$= 2K^2(1-2f_1^4 + f_2)\ \bar{N}$$

with $\bar{N} = \sum_{<i'j'>}$ = number of n.n. cells in the system.

The other possible arrangements are of the type

$$<V_{i'j'}\ V_{j'k'}>^c_o = <V_{i'j'}\ V_{j'k'}>_o - <V_{i'j'}>_o<V_{j'k'}>_o$$

and are depicted in Fig. 13.

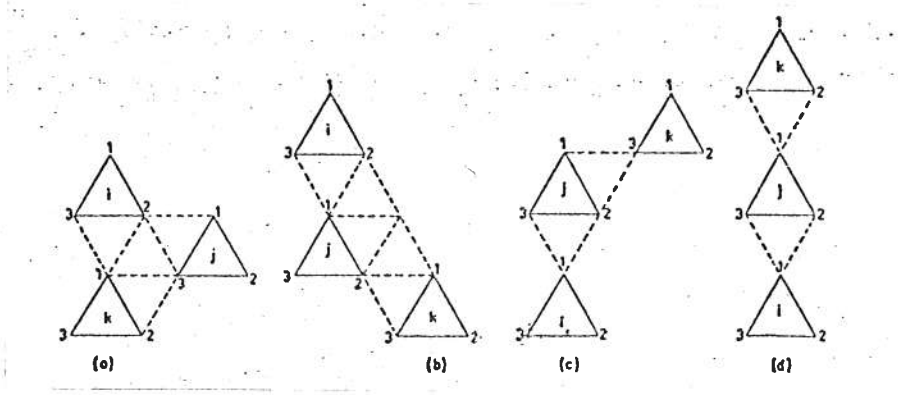


Fig. 13

It follows that apart from the n.n. coupling there is generated a next n.n. interaction L' and a third neighbor coupling M':

configuration	interaction
a	n.n
b	n.n.n
c	n.n.n
d	3rd n.n .

A straightforward calculation of the corresponding terms leads to the 2nd order contributions

$$(K')_2 = 4(1-2f_1^2 + f_2)f_1^2 K^2$$

$$(L')_2 = (1-8f_1^2 + 7f_2)\ f_1^2 K^2 \qquad (8.27)$$

$$(M')_2 = 4(-f_1^2 + f_2)\ f_1^2 K^2$$

Thus from the n.n coupling the RG generates in second order new "second order" couplings L' and M'. K' is considered as a first order quantity.

If 2nd n.n and 3rd n.n couplings are present in the site spin system they contribute also to first order $\langle V\rangle_o$. For the configuration Fig. 12 one finds:

$$-V_{i'j'} = L(s^1_{i'}\ s^1_{j'} + s^2_{i'}\ s^2_{j'} + s^3_{i'}\ s^3_{j'}) + M(s^2_{i'}\ s^3_{j'} + s^3_{i'}\ s^2_{j'}). \qquad (8.28)$$

Accordingly one evaluates the additional contributions

$$(K')_1 = 3f_1^2\ L + 2f_1^2\ M$$

$$(L')_1 = \qquad f_1^2\ M \qquad (8.29)$$

$$(M')_1 = o$$

Collecting the various contributions the full second order RG-transformation reads

$$
\begin{aligned}
K' &= 2f_1^2\, K + 4(1-2f_1^2 + f_2)f_1^2\, K^2 + 3f_1^2\, L + 2f_1^2\, M \\
L' &= \qquad\qquad (1-8f_1^2 + 7f_2)f_1^2 K^2 \qquad\qquad + f_1^2\, M \\
M' &= \qquad\qquad 4(-f_1^2 + f_2)f_1^2\, K^2
\end{aligned}
\tag{8.30}
$$

Fixed points:

The fixed points of the transformations (8.6)

$$K_\alpha^* = K_\alpha^*(K^*) \tag{8.31}$$

can be calculated (numerically) for the 1st and 2nd order approximations to the RG-equations (8.19) and (8.30). In both cases one finds a nontrivial fixed point (we do not consider the f_o' term contributing only to the regular part of the free energy density).

	1st order	2nd order	exact Ising
K^*	0.3356	0.2789	$\ln 3/_4 = 0.2744$
L^*	0	-0.0143	0
M^*	0	-0.0152	0
h^*	0	0	0

(8.32)

Linearization, critical exponents:

We now linearize the RG-transformation (8.6) near the fixed point (8.31)

$$K_\alpha' = K_\alpha^* + \sum_\beta \left(\frac{\partial K_\alpha}{\partial K_\beta}\right)_{K=K^*} (K_\beta - K_\beta^*) + O\left((K-K^*)^2\right). \tag{8.33}$$

The matrix

$$T_{\alpha\beta} = \left(\frac{\partial K'_\alpha}{\partial K_\beta}\right)_{K=K^*} = T^e_{\alpha\beta} + T^o_{\alpha\beta} \tag{8.34}$$

consists of an even (zero external magnetic field) and an odd (external magnetic field) part as the RG transforms even into even odd into odd terms. $T_{\alpha\beta}$ can be calculated explicitly from (8.19), (8.25), (8.30) with (8.32).

$$T^{e(1)}_{\alpha\beta} = (1.634) \; ; \; T^{o(1)}_h = (3.03)$$

$$T^{e(2)}_{\alpha\beta} = \begin{pmatrix} 1.8966 & 1.3446 & 0.8964 \\ -0.0403 & 0.0 & 0.4482 \\ -0.0782 & 0.0 & 0.0 \end{pmatrix}. \tag{8.35}$$

The matrix $T_{\alpha\beta}$ can be diagonalized by a change of parameters. Let

$$\begin{aligned} g_\beta &= \sum_\alpha C_{\beta\alpha} \; (K_\alpha - K^*_\alpha) \\ g'_\beta &= \sum_\alpha C_{\beta\alpha} \; (K'_\alpha - K^*_\alpha) \end{aligned} \tag{8.36}$$

be a linear transformation in parameter space. Choose $C_{\beta\alpha}$ such that (eigenvalue problem)

$$\sum_\alpha C_{\beta\alpha} \; T_{\alpha\gamma} = \lambda_\beta C_{\beta\alpha} \tag{8.37}$$

i.e. $C_{\beta\alpha}$ are the (left) eigenvectors and λ_β the corresponding eigenvalues of the matrix T. We will normalize the eigenvectors by

$$C_{\alpha\alpha} = 1 \; . \tag{8.37a}$$

Then the parameters g transform in the linear approximation

$$g'_\alpha = \lambda_\alpha \; g_\alpha \quad . \tag{8.38}$$

The eigenvalues and eigenvectors for the above transformations are

$$\lambda_E^{(1)} = 1.634 \qquad c_{E\alpha}^{(1)} = 1$$

$$\lambda_\sigma^{(1)} = 3.03 \qquad c_{\sigma\alpha}^{(1)} = 1$$

$$\lambda_E^{(2)} = 1.7835 \qquad c_{1\alpha}^{(2)} = (1,\ 0.7539,\ 1.0961)$$

$$\lambda_2^{(2)} = 0.2286 \qquad c_{2\alpha}^{(2)} = (\ldots\ldots,1,\ldots\ldots) \tag{8.39}$$

$$\lambda_3^{(2)} = -0.1156 \qquad c_{3\alpha}^{(2)} = (\ldots\ldots,\ \ldots\ldots,\ 1) \quad .$$

This result confirms our earlier discussion on the Ising model. There is (in 1st and 2nd order approximation) only one even eigenvalue bigger than one and there is one odd eigenvalue bigger then one. They correspond to the two relevant operators $\mathcal{O}_E = \sum_{n.n} s_i s_j$ and $\mathcal{O}_\sigma = \sum s_i$ (see chapter 7). The other terms are irrelevant (to this order).

Because of the semigroup property we must have

$$\lambda_\alpha = l^{y_\alpha} \tag{8.40}$$

where $l = \sqrt{3}$ is the dilatation factor. Thus

$$y_\alpha = \frac{\ln\lambda_\alpha}{\ln l} \tag{8.41}$$

and the two independent exponents $x = y_\sigma$ and $y = y_E$ (see sections 6 and 7) are determined by λ_σ and λ_E.

	1st order	2nd order	exact
x	2.018	-	$\frac{15}{8} \simeq 1.875$
y	0.894	1.053	1

(8.42)

All the critical exponents may be determined then, using the scaling relations (7.14).

Critical surface:

As mentioned above the matrix $T_{\alpha\beta}$ breaks up into an even-even block and an odd-odd block:

$$T \begin{pmatrix} \Delta K^e \\ \\ \Delta K^o \end{pmatrix} = \begin{pmatrix} T^e & o \\ \\ o & T^o \end{pmatrix} \begin{pmatrix} \Delta K^e \\ \\ \Delta K^o \end{pmatrix} = \begin{pmatrix} T^e & \Delta K^e \\ \\ T^o & \Delta K^o \end{pmatrix} . \tag{8.43}$$

The critical surface is defined by

$$g_E = o \quad \text{and} \quad g_\sigma = o \tag{8.44}$$

corresponding to zero reduced temperature and zero external field. All other parameters are irrelevant. We may write the Hamiltonian near the fixed point in terms of the scaling parameters g_α:

$$\begin{aligned} H(s) &= H^*(s) + \sum_\alpha (K_\alpha - K_\alpha^*) \sum_{A_\alpha} S_A + O(\Delta K^2) \\ &= H^*(s) + \sum_\beta g_\beta \, \mathcal{O}_\beta^* + O(g^2) \end{aligned} \tag{8.45}$$

where we introduced eigenoperators

$$\mathcal{O}_\beta^* = \sum_\alpha C_{\alpha\beta}^{-1} \sum_{A_\alpha} S_{A_\alpha} \tag{8.46}$$

$C_{\alpha\beta}^{-1}$ is the inverse of the matrix $C_{\alpha\beta}$ in (8.36) and

$$K_\alpha - K_\alpha^* = \sum_\beta C_{\alpha\beta}^{-1} g_\beta . \tag{8.47}$$

The critical Hamiltonian is

$$H(s)_c = H^*(s) + \sum_\beta g_\beta \, \mathcal{O}_\beta^* + O(g^2) \tag{8.48a}$$

with $g_E = g_\sigma = o$ such that

$$R_1 \; H_c = H^* + \sum_\beta \lambda_\beta \; g_\beta \; \mathcal{O}^*_\beta + O(g'^2) \tag{8.48b}$$

with $|\lambda_\beta| < 1$ all β and

$$(R_1)^n \; H_c = H^* + \sum_\beta \; (\lambda_\beta)^n \; g_\beta \; \mathcal{O}^*_\beta + O(g^{(n)2}) \underset{(n\to\infty)}{\to} H^* \quad . \tag{8.48c}$$

In the "physical" parameter space K_α the critical surface is in linear approximation

$$\begin{aligned} g_E &= o = \sum_\alpha C^e_{E\alpha} \; (K_\alpha - K^*_\alpha) \\ g_\sigma &= o = \sum_{\alpha \text{ odd}} C^o_{\sigma\alpha} (K_\alpha - K^*_\alpha) \quad . \end{aligned} \tag{8.49}$$

The second condition requires h = o. In the even coupling space (8.49) contains the variation of the critical temperature with the interaction parameters. g_E is a generalized reduced temperature.

For the n.n. system K_α = o except for α = n.n.; then the condition (8.49) reads:

$$K_c = \sum_\alpha C^e_{E\alpha} \; K^*_\alpha = \begin{cases} 0.3356 & 1^{st} \text{ order} \\ 0.251 & 2^{nd} \text{ order} \end{cases} \quad . \tag{8.50}$$

The critical temperature T_c for the n.n. system is defined by

$$K_c = \frac{J}{k_B T_c} \; ; \; J = \text{n.n Ising coupling.} \tag{8.51}$$

We define $T_c(J)$ by

$$K^*_\alpha = \frac{J_\alpha}{k_B T_c(J)} \quad . \tag{8.52}$$

We can thus find the critical temperature $T_c(J)$ of a general (not n.n.) system in the linear approximation:

$$T_c(J) = T_c \; \{1 + \sum_{\alpha \neq \text{n.n}} C^e_{E\alpha} \; \frac{J_\alpha}{J} \} \quad . \tag{8.53}$$

Near the critical point in the n.n. system

$$g_E = K_{n.n} - \sum_\alpha C^e_{E\alpha} K^*_\alpha = K_{n.n} - K_c$$
$$= - \frac{J}{k_B T} \frac{T-T_c}{T_c} . \qquad (8.54)$$

i.e. in the Ising system g_E is proportional to the reduced temperature. For the even couplings subspace we depict the critical surface in Fig. 14.

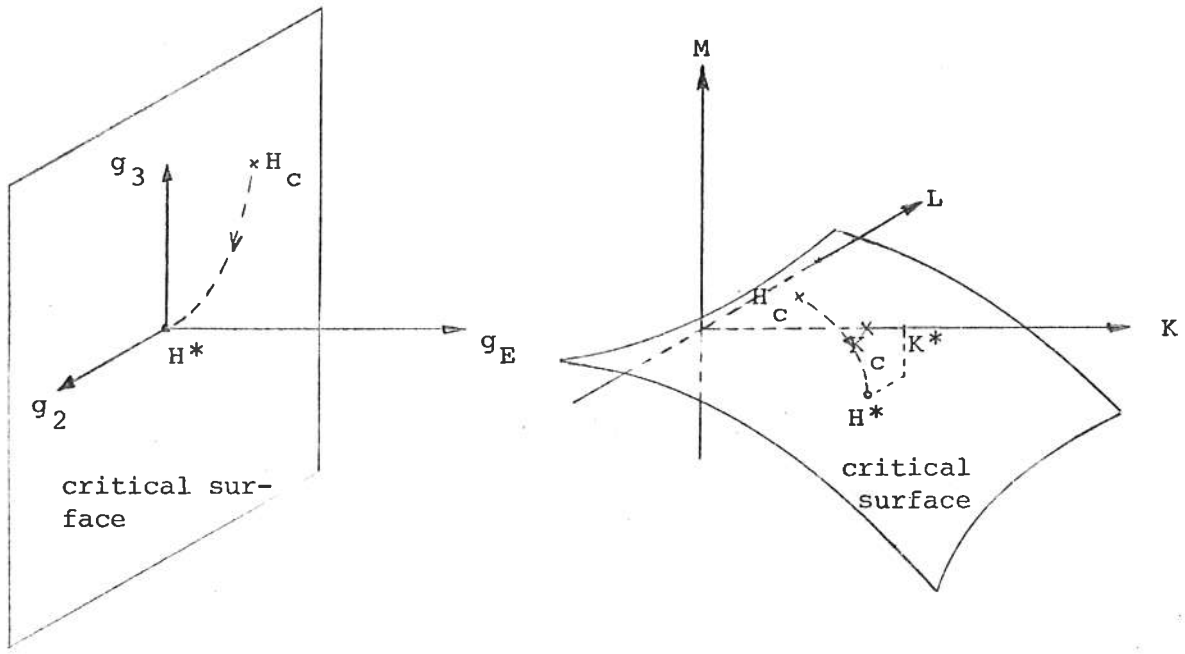


Fig. 14

In the simple perturbation expansion considered so far one essentially uses K^* as an expansion parameter. As the values for K^* are not really small one cannot expect fast convergence of this expansion. Niemeyer-van Leeuwen therefore developed a cluster approximation not based on the smallness of K but on the fact that the short ranged interactions are most important. Roughly speaking one does not expand $\langle \exp -V \rangle_o$ in (8.9) in powers of V

but instead tries to calculate exactly the effective Hamiltonian

$$e^{-H'(s'_{i'_1},\ldots,s'_{i'_n})} = \sum_{\sigma_{i'_1},\ldots,\sigma_{i'_n}} e^{-H(s'_{i'_1}\sigma_{i'_1},\ldots,s'_{i'_n}\sigma_{i'_n})} \tag{8.55}$$

$$= z_o^n \langle \exp -V_n \rangle_o$$

for a cluster of n adjacent cells. Due to combinatorial complications some systematic approximations have to be made and are possible. We give the result in Tab. 4.

# of cells in cluster	λ_T	λ_H	K_C
2	1.544	3.036	0.365
3	1.501	2.501	0.255
4	1.567	2.497	0.253
5	1.782	2.850	0.281
7	1.7590	2.8024	0.27416
Exact (Ising)	$3^{1/2} = 1.73205$	$3^{15/16} = 2.80092$	0.27465

On the basis of the λ_T and λ_H in the five-cell cluster approximation we find e.g., the critical exponents $\nu = 0.973$ and $\delta = 15.017$

Tab. 4

The results obtained by this method are remarkably encouraging and confirm many of the intuitive ideas on critical behavior and the renormalization group.

Notice that the critical exponents are independent of the point in the critical surface. Also they coincide within our approximation with square lattice Ising values. These facts are a manifestation of universality.

The methods used here to investigate critical behavior do not make use of the special properties that make a model exactly solvable and hence can in principle be directly applied to a large class of models.

We should also keep in mind that the methods using the exact solution of Onsager for the evaluation of say the spin-spin correlation at the critical point are extremely involved (see e.g. McCoy and Wu). In contrast the RG method seems to lead to rather good approximate results at much less technical expense and this not only for the n.n. Ising model.

9. RG-Calculations for Other Ising Models

We will not comment on the diverse RG-investigations of the one dimensional Ising model. The d=1 Ising system with short range interactions has a phase transition (of peculiar structure) only at T = o. More interesting are the d=1 Ising chains with long range interactions in particular the so called hierarchical models which are equivalent to Wilson's approximate RG models (field theory version).

Of great importance are the calculations on the planar (d=2) square lattice Ising models as they provide an important test of methods that may be used for d=3 Ising systems (and possibly also for rotator and Heisenberg models). For geometrical reasons the Δ-lattice calculations cannot be directly generalized to include other models.

By universality critical exponents for Δ- and □-lattices in d=2 should coincide not, however, the values for fixed point couplings etc..

We now briefly mention some results obtained from RG analysis of planar square lattice systems.

A straightforward generalization of the Niemeyer-van Leeuwen technique from planar Δ- to □-lattice was investigated by Nauenberg-Nienhuis. The 2x2-block spin variables are defined as follows:

For the ten configurations with $\sum s_i \neq o$:

$$s'_{i'} = \text{sign} \sum s_i \quad \text{for} \quad \sum s_i \neq o.$$

There are however six configurations with $\sum s_i = o$. By definitions one assigns to three of them (there are four equivalent choices) $s'_{i'} = 1$ to the others $s'_{i'} = -1$. e.g.

$$(+\ +\ -\ -)\ ,\ (+\ -\ +\ -)\ ,\ (+\ -\ -\ +) \Rightarrow s'_{i'} = 1$$

$$(-\ -\ +\ +)\ ,\ (-\ +\ -\ +)\ ,\ (-\ +\ +\ -) \Rightarrow s'_{i'} = -1.$$

For a finite 4x4 square Ising lattice Nauenburg-Nienhuis numerically calculated the RG-transformation. The result obtained is

Fixed point: $K_1^* = 0.307$, $K_2^* = 0.084$, $K_3^* = -0.004$

Critical Ising coupling: $K_* = 0.420$, $K_2 = 0$, $K_3 = 0$

Eigenvalues: $\lambda_1 = 1.914$, $\lambda_2 = 0.248$, $\lambda_3 = 0.137$

with K_1 n.n K_2 n.n.n and K_3 four spin coupling.

The results on numerical calculations of the free energy density and the first two derivatives are plotted in Fig. 15. The critical surface in the $(K_1K_2K_3)$-space is shown in Fig. 16.

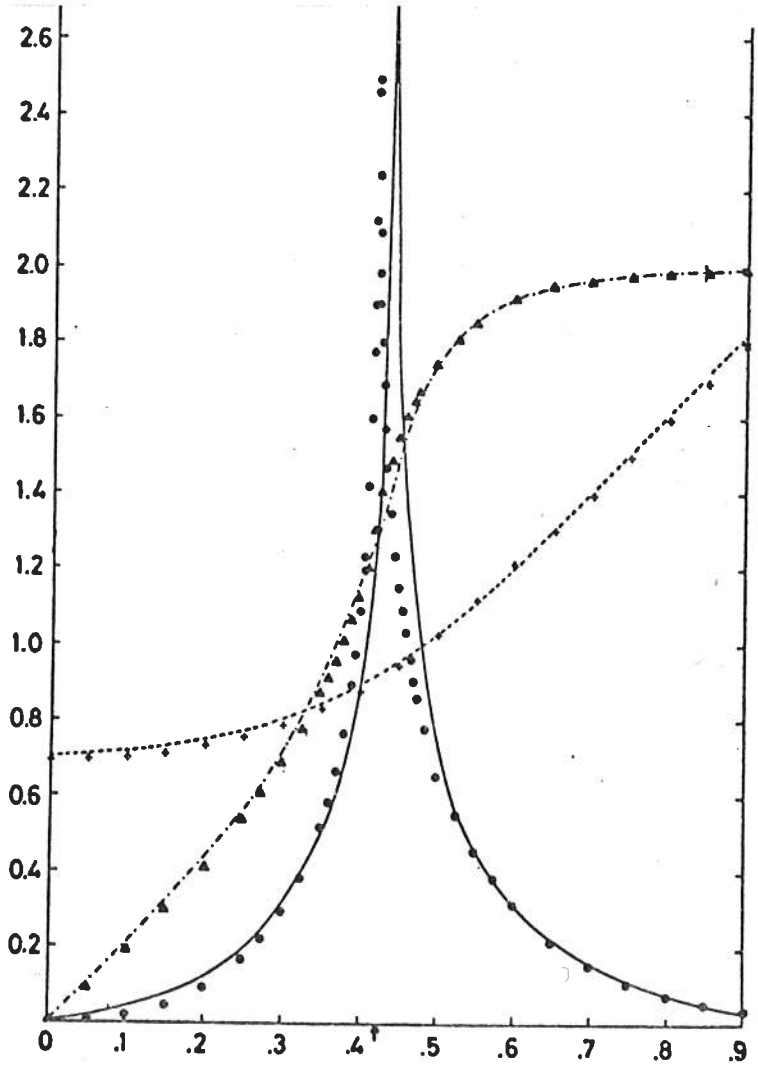


Fig. 15

dashed curve ----		Onsager's free energy
crosses +	$f(K_1)$	free energy
dashed-dot curve -·-·-		Onsager's energy
triangles ▲	$\frac{\partial f(K_1)}{\partial K_1}$	energy from first derivative
solid curve ——		Onsager's specific heat
dots •	$K_1^2 \frac{\partial^2 f(K_1)}{\partial K_1^2}$	specific heat from second derivative

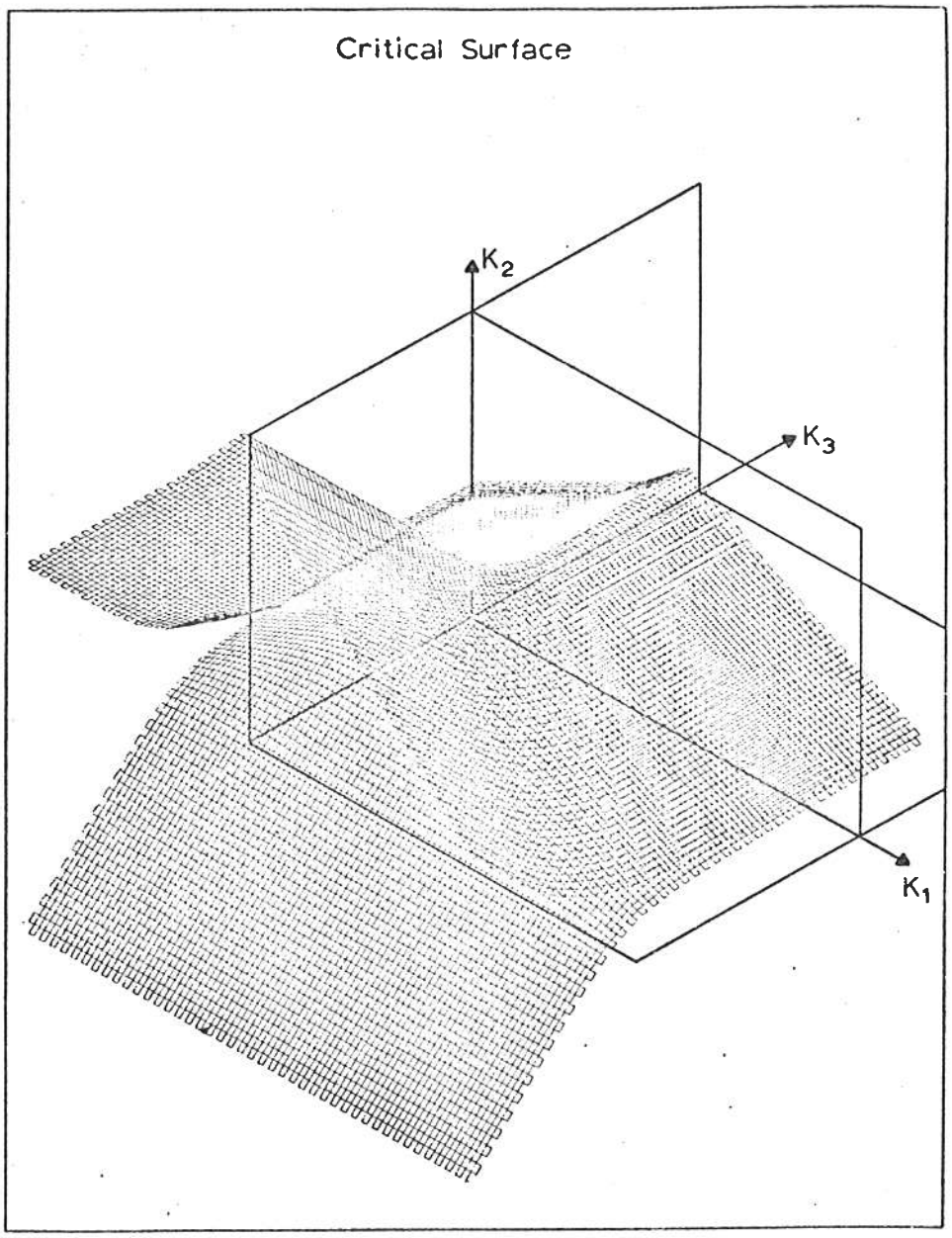


Fig. 16

Critical surface in the range $-2 \leq K_1, K_3 \leq +2$ seen along the direction $K_1 = 1$, $K_2 = 1$ and $K_3 = -1$.

This RG result obtained by looking at 4x4 spins! only should be compared with the result obtained by direct calculation shown in Fig. 7.

Kadanoff-Houghton have calculated critical exponents from a 2x2 block spin RG-transformation choosing as block spin variables

$$s'_{i'} = \tan h\, K \sum_{\text{cell}_{i'}} s_i \; .$$

Couplings up to four (even) resp. five (odd) spin interactions have been estimated. The values obtained (as in all such calcu-

lations there is no control on errors made in the approximations) are quite impressive:

$$1/\nu = 0.999964 \qquad \text{(exact: 1)}$$

$$\beta/\nu = 0.12447 \qquad \text{(exact: 0.125)} .$$

Introducing a variational principle Kadanoff has improved the technique of approximate lattice RG calculations. By use of these variational methods the first lattice RG calculations for the d=3 Ising lattice have provided rather good numerical results for critical indices (see Tab. 5).

Exponent	d = 2		d = 3	
	calculated	exact	calculated	"accepted"
δ	15.04	15.0	4.818	5.0 ± 0.2
$2-\alpha$	1.998	2.0	1.887	1.875

Tab. 5

Kadanoff's variational principle may be sketched as follows: The RG-transformation may be formally written as

$$H'(s') = - \ln \sum_{\{s\}} \exp\{S(s',s) - H(s)\}$$

where the block spin transformation kernel exp S obeys the restriction: S has the same symmetries as H(s) and

$$\sum_{\{s'\}} \exp S(s',s) = 1.$$

One introduces appropriately chosen variational functions $H_o(s',s)$ and $V(s',s) = V_1(s) + V_2(s')$ and defines upper and

lower bound effective Hamiltonians by

$$H^U(s') = - \sum_{\{s\}} \exp H_o(s',s) \{S(s',s) - H(s) - H_o(s',s)\}$$

and

$$H^L(s') = - \ln \sum_{\{s\}} \exp \{S(s',s) - H(s) + V(s',s)\}.$$

For fixed parameters $\{K\}$ and fixed S we have

1° $$H'(s') \leq H^U(s')$$

for all configurations $\{s'\}$ provided

$H_o(s',s)$ has the same symmetries as $H(s)$

and

$$\sum_{\{s\}} \exp H_o(s',s) = 1.$$

2° $$H^L(s') \leq H'(s')$$

for all configurations $\{s'\}$ provided

$V(s',s)$ is a sum of terms which are odd under a lattice symmetry operation and such that $H^L(s')$ has the same symmetries as $H(s)$.

(Choose $V_1(s)$ odd and $V_2(s')$ such that $H^L(s')$ preserves symmetry.)

The associated transformations in parameter space denoted by R^U and R^L have the property that the total free energy satisfies

$$F_{N'}(R^U\{K\}) \geq F_{N'}(R\{K\}) \geq F_{N'}(R^L\{K\}).$$

These estimates hold for the iterated transformations also in the limit $\lim_{n\to\infty} R^n$. Choosing manageable H_o and V one can vary the parameters in $H(s)$ and $S(s',s)$ so as to obtain optimum bounds on $F_N(\{K\}) = F_{N'}(R\{K\})$.

Part III. Wilson's RG and the ε-Expansion

The block spin lattice RG's are in general rather complicated (see e.g. the Kadanoff-Houghton RG for the d = 2 square lattice Ising model). Using the fact that short range details do not affect the universal critical properties one might try to study somewhat simpler models belonging to the same universality class. We know that quantities like the critical temperature are not universal. The value of the critical temperature depends on the short range properties of the system.

Different physical systems with the same critical long range behavior (critical exponents) are located in general at different points in the critical subspace of physical parameters. Only part of the change in critical temperature can be accounted for by a change of the coupling strength J i.e. $K_c = {}^{J}/_{k_B T_c}$ in general differs from system to system (see e.g. (8.53)). Also if we are able to calculate the fixed point value K* for a good microscopic model of a system in general $(K_c - K^*)$ will not be small and we will not be able to calculate K_c in the curved critical surface from the linear approximation with reasonable accuracy.

If we restrict ourselves to study only the universal long range properties of the system we may "simplify" the model by changing its short range properties. In doing this we have to take care that the long range behavior is not affected.

The modified model (e.g. the Landau-Ginzburg model in the Ising case) has to be considered as an effective theory (<u>phenomenological model</u>) for the long range properties of the original model. The related models lie in the same (generalized) critical surface although they may look very different globally.

We will now elaborate the connection between the Landau-Ginzburg-Wilson model (Euclidean S^4-cut-off field theory) and the Ising model. The two models have the same critical behavior. For the field theoretic version other techniques will be available to study the critical point.

10. The Landau-Ginzburg-Wilson Model

(Euclidean S^4-cut-off field theory)

We start from the d-dimensional ferromagnetic n.n. Ising model. On a regular lattice G_a the classical spins σ_x have distribution

$$\rho(\sigma_x^2) = \delta(\sigma^2 - 1) \quad \text{i.e. } \sigma_x = \pm 1 \tag{10.1}$$

and interaction

$$H(\sigma) = - \sum_{x,y} \sigma_x K_{x-y} \sigma_y \tag{10.2}$$

with (we assume periodic boundary conditions)

$$K_{x-y} = \begin{cases} k > o & \text{for } |x-y| = a \\ o & \text{otherwise} \end{cases} \quad . \tag{10.3}$$

The partition function is

$$Z = \int \prod_{x \in V_a} d\sigma_x \, \rho(\sigma_x^2) \, e^{-H(\sigma)} \quad . \tag{10.4}$$

The correlation functions may be obtained by differentiation with respect to the x-dependent external magnetic field J_x from the generating functional

$$Z\{J\} = Z^{-1} \int \prod_{x \in V_a} d\sigma_x \, \rho(\sigma_x^2) \, e^{-H(\sigma)+(J,\sigma)} \tag{10.5}$$

i.e.

$$\langle \sigma_{x_1} \ldots \sigma_{x_p} \rangle = \frac{\delta^p Z\{J\}}{\delta J_{x_1} \ldots \delta J_{x_p}} \Big|_{J=o} \quad . \tag{10.6}$$

We have introduced the "scalar product"

$$(J,\sigma) = \sum_x J_x \sigma_x \quad . \tag{10.7}$$

We now approximate the Ising model by replacing the discrete spin distribution

$$\rho(\sigma_x^2) = \delta(\sigma_x^2-1)$$

by a continuous spin distribution

$$\rho(\sigma_x^2) = \sqrt{\frac{u_o}{\pi}}\, e^{-u_o(\sigma_x^2-1)^2} \tag{10.8}$$

with $u_o >> 1$ (see Fig.17).

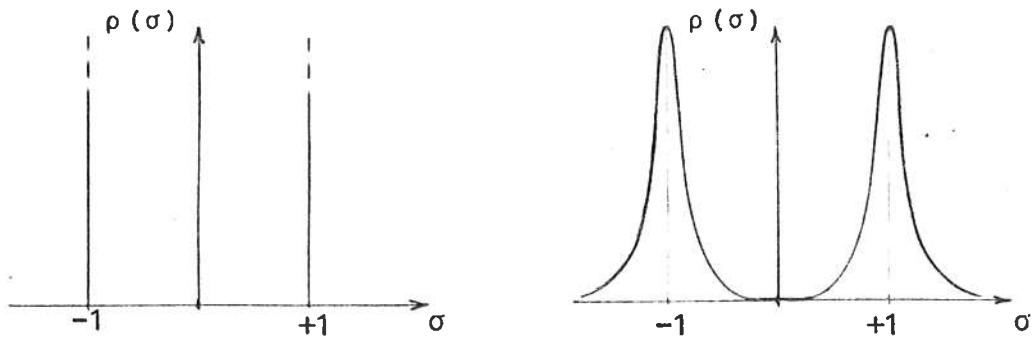


Fig. 17

We notice that

$$\delta(\sigma_x^2 - 1) = \lim_{u_o\to\infty} \sqrt{\frac{u_o}{\pi}}\, e^{-u_o(\sigma_x^2-1)^2} \tag{10.9}$$

i.e. we recover the exact Ising system from its continuous approximation in the limit $u_o \to \infty$.

For the continuous model we may write

$$Z = \left(\frac{u_o}{\pi}\right)^{N/2} e^{-Nu_o} \int \prod_x d\sigma_x\, e^{-\bar{H}(\sigma)} \tag{10.10}$$

with

$$\bar{H}(\sigma) = \bar{H}_o(\sigma) + \bar{H}_{int}(\sigma)$$

$$\bar{H}_o(\sigma) = -\sum_{x,y} \sigma_x (K_{x-y} - 2u_o\, \delta_{x,y})\, \sigma_y \tag{10.11}$$

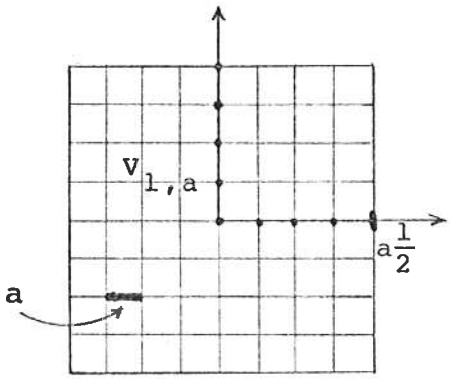
$$\bar{H}_{int}(\sigma) = u_o \sum_x \sigma_x^4 \quad .$$

As: $\prod_x \exp -u_o(\sigma_x^2-1)^2 = \exp(-u_o \sum_x \sigma_x^4 + 2u_o \sum_x \sigma_x^2) \exp - Nu_o$

and $\sum_{x,y} \sigma_x \delta_{x,y} \sigma_y = \sum_x \sigma_x^2$

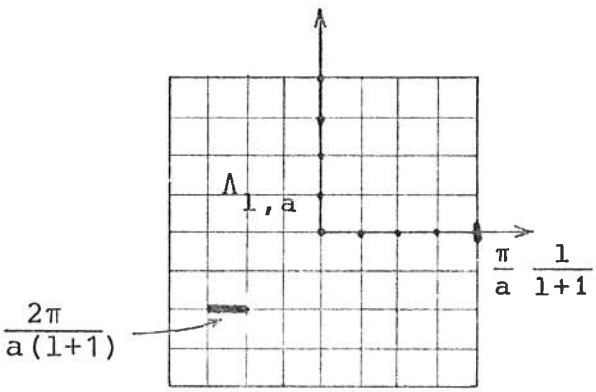
Let us express the Hamiltonian $\bar{H}$ in Fourier space:

We consider the system to be centered in configuration space round the origin



with box size $L = a \cdot l$ (l even). The first Brillouin zone is then

$$\Lambda_{l,a} = \{q \; ; \; q_i = \frac{2\pi}{a(l+1)} \; (-\frac{l}{2} + n) \; ; \; n = o,1,\ldots,l\} \; .$$



On the first Brillouin zone (with $N = (l+1)^d$)

$$a^d \sum_{x \in V_{l,a}} e^{-iqx} = a^d N \, \delta^{(d)}_{q,o} \; ; \; q \in \Lambda_{l,a}$$

and (10.12)

$$\frac{1}{Na^d} \sum_{q \in V_{l,a}} e^{iqx} = a^{-d} \, \delta^{(d)}_{x,o} \; ; \; x \in V_{l,a} \; .$$

We define the Fourier transformation from $V_{l,a}$ to $\Lambda_{l,a}$ by

$$\tilde{\sigma}_q = a^d \sum_{x \in V_{l,a}} e^{-iqx} \, \sigma_x$$

such that (10.13)

$$\sigma_x = \frac{1}{Na^d} \sum_{q \in \Lambda_{l,a}} e^{iqx} \, \tilde{\sigma}_q \; .$$

For real σ_x we have $\tilde{\sigma}^*_q = \tilde{\sigma}_{-q}$.

Inserting (10.13) into (10.11) we obtain

$$\bar{H}_o(\sigma) = \frac{Z_a}{2} \frac{1}{Na^d} \sum_{q \in \Lambda_{l,a}} |\tilde{\sigma}_q|^2 \, \tilde{G}^{-1}_{a,q}$$

(10.14)

$$\bar{H}_{int}(\sigma) = u \, Z_a^2 \frac{1}{(Na^d)^4} \sum_{q_i \in \Lambda_{l,a}} \tilde{\sigma}_{q_1} \, \tilde{\sigma}_{q_2} \, \tilde{\sigma}_{q_3} \, \tilde{\sigma}_{q_4} \, Na^d \, \delta^{(d)}_{\Sigma q_i, o} \; .$$

with

$$Z_a = 2k \, a^{2-d} > o$$

$$\hat{m}^2 = -2a^{-2} \left(\frac{u_o}{k} + d\right) < o \qquad (10.15)$$

$$u = \frac{u_o}{4k^2} \, a^{d-4} > o$$

and

$$\tilde{G}^{-1}_{a,q} = \hat{m}^2 + 4a^{-2} \sum_{i=1}^{d} \sin^2 \frac{aq_i}{2} \; . \qquad (10.16)$$

We used ($K_x = k$, $|x| = a$ and $K_x = o, |x| \neq a$)

$$\tilde{K}_q = a^d \sum_x K_x \, e^{-iqx} = a^d k \sum_{i=1}^{d} (e^{-iq_i a} + e^{iq_i a})$$

$$= 2ka^d \sum_{i=1}^{d} \cos q_i a \quad ; \quad 1-\cos q_i a = 2 \sin^2 \frac{aq_i}{2} \, .$$

The reason for the choice of diverse factors will become clear immediately. If we now <u>renormalize</u> the spins and field variables as

$$S_x = \sqrt{Z_a} \, \sigma_x$$

$$\bar{J}_x = \frac{a^{-d}}{\sqrt{Z_a}} \, J_x \tag{10.17}$$

the Hamiltonian takes the simple form:

$$\bar{H}_o(S) = \frac{1}{2} \frac{1}{Na^d} \sum_q |\tilde{S}_q|^2 \, \tilde{G}^{-1}_{aq}$$

$$\bar{H}_{int}(S) = ua^d \sum_x S_x^4 \quad . \tag{10.18}$$

The partition function is

$$Z = C \cdot \bar{Z} \tag{10.19}$$

with

$$C = \left(\frac{u_o Z_a^{-1}}{\pi}\right)^{N/2} e^{-Nu_o}$$

and

$$\bar{Z} = \int \prod_x dS_x \, e^{-H(S)} \, . \tag{10.20}$$

The correlation functions are

$$<\sigma_{x_1} \ldots \sigma_{x_p}> = (Z_a)^{-p/2} <S_{x_1} \ldots S_{x_p}> \tag{10.21}$$

and the generating functional for the S-correlations is

$$Z\{\bar{J}\} = \bar{Z}^{-1} \int \prod_x dS_x \; e^{-\bar{H}(S) + (\bar{J},S)} \; . \tag{10.22}$$

In this representation our continuous spin model is what field theorists call a lattice approximation of an Euclidean Bose field theory. The close connection to field theory can be seen as follows:

1. Gaussian model

Let us set u = o and choose

$$m^2 = \hat{m}^2 \Big|_{u_o = -b} > o \text{ fixed.}$$

This corresponds to a choice of a Gaussian spin distribution

$$\rho(\sigma_x^2) = \sqrt{\frac{2b}{\pi}} \; e^{-2b\sigma_x^2} \tag{10.23}$$

in the original model (Fig. 18)

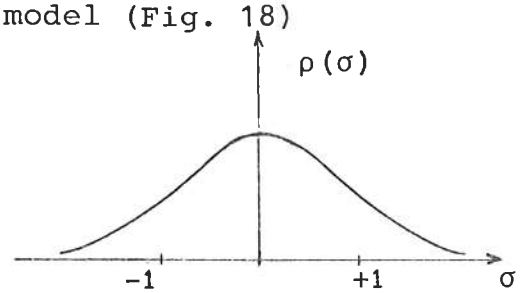


Fig. 18

and as a varies the Ising coupling must change as

$$k = k(a) = \frac{b}{d+m^2 \frac{a^2}{2}} \quad \text{with } k(o) = \frac{b}{d} \; . \tag{10.24}$$

We easily calculate the Gaussian integrals

$$\bar{Z}_0 = \int \prod_x dS_x \, e^{-H_0(S)} = N^{-N/2} a^{-dN} \int \prod_q d\tilde{S}_q \, e^{-H_0(S)}$$

$$\bar{Z}_0 = \int \prod_q d\tilde{S}_q \, e^{-\frac{1}{2}\frac{1}{Na^d}\sum_q |\tilde{S}_q|^2 \tilde{G}^{-1}_{a,q}} \cdot N^{-N/2} \cdot a^{-dN} \tag{10.25}$$

$$= (2\pi)^{N/2} (Na^d)^{+N/2} (\det \tilde{G}^{-1}_{aq})^{-1/2} \cdot N^{-N/2} \cdot a^{-dN}$$

$$= (2\pi)^{N/2} a^{-dN/2} (\det \tilde{G}^{-1}_{aq})^{-1/2}$$

and

$$Z_0\{\bar{J}\} = \exp \tfrac{1}{2}(\bar{J}, G_a \bar{J}) \tag{10.26a}$$

with

$$(\bar{J}, G_a \bar{J}) = \frac{1}{Na^d} \sum_q |\tilde{\bar{J}}_q|^2 \tilde{G}_{a,q} \quad . \tag{10.26b}$$

The two point correlation is

$$\langle S_x S_y \rangle = \left.\frac{\delta^2 Z\{\bar{J}\}}{\delta \bar{J}_x \delta \bar{J}_y}\right|_{\bar{J}=o} = G_a(x-y) = \frac{1}{Na^d} \sum_q e^{iq(x-y)} \tilde{G}_{a,q} . \tag{10.27}$$

All higher correlations are products of two point correlations i.e. all higher "connected" correlations are zero.

Now we may perform the infinite volume limit $N \to \infty$

$$\frac{1}{Na^d} \sum_q \ldots \underset{N\to\infty}{\to} \frac{1}{(2\pi)^d} \int_{-\pi/a}^{+\pi/a} d^d q \ldots$$

and the continuum limit (field theory limit) $a \to o$

$$\frac{1}{(2\pi)^d} \int_{-\pi/a}^{\pi/a} d^d q \ldots \to \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} d^d q \ldots$$

and

$$G_{aq} = \frac{1}{m^2 + 4a^{-2} \sum_{i=1}^{d} \sin^2 \frac{aq_i}{2}} \to \frac{1}{m^2 + q^2 + O(a^2 q^4)}$$

The limits exist for

$$Z_o\{\bar{J}\} = \exp \frac{1}{2}(\bar{J}, G\,\bar{J}) \tag{10.28a}$$

with

$$(\bar{J}, G\,\bar{J}) = \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} d^d q\ |\tilde{\bar{J}}(q)|^2 \frac{1}{q^2+m^2} \tag{10.28b}$$

the two point function

$$\langle S_x\ S_y \rangle = G(x-y) = \frac{1}{(2\pi)^d} \int d^d q\ \frac{e^{iq(x-y)}}{q^2+m^2} \tag{10.29}$$

and the Hamiltonian

$$\bar{H}_o(S) = \frac{1}{2} \frac{1}{(2\pi)^d} \int d^d q\ |\tilde{S}(q)|^2\ (q^2+m^2)\ . \tag{10.30}$$

This identifies the renormalized Gaussian spin variable in the limit $a \to o$ as a free Euclidean Bose field. In coordinate space we have

$$\bar{H}_o(S) = \frac{1}{2} \int d^d x\ \{S(x)\ (-\Delta+m^2)\ S(x)\}\ . \tag{10.30a}$$

where Δ is the Laplacian. m is the bare mass of field S.

2. Non Gaussian model

We now consider the non Gaussian system defined by the Hamiltonian (10.18) . We expect the infinite volume limit of the $Z\{\bar{J}\}$ (see (10.22)) and the correlation functions to exist whereas $\bar{Z} \propto \exp - \bar{f}N$ with finite free energy density $\bar{f}$.

The continuum limit in general (for d>2) does not exist (renormalization problem) although formally we may write the Hamiltonian as

$$\bar{H}(S) = \int d^d x\ \{\frac{1}{2}\partial S(x)\,\partial S(x) + \frac{m^2}{2}\ S^2(x) + uS^4(x)\}\ . \tag{10.31}$$

Notice that in the field theory language the terms "free" and "interaction" are essentially interchanged (up to S^2-terms):

$S^4(x)$	σ_x^4
scalar self interaction	part of free site spin distribution
$(\partial S)^2(x)$	$\sigma_x \sigma_{x+a}$
free scalar field	n.n. spin interaction.

It is a (quite depressing) fact that we can do explicit calculations in non trivial field theories in general only by perturbation theory. So we have to say a few words on it.

3. Perturbation expansion

We consider m^2 and u as independent parameters and $m^2 > o$ as in the Gaussian model. After summing up the perturbation expansion we can then set $m^2 = \hat{m}^2 < o$ as given by (10.5).

The perturbation expansion is a formal power series expansion in u:

$$\bar{Z} = \sum_{j=o}^{\infty} \frac{(-u)^j}{j!} a^{jd} \sum_{y_1 \cdots y_j} \int \prod_x dS_x \, S_{y_1}^4 \cdots S_{y_j}^4 \, e^{-\bar{H}_o(S)} \tag{10.32a}$$

and

$$\bar{Z} \langle S_{x_1} \cdots S_{x_p} \rangle = \sum_{j=1}^{\infty} \frac{(-u)^j}{j!} a^{jd} \sum_{y_1 \cdots y_j} \int \prod_x dS_x \, S_{x_1} \cdots S_{x_p} \, S_{y_1}^4 \cdots S_{y_j}^4 \, e^{-\bar{H}_o(S)} . \tag{10.32b}$$

Notice that by (10.22)

$$\bar{Z} \, Z\{J\} = \sum_{j=o}^{\infty} \frac{1}{j!} \int \prod_x dS_x (\bar{H}_{int}(S))^j \, e^{-\bar{H}_o(S) + (\bar{J},S)} \tag{10.33a}$$

$$= \sum_{j=o}^{\infty} \frac{1}{j!} \left(\bar{H}_{int}\left(\frac{\delta}{\delta \bar{J}}\right)\right)^j \int \prod_x dS_x \, e^{-\bar{H}_o(S) + (\bar{J},S)}$$

where the last integral is the Gaussian functional i.e. by (10.25) and (10.26)

$$\bar{Z}\ Z\{\bar{J}\} = \bar{Z}_o \sum_{j=o}^{\infty} \frac{(-1)^j}{j!} \left(\bar{H}_{int}\left(\frac{\delta}{\delta\bar{J}}\right)\right)^j \exp \tfrac{1}{2}(\bar{J},G_a\ \bar{J}) \quad . \tag{10.33b}$$

The Gaussian integrals

$$[S_{x_1},\ldots,S_{x_p},\ S^4_{y_1}\ldots S^4_{y_j}] = \bar{Z}_o^{-1}\int \prod_x dS_x\ S_{x_1}\ldots S^4_{y_j}\ e^{-\bar{H}_o(S)} \tag{10.34a}$$

are easily evaluated from the formulae (10.33)

$$[S_{x_1},\ldots,S^4_{y_j}] = \frac{\delta}{\delta\bar{J}_{x_1}}\ \ldots\left(\frac{\delta}{\delta\bar{J}_{y_j}}\right)^4 \exp \tfrac{1}{2}(\bar{J},G_a\ \bar{J})\Big|_{\bar{J}\ =\ o}$$

$$= \frac{\delta}{\delta\bar{J}_{x_1}}\ \ldots\ldots\left(\frac{\delta}{\delta\bar{J}_{y_j}}\right)^4 \left(\tfrac{1}{2}\right)^{\frac{p+4j}{2}} \frac{1}{\left(\frac{p+4j}{2}\right)!}\ (\bar{J},G_a\ \bar{J})^{\frac{p+4j}{2}} \quad . \tag{10.34b}$$

We then have

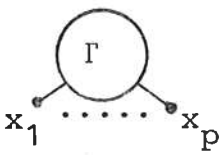
$$\bar{Z} = \bar{Z}_o \sum_{j=o}^{\infty} \frac{(-u)^j}{j!}\ a^{jd} \sum_{y_1\cdot\cdot y_j} [S^4_{y_1}\ldots S^4_{y_j}] \tag{10.35a}$$

and

$$\bar{Z}\ \langle S_{x_1}\ldots S_{x_p}\rangle = \bar{Z}_o \sum_{j=o}^{\infty} \frac{(-u)^j}{j!}\ a^{jd} \sum_{y_1\ldots y_j} [S_{x_1}\ldots S^4_{y_j}] \quad . \tag{10.35b}$$

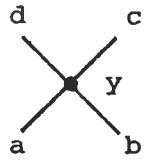
A convenient book-keeping of the contributions to (10.34) are the Feynman rules i.e.

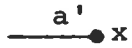
$$[S_{x_1}\ldots S^4_{y_j}] = \sum_{\Gamma} \prod G(z_i - z_k) = \sum_{\Gamma}$$



(10.36)

where the sum extends over all possible Feynman graphs Γ with

vertices 



and

lines

$z_i, z_k \in (x_1 \ldots y_j)$.

To each line joining two vertices z_i and z_k there corresponds a factor $G(z_i - z_k)$.

Counting only differently looking graphs we obtain

$$\bar{Z}\,\bar{Z}_o^{-1} \langle S_{x_1} \ldots S_{x_p} \rangle = \sum_{\Gamma} \frac{1}{\gamma(\Gamma)} \; I_\Gamma(x_1, \ldots, x_p)$$

with (10.37)

$$I_\Gamma(x_1, \ldots, x_p) = (-4u)^{j_\Gamma} \, a^{j_\Gamma d} \sum_{y_1 \ldots y_{j_\Gamma}} \prod_{z_i, z_k} G(z_i - z_k)$$

where j_Γ is the perturbation order of Γ.
The combinatorial factor $\gamma(\Gamma)$ is given by

$$\gamma(\Gamma) = s \cdot 2^\alpha 2^\beta (3!)^\gamma$$

where

α number of (single lines)

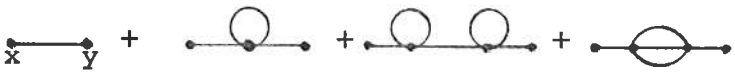
β number of (double lines)

γ number of (triple lines)

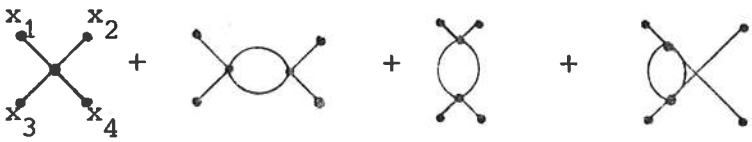
and s the number of permutations of y vertices leaving Γ invariant.

Examples: (even Hamiltonian)

$\bar{Z}\,\bar{Z}_o^{-1}$ = + ...

$\langle \sigma_x \sigma_y \rangle$ = 

+ ...

$\langle \sigma_{x_1} \ldots \sigma_{x_4} \rangle$ = 

+

+ disc. + ...

In terms of graphs

i) division by $\bar{Z}\,\bar{Z}_o^{-1}$ is equivalent to the omission of all vacuum graphs ($\boxtimes$)

$$Z\{J\} = \bar{Z}^{-1}\int \Pi\, dS_x\; e^{-\bar{H}(S)+(\bar{J},S)} = \int_{\boxtimes} \prod dS_x \ldots$$

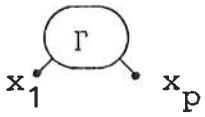
ii) taking the logarithm is equivalent to the omission of all disconnected graphs (conn)

$$\ln Z\{J\} = \ln \int \Pi dS_x\; e^{-\bar{H}(S)+(\bar{J},S)} - \ln \bar{Z} = \int_{\boxtimes,\mathrm{conn}} \prod dS_x \ldots$$

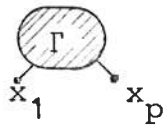
(a vacuum graph has at least one part which is not connected to an external leg).

Thus in terms of graphs:

$$\langle S_{x_1} \ldots S_{x_p} \rangle = \sum_{\substack{\Gamma \\ \boxtimes}}$$



$$\langle S_{x_1} \ldots S_{x_p} \rangle^{\mathrm{conn}} = \sum_{\substack{\Gamma \\ \boxtimes,\mathrm{conn}}}$$



(10.38)

In momentum space the connected correlation functions take the form

$$\langle \tilde{S}_{p_1} \ldots \tilde{S}_{p_r} \rangle^{\mathrm{conn}} = \sum_{\Gamma} \frac{1}{\gamma(\Gamma)}\, \tilde{I}_{\Gamma}(p_1,\ldots p_r)^{\mathrm{conn}}$$

with

$$\tilde{I}_{\Gamma}(p_1\ldots p_r)^{\mathrm{conn}} = (-4u)^{j_\Gamma}\, Na^d \delta^{(d)}_{\Sigma p_i,o}\, \frac{1}{(Na^d)^l} \sum_{k_1,\ldots,k_l} \prod_{n=1}^{L} \frac{1}{m^2+4a^{-2}\sum_{i=1}^{d} \sin^2 \frac{a(q_n)_i}{2}} .$$

Here $(p_1,\ldots,p_r)$ are the p external momenta $(k_1,\ldots,k_l)$ a set of independent internal (loop) momenta and the line momenta of the L lines $(q_1,\ldots,q_L)$ are linear combinations of the p's and k's such that at each vertex the momentum is conserved.

4. Landau-Ginzburg-Wilson model

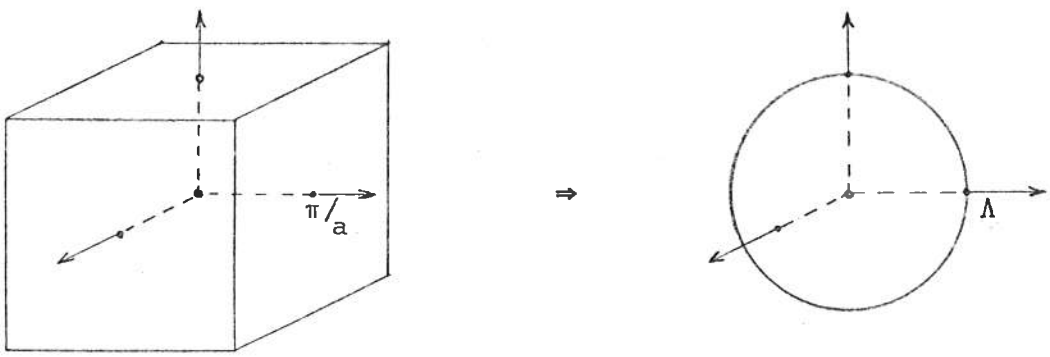
It is almost impossible to calculate the sums (10.39) even in the thermodynamic limit. Therefore we have to approximate the lattice system by a model having identical long range properties and being manageable in perturbation theory.

The following changes (approximations) will be made

i) the propagator is replaced by its small momentum (long range part

$$\frac{1}{m^2+4a^{-2}\sum_{i=1}^{d}\sin^2\frac{aq_i}{2}} \quad \Rightarrow \quad \frac{1}{m^2+q^2}$$

ii) in the infinite volume limit we replace in momentum space the box of length $2\pi/a$ by a ball of radius $\Lambda \simeq \frac{\pi}{a}$ (again the small momenta are not affected).



The model obtained is the Landau-Ginzburg-Wilson model. It is defined by the following Hamiltonian (we write it formally in the infinite volume limit):

$$H_\Lambda(S) = H_{o\Lambda}(S) + H_{int\Lambda}(S)$$

$$H_{o\Lambda}(S) = \frac{1}{2}\int_{\substack{q\\|q|\le\Lambda}} |\tilde{S}_q|^2\,(q^2+m^2) \tag{10.40}$$

$$H_{int\Lambda}(S) = \sum_{n=2}^{\infty}\int_{\substack{q_1\\|q_i|\le\Lambda}}\cdots\int_{q_{2n}} \tilde{S}_{q_1}\cdots\tilde{S}_{q_{2n}}\,(2\pi)^d\,\delta^{(d)}(\Sigma q_i)\;u_{2n}(q_i)$$

The generating functional is

$$Z_\Lambda\{J\} = \int_{\boxtimes} \prod_q d\tilde{S}_q \exp - H_\Lambda(S) + (\bar{J},S)$$

$$\doteq \exp - H_{int\Lambda}\left(\frac{\delta}{\delta\bar{J}}\right) \exp \frac{1}{2}(\bar{J},G\,\bar{J})\Big|_{\boxtimes} \tag{10.41}$$

with

$$(\bar{J},G\,\bar{J}) = \int_{\substack{q \\ |q|\leq\Lambda}} |\tilde{\bar{J}}(q)|^2 \frac{1}{q^2+m^2}$$

(we abbreviate $(2\pi)^{-d}\int d^dq \ldots \equiv \int_q \ldots$).

In general we will take $u_4 = u$ and $2_{un} = o$, $n > 2$.

The Gaussian integrals are evaluated by exactly the same rules as above with the replacements

$$\frac{1}{Na^d}\sum_k \quad \to \quad \int_q$$

$$Na^d\,\delta^{(d)}_{q_i,o} \quad \to \quad (2\pi)^d\delta^{(d)}(\Sigma q_i) \tag{10.42}$$

$$\frac{1}{m^2+4a^{-2}\sum_{i=1}^{d}\sin^2\frac{aq_i}{2}} \quad \to \quad \frac{1}{m^2+q^2}$$

in formula (10.39).

Remark for skeptics:

The Landau-Ginzburg-Wilson model can be related to the continuous spin lattice model in an exact manner as follows

i) Expand the propagator in $H_o(S)$:

$$\bar{H}_o(S) = \frac{1}{2}\frac{1}{Na^d}\sum_q |\tilde{S}_q|^2(m^2+q^2+a^2q^4+\ldots\ldots)$$

and do the perturbation expansion with respect to the simpler

free Hamiltonian

$$\bar{H}_o'(S) = \frac{1}{2}\frac{1}{Na^d}\sum_q |\tilde{S}_q|^2(m^2+q^2).$$

The rest of $\bar{H}_o$ being incorporated into $\bar{H}_{int}'$

$$\bar{H}_{int}'(S) = \bar{H}_{int}(S) + (\bar{H}_o(S) - \bar{H}_o'(S))$$

where

$$\bar{H}_o - \bar{H}_o' = \frac{1}{2}\frac{1}{Na^d}\sum_q |\tilde{S}_q|^2(a^2q^4 + \text{higher terms}).$$

ii) Perform a transformation of variables (Fig. 19) (into relevant and irrelevant fluctuations)

$$\tilde{S}_q = \tilde{S}_{oq} + \tilde{S}_{1q} \; ; \quad q \in \Lambda_{l,a}$$

$$\tilde{S}_{oq} = \begin{cases} \tilde{S}_q & |q| \le \Lambda < \frac{\pi}{a}\frac{l}{l+1} \\ o & \text{otherwise} \end{cases}$$

$$\tilde{S}_{1q} = \begin{cases} o & |q| \le \Lambda \\ \tilde{S}_q & |q| > \Lambda \; ; \; |q_i| \le \frac{\pi}{a}\frac{l}{l+1} \end{cases}.$$

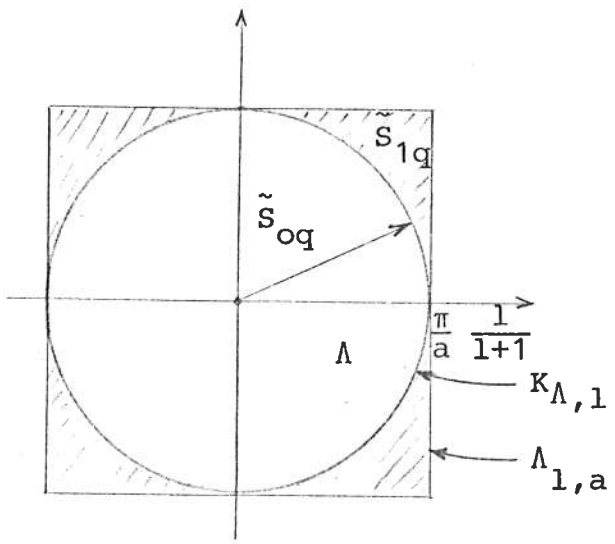


Fig. 19

iii) Eliminate the $\tilde{S}_{1q}$ fluctuations as these are expected to be irrelevant in the long range region. By this elimination we may define a new effective Hamiltonian

$$\exp - H_\Lambda^{eff}(S') = \int_{\otimes} \prod_{q\in\Lambda_{1,a}} d\tilde{S}_{1a} \exp - \bar{H}_{1,a}(S_o+S_1)$$

α is chosen such that $|\tilde{S}'_q|^2 q^2$ term in H_Λ^{eff} has coefficient 1.

If we write

$$\bar{H}_{1,a}(S) = \sum_{n=1}^{\infty} \frac{1}{(Na^d)^{2n}} \sum_{\substack{q_1\ldots q_{2n}\\ \in\Lambda_{1,a}}} \tilde{S}_{q_1}\ldots\tilde{S}_{q_{2n}} Na^d \delta^{(d)}_{\Sigma q_i,o}\, u_{2n}(q_i;a,m^2,u$$

and

$$H_\Lambda^{eff}(S') = \sum_{n=1}^{\infty} \frac{1}{(N_\Lambda \Lambda^{-d})^{2n}} \sum_{\substack{q_1\ldots q_{2n}\\ \in K_{\Lambda,1}}} \tilde{S}'_{q_1}\ldots\tilde{S}'_{q_{2n}} N_\Lambda \Lambda^{-d} \delta^{(d)}_{\Sigma q_i,o} \times$$

$$\times\ u'_{2n}(q_i;\Lambda,a,m^2,u)$$

the transformation into the Landau-Ginzburg-Wilson form of the model is characterized by a transformation in the parameter space

$$u'_{2n} = (R^{eff}u)_{2n}.$$

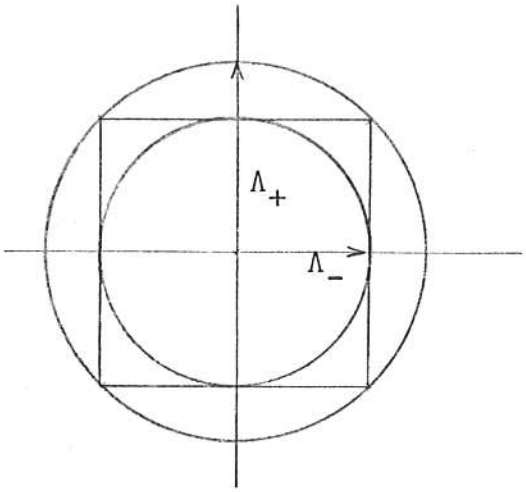
This relation can be calculated in perturbation theory.

Notice that we may give upper and lower estimates on the lattice model in terms of cut-off models also if we perform the transformations ii) and iii) using the original free Hamiltonian $\bar{H}_o$ together with the estimate

$$\frac{1}{m^2+q^2} \le \tilde{G}_{aq} \le \frac{\pi^2}{4} \quad \frac{1}{\frac{m^2\pi^2}{4} + q^2}$$

and the upper and lower cut-offs.

$$\Lambda_+ = \sqrt{d}\,\frac{\pi}{a} \quad \text{and} \quad \Lambda_- = \frac{\pi}{a} \quad .$$



We conclude that the cut-off system (10.40) representing the lattice system (10.18) in the region $|q| \leq \Lambda$ exactly will have an infinite sequence of non vanishing couplings u_{2n}. The u_{2n}, $n>2$ will, however, turn out to be irrelevant in agreement with the heuristic arguments.

11. Wilson's RG

The RG for the Landau-Ginzburg-Wilson model is defined in essentially the same way as the block spin RG (see chapter 3.2). Only the transformation of the spin variables is changed (simplified!). Wilson's RG is formulated in momentum space: relevant are the small momentum (long range) fluctuations irrelevant the large momentum (small distance) fluctuation.

Definition of R_s:

A. Enlargement of the system

$$N = N' = s^d N \tag{11.1}$$

This is trivial in the thermodynamic limit $N \to \infty$.

B. Transformation of spin variables (Fig.20)

$$\tilde{S}_q = \tilde{S}_{oq} + \tilde{S}_{1q}$$

$$\tilde{S}_{oq} = \begin{cases} \tilde{S}_q & |q| \leq \frac{\Lambda}{s} \\ o & \frac{\Lambda}{s} < |q| \leq \Lambda \end{cases} \qquad \text{relevant part} \tag{11.2}$$

$$\tilde{S}_{1q} = \begin{cases} o & |q| \leq \frac{\Lambda}{s} \\ \tilde{S}_q & \frac{\Lambda}{s} < |q| \leq \Lambda \end{cases} \qquad \text{irrelevant part}$$

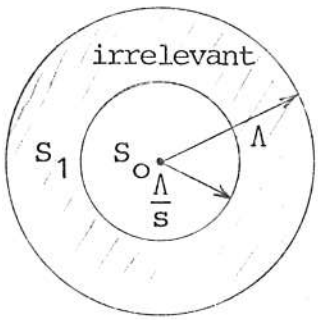


Fig. 20

C. Renormalization

$$\tilde{S}'_q = \alpha_s^{-1}\, \tilde{S}_{oq} \,. \tag{11.3}$$

D. Elimination of irrelevant fluctuations

$$\begin{aligned} Z\{J\} &= \int_{\tilde{S}_{oq}\,\boxtimes} \int_{\tilde{S}_{1q}} \exp\{-H_\Lambda(S_o+S_1) + (J,S_o+S_1)\} \\ &= \int_{\boxtimes\,\tilde{S}_{oq}} \exp\{-H'_{\frac{\Lambda}{s}}(S_o) + (J,S_o)\} \end{aligned} \tag{11.4}$$

provided $\tilde{J}_q$ has support in $|q| \leq \frac{\Lambda}{s}$. (We abbreviate $\int \Pi d\tilde{S}_q \ldots \equiv \int_S \ldots$)

E. Dilatation of the system:

$$\begin{aligned} sq &= q' \\ \text{i.e.} \quad \Lambda' &= s\,\frac{\Lambda}{s} = \Lambda. \end{aligned} \tag{11.5}$$

Together with C. we thus have to substitute

$$\tilde{S}_{oq} = \alpha_s\, \tilde{S}'_{q'=sq}$$

in (11.4).

These transformations define the Wilson RG

$$H'_\Lambda(S') = R_s H_\Lambda(S\ ) = -\ln \int_{\boxtimes\,\tilde{S}_1} \exp - H_\Lambda(S)\Big|_{\tilde{S}_{oq}=\alpha_s\,\tilde{S}'_{q'=sq}} \tag{11.6}$$

R_s has the semi-group property

$$R_{s_1} \cdot R_{s_2} = R_{s_1 s_2} \quad ; \quad s_1 s_2 \in R^+ \tag{11.7a}$$

provided

$$\alpha_{s_1} \cdot \alpha_{s_2} = \alpha_{s_1 \cdot s_2} \tag{11.7b}$$

which requires

$$\alpha_s = s^x \quad (x \text{ independent on } s). \tag{11.7c}$$

If the Hamiltonian

$$H_\Lambda(S) = \sum_{n=1}^{\infty} \int_{q_1} \ldots \int_{q_{2n}} \tilde{S}_{q_1} \ldots \tilde{S}_{q_{2n}} \, (2\pi)^d \delta^{(d)}(\Sigma q_i) \; u_{2n}(q_i) \tag{11.8a}$$

is of short range interaction type the $u_{2n}(q_i)$ must be analytic functions in the momenta. In perturbation theory then the transformed Hamiltonian has the form

$$H'_\Lambda(S') = \sum_{n=1}^{\infty} \int_{q'_1} \ldots \int_{q'_{2n}} \tilde{S}'_{q'_1} \ldots \tilde{S}'_{q'_{2n}} \, (2\pi)^d \delta^{(d)}(\Sigma q'_i) \; u'_{2n}(q'_i;u) \tag{11.8b}$$

where the couplings u'_{2n} up to Θ-functions are analytic in the q'_i. We may write

$$u'_{2n} = (R_s u)_{2n} \; . \tag{11.9a}$$

Splitting off eventual products of Θ-functions (appearing explicitly in the perturbation expansion see below)

$$\begin{aligned} u_{2n}(q) &= \prod \Theta_{q_i} \; \hat{u}_{2n}(q) \\ u'_{2n}(q') &= \prod \Theta_{q'_i} \; \hat{u}'_{2n}(q') \end{aligned} \tag{11.10a}$$

we may expand the couplings $\hat{u}$ in a Taylor series about $q = o$

$$\hat{u}_{2n}(q) = \sum_{n,l,\alpha} P_\alpha(q_1^{l_1} \ldots q_{2n}^{l_{2n}}) g_{nl\alpha} \tag{11.10b}$$

and the RG takes the form

$$g'_{n'l'\alpha'}(g,\Lambda,s) = \sum_{n,l,\alpha} R_s(g,\Lambda)_{n'l'\alpha',nl\alpha} g_{nl\alpha}(\Lambda) . \tag{11.9b}$$

The P_α are polynomials in $(q_1 \ldots q_{2n})$ of degree

$$l = l_1 + l_2 + \ldots + l_{2n} \quad \text{(even)}$$

α labelling the different polynomials of degree l.

Transformation of correlations

For the Wilson RG the transformation of correlation functions is formally rather simple. In the restricted (long range) region $|q_i| \leq \frac{\Lambda}{s}$ we find

$$\langle \tilde{S}(q_1) \ldots \tilde{S}(q_n) \rangle_u = s^{nx} \langle \tilde{S}'(sq_1) \ldots \tilde{S}'(sq_n) \rangle_{u'=R_s u} \quad . \qquad (11.11)$$

As in the region $|q_i| \leq \frac{\Lambda}{s}$ $\tilde{S}_{oq} \equiv \tilde{S}_q$ we immediately obtain

$$\int_{\boxtimes} \tilde{S}\, \tilde{S}_{q_1} \ldots \tilde{S}_{q_n}\, e^{-H_\Lambda(S)}$$

$$= \int_{\boxtimes} \tilde{S}_o\, \tilde{S}_{oq_1} \ldots \tilde{S}_{oq_n} \int_{\boxtimes} \tilde{S}_1\, e^{-H_\Lambda(S_o+S_1)}$$

$$= \int_{\boxtimes} \tilde{S}_o\, \tilde{S}_{oq_1} \ldots \tilde{S}_{oq_n}\, e^{-H'_{\Lambda/s}(S_o)} \Big|_{S_{oq}=\alpha_s \tilde{S}'_{sq}}$$

$$= \alpha_s^n \int_{\boxtimes} \tilde{S}'\, \tilde{S}'_{sq_1} \ldots \tilde{S}'_{sq_n}\, e^{-H'_\Lambda(S')} \quad .$$

this proves (11.11). Equation (11.11) is the analogue to the block spin relation (3.14).

Remark: Formula (11.11) is used to calculate the original correlation functions in the restricted region as follows:

i) calculate $\langle \tilde{S}'_{q'_1} \ldots \tilde{S}'_{q'_n} \rangle_{u'}$ in $|q'_i| \leq \Lambda$

ii) use the "scaling formula" (11.11) to calculate $\langle \tilde{S}_{q_1} \ldots \tilde{S}_{q_n} \rangle_u$ in $|q_i| \leq \frac{\Lambda}{s}$.

The sense of this procedure is that the asymptotic part of the original correlation functions in the limit $s \to \infty$ at the critical point is calculable from non asymptotic (finite distance) correlations with a "simple" fixed point Hamiltonian H*: i.e. if we choose s such that $sq_i = q_{oi}$ fixed we have as $q_i \to o$

$$\langle \tilde{S}(q_1) \ldots \tilde{S}(q_n) \rangle_u \simeq \left| \frac{q}{q_o} \right|^{nx} \langle \tilde{S}'(q_{o1}) \ldots \tilde{S}(q_{on}) \rangle_{u*}$$

thereby no subsum of the momenta q_1 to q_n should vanish.

Transformation of free energy density:

For $s \gg 1$ the energy density can be written as

$$f(u) = g(u) + s^{-d} f'(u') \tag{11.12}$$

where the regular part $g(u)$ is the constant (S_o-independent) term appearing by the elimination of $\tilde{S}_{1q}$ fluctuations and the singular part $f'(u')$ is the contribution coming from the transformed Hamiltonian.

For the finite system we easily derive a scaling formula for the free energy density. The free energy

$$F_{N'} = -\ln \int \prod_q d\tilde{S}_q \exp - H_\Lambda(S) = - \sum_{\substack{\Gamma \\ \text{conn}}}$$

is given by the sum of vacuum diagrams. By the RG transformation we obtain

$$F_{N'} = G_{N'-N} + F'_N + A_N$$

with contributions

$$G_{N'-N} = - \ln \int \prod_q d\tilde{S}_{1q} \exp - H_\Lambda(S_o+S_1) \Big|_{S_o=o} = - \sum_{\substack{\Gamma \\ \text{conn}}}$$

the vacuum contributions omitted in (11.6)

$$F'_N = - \ln \int \prod_{q'} d\tilde{S}'_{q'} \exp - H'_\Lambda(S') = - \sum_{\substack{\Gamma \\ \text{conn}}}$$

the free energy part given by the transformed Hamiltonian and

$$A_N = - \ln \prod_{o<q\leq\Lambda/s} \alpha_s = - N \times \ln s \; ; \; N = \sum_{o<q\leq\Lambda} \; ; \; N' = \sum_{o<q\leq\Lambda/s} = s^d N$$

is the part coming from the change of phase space

$$\prod d\tilde{S}_{oq} = \alpha_s^N \prod d\tilde{S}'_{q'} \; .$$

The free energy density thus transforms as

$$f(u) = g(\overset{(u,s)}{\cancel{u}})(1-s^{-d}) + x\, s^{-d} \ln s + s^{-d} f'(u'=R_s u) .$$

This proves (11.12) for s large.

Perturbation expansion for $H'_\Lambda(S')$:

We now expand the RG-transformation (11.6) in a formal power series of $H_{int\Lambda}$. From the support poperties of S_o and S_1 (e.g. $\int_q \tilde{S}_{oq} \tilde{S}_{1q} = o$) it follows

$$H_{o\Lambda}(S_o+S_1) = H_{o\Lambda}(S_o) + H_{o\Lambda}(S_1) . \tag{11.13}$$

Thus

$$H'_\Lambda(S') = H_{o\Lambda}(S_o) - \ln \int_{\otimes} S_1 \; e^{-H_{o\Lambda}(S_1)} \; e^{-H_{int\Lambda}(S_o+S_1)} \Big|_{\tilde{S}_{oq}=\alpha_s \tilde{S}'_{q'}=sq}$$

$$= H_{o\Lambda}(S_o) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} H^n_{int\Lambda}\left(S_o + \frac{\delta}{\delta J_1}\right) \exp \frac{1}{2}(J_1, G_1 J_1) \Big|_{J_1=o} \; . \tag{11.14}$$

$$\otimes_{\substack{\text{conn} \\ \tilde{S}_{oq}=\alpha_s S'_{sq}}}$$

Here we used the analogue formula to (10.33). The omission of vacuum graphs ⊠ means that we have no constant term in H'(S') (regular contribution to free energy). Omission of disconnected graphs (conn) means taking the logarithm.

The "propagator" $\tilde{G}_{1q}$ related to $H_{o\Lambda}(S_1)$ has support in the shell $\Lambda/s < |q| \leq \Lambda$ i.e.

$$(J_1, G_1 J_1) = \int_{\substack{q \\ \Lambda/s<|q|\leq\Lambda}} |\tilde{J}_{1q}|^2 (q^2+m^2)^{-1}$$

and (11.15)

$$G_{1q} = \frac{\Theta(\Lambda-|q|)\ \Theta(|q|-\Lambda/s)}{q^2+m^2}$$

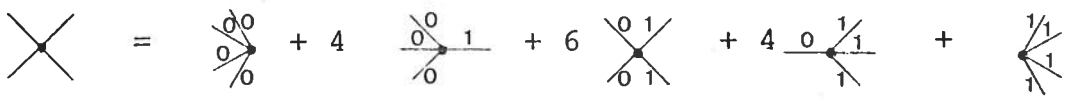
where

$$\Theta(x) = \begin{cases} 1 & x > o \\ o & x < o \end{cases} .$$

The Gaussian integrals in (11.14) are evaluated according to the rules given in section 10.3 with the following modifications:

Internal lines ($\frac{\delta}{\delta J_1}$ terms from H_{int}) have associated a propagator $\tilde{G}_{1q}$ with momentum in a shell $\Lambda/s < |q| \leq \Lambda$. External lines ($S_o$ terms from H_{int}) have associated a factor $\tilde{S}_{oq}$ with momentum in a ball $|q| \leq \Lambda/s$. To the S^4 interaction term there are 5 types of vertices

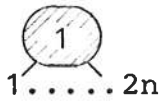
$$(S_o+\frac{\delta}{\delta J_1})^4 = S_o^4 + 4S_o^3 \frac{\delta}{\delta J_1} + 6S_o^2(\frac{\delta}{\delta J_1})^2 + 4S_o(\frac{\delta}{\delta J_1})^3 + (\frac{\delta}{\delta J_1})^4$$



Only the 1-lines are contracted.

Feynman rules for $H'_\Lambda(S')$:

We may write H' as a sum

$$-H'_\Lambda(S') = \sum_{\Gamma_n} J_{\Gamma_n}(\tilde{S}'_{q'}) = \sum_{\substack{\Gamma_n \\ \boxtimes \\ \text{conn}}}$$ 

(11.16)

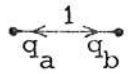
where each contribution J_{Γ_n} is represented by a graph Γ_n

$J_{\Gamma_n}(\tilde{S}'_{q'}) \doteq$

.

1° The graphs Γ_n are all possible connected non vacuum graphs with

vertices

internal lines 

external lines

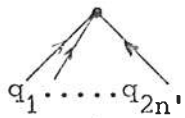
2° The lines at each vertex are labelled by $q_1,\ldots,q_{2n'}$ incoming momenta

internal momenta range: $\Lambda/s < |q_a| \leq \Lambda$
(S_1-lines)

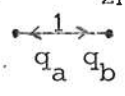
external momenta range: $|q_e| \leq \Lambda/s$.
(S_o-lines)

The contribution J_{Γ_n} corresponding to Γ_n is then given by the following assignement:

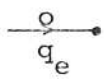
3° Associate to each

vertex : 

$= (2\pi)^d \delta^{(d)}(q_1+\ldots+q_{2n'})\, u_{2n'}(q)$

internal line : 

$= (2\pi)^d \delta^{(d)}(q_a+q_b)\ \dfrac{1}{q_a^2+m^2}$

external line : 

$= \tilde{S}_{oq_e}$

4° Integrate over all momenta

$$\frac{1}{(2\pi)^d} \int_{\Lambda/s<|q_a|\leq\Lambda} d^d q_a \ldots\ldots \qquad (S_1\text{-lines})$$

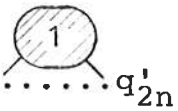
$$\frac{1}{(2\pi)^d} \int_{|q_e|\leq\Lambda/s} d^d q_e \ldots\ldots \qquad (S_o\text{-lines})$$

5° substitute:

$$\tilde{S}_{oq} = \alpha_s \tilde{S}'_{osq} \quad \text{and} \quad sq = q' \ .$$

Omitting the factors $\tilde{S}'_{q_e}$, the integrations over the external momenta q'_e, and the overall δ-function $(2\pi)^d \delta^{(d)}(q'_{e1}+..+q'_{e2n})$ (we denote this omission by a(!)) in these rules, we obtain the Feynman rules for the perturbation expansion of the transformed couplings:

$$u'_{2n}(q'_1,\ldots q'_{2n}) = (R_s u)_{2n} = \sum_{\substack{\boxtimes \\ \text{conn} \\ (!)}} \quad q'_1 \ldots\ldots q'_{2n} \ . \qquad (11.17)$$



Before we are going to perform the lowest order perturbation calculations for R_s we have to investigate under what circumstances we can expect the perturbation expansion to make sense. This will be done in the next section.

12. Power Counting, Motivation of the ε-Expansion

Disregarding convergence problems with infinite series we may estimate the behavior of the transformed couplings $u'_{2n}(q')$ for large s (power counting) in the perturbation expansion.

Let us introduce in virtue of (11.8) and (11.10) the operators:

$$\mathcal{O}_{nl\alpha}(\tilde{S}_q) = \prod \theta_{q_i} \; P_\alpha(q_1^{l_1},\ldots q_{2n}^{l_{2n}})\tilde{S}_{q_1}\ldots\tilde{S}_{q_{2n}} \tag{12.1}$$

such that we may write

$$H_\Lambda(S) = \sum_{n,l,\alpha} g_{nl\alpha} \int_{q_1}\ldots\int_{q_{2n}} (2\pi)^d \delta^{(d)}(\Sigma q_i)\; \mathcal{O}_{nl\alpha}(\tilde{S}_q)$$

$$H'_\Lambda(S') = \sum_{n,l,\alpha} g'_{nl\alpha} \int_{q'_1}\ldots\int_{q'_{2n}} (2\pi)^d \delta^{(d)}(\Sigma q'_i)\; \mathcal{O}_{nl\alpha}(\tilde{S}'_{q'}) \tag{12.2}$$

with

$$g' = R_s g \; . \tag{12.3}$$

By definition of the RG-transformation g' picks up powers in s from the substitution

$$\tilde{S}_{oq} = s^x \tilde{S}'_{q'=sq}$$

with

$$x = \frac{d+2-\eta}{2} \qquad \text{(see (4.11))}.$$

We have

(i) by the substitution as $\tilde{S}_{oq} \to s^x \tilde{S}'_{q'}$ and $q \to s^{-1}q'$

$$\mathcal{O}_{nl\alpha}(\tilde{S}_{oq}) = s^{2nx}\, s^{-l}\; \mathcal{O}'_{nl\alpha}(\tilde{S}'_{q'}) \tag{12.4}$$

a factor s^{2nx-l} (s^{2nx} renormalization factor, s^{-l} dilatation factor)

(ii) by the change of integration variables

$$\int d^d_{q_1}\ldots\int d^d_{q_{2n}}\; \delta^{(d)}(\Sigma q_i)\ldots = s^{-(2n-1)d} \int d^d_{q'_1}\ldots\int d^d_{q'_{2n}}\; \delta^{(d)}(\Sigma q'_i)\ldots$$

a factor $s^{-(2n-1)d}$ (phase space factor) (12.5)

and thus

$$g'(s) \propto s^{\omega} \tag{12.6}$$

with

$$\omega = 2nx - (2n-1)d - 1 = d - 2nd_S - 1 \ . \tag{12.7a}$$

In this formula we introduced the dynamical dimension of S (see (4.10) and (4.11))

$$d_S = d-x = \frac{d-2+\eta}{2} \ . \tag{12.7b}$$

According to the definition (5.6) the "operators" $\mathcal{O}_{nl\alpha}$ resp. the couplings $g_{nl\alpha}$ are classified as:

$$\begin{array}{lll} \text{relevant} & \omega(n,l) > o \ ; \ g'_{nl\alpha} \propto s^{\omega} \to \infty & (s\to\infty) \\ \text{marginal} & \omega(n,l) = o \ ; \ g'_{nl\alpha} \ \text{(const. or slowly variing)} & \\ \text{irrelevant} & \omega(n,l) < o \ ; \ g'_{nl\alpha} \propto s^{\omega} \to o & (s\to\infty) . \end{array} \tag{12.8}$$

We will <u>assume</u> that the terms irrelevant order by order in the perturbation expansion are irrelevant also in the summed up theory. Irrelevant terms will be ignored as they are transformed away by the RG-transformation.

What are the non irrelevant "operators"?
From positivity of the two point correlation η must be non negative

$$\eta \geq o \ .$$

We further expect (see below and from known results for the Ising system)

$$\eta = o \qquad d \geq 4$$
$$\eta = \frac{1}{4} \qquad d = 2 \ \text{(Ising)} .$$

There are also indications that $\eta(d)$ is a monotonic continuous function i.e.

$$\frac{1}{4} \geq \eta \geq o \ \text{for} \ d \geq 2 \ .$$

As we will see we may use for $d>2$ the canonical classification setting $\eta = o$ (rough estimate).

The relevant and marginal terms can be read off from the following table where we list the values of $\omega(n,l)$ and the corresponding fields (in brackets the dimension for which the corresponding field is canonically marginal or gets relevant by lowering the dimension):

l \ n	1	2	3	4
o	S^2 relevant $2-\eta>o$	S^4 $4-d-2\eta$ $(d=4)$	S^6 $6-2d-3\eta$ $(d=3)$	S^8 $8-3d-4\eta$ $(d=2,66..)$
2	$S\,\partial^2\,S$ (marginal) $-\eta$	$S^3\,\partial^2\,S$ $2-d-2\eta$ $(d=2)$	$S^5\,\partial^2\,S$ $4-2d-3\eta$ $(d=2)$	$S^7\,\partial^2\,S$ $6-3d-4\eta$ $(d=2)$
4	$S\,\partial^4\,S$ $-2-\eta<o$	$S^3\,\partial^4\,S$ irrelevant $-d-2\eta<o$	$S^5\,\partial^4\,S$ $2-2d-3\eta<o$	$S^7\,\partial^4\,S$ $4-3d-4\eta<o$

We summarize the canonical classification:

1) $d > 4$: Only $S^2(x)$ and $S\partial^2S(x)$ are not irrelevant i.e. the critical theory is a free field theory (Gaussian model)

2) $4 \geq d > 3$: The $S^4(x)$ term appears as an additional relevant ($d=4$: marginal) field

3) $3 \geq d > 2$: More and more terms $S^{2n}(x)$ $(n=3,4,..)$ $(n \leq \frac{d}{d-2})$ appear (at $d \leq \frac{2n}{n-1}$) to be not irrelevant as d decreases. So $S^6(x)$ in $d=3$ $S^8(x)$ in $d=2,66..$ etc.

4) $d = 2$: Canonically $\omega = 2-l$ i.e. it appears that all terms $S^{2n}(x)$ are relevant and $S^{2n-3}\partial^2S(x)$ are marginal. Dynamically, however, with the Ising value $\frac{1}{4}$ for η only the terms $S^{2n}(x)$ with $n \leq 8$ are not irrelevant. In any case the term with the largest $\omega(n,l)$ dominates provided the corresponding coupling is non vanishing.

The conclusion we have to draw from the power counting consideration is the following: In $d \geq 4$ dimensions the critical theory is Gaussian i.e. a free field theory. A non trivial theory is expected for $d < 4$ dimensions. It looks plausible that the critical theory (e.g. critical exponents etc.) depends continuously (even monotonically?) on d. Wilson and Fisher proposed on this basis the ε-expansion. This expansion in the dimensionality $\varepsilon = 4-d > o$ of the system around $d = 4$ appears as an asymptotic expansion around the Gaussian model. It is in this case that we expect perturbation theory to be applicable.

By our consideration (true to all orders in the perturbation expansion according to power counting) Kadanoff's picture on critical behavior is confirmed for the Euclidean cut-off field theory model (and to the extent of its connection with the Ising model also for the later one). According to this picture the effective Hamiltonian describing the critical long range behavior is "simple" in the sense that only a few couplings are relevant and ∞-many irrelevant couplings do not contribute (universality).

Important remark:
The Ising energy density $\mathcal{O}_E$ defined in (6.2) has in the continuous spin model defined in section 10 the form

$$\mathcal{O}_E(x) = - \sum_y \sigma_x K_{x-y} \sigma_y = \frac{1}{2} S(x) (-\Delta_a + m_o^2) S(x) \qquad (12.9)$$

with $m_o^2 = -2a^{-2}d$. By the power counting argument S^2 is relevant and $S\Delta S$ marginal irrespective of the dimension i.e. the S^2 term is <u>strongly dominating</u> $S\Delta S$ in the critical region. Therefore

$$E(x) = \frac{1}{2} S^2(x) \qquad (12.10)$$

represents the renormalized (by m_o^2) energy density near the critical point.

13. Calculation of Fixed Points and Critical Exponents in 4-ε (ε > o) Dimensions

The analysis of the last section suggests that an approximate calculation of R_s by the perturbation expansion (in H_{int}) is possible for large enough dimensions (at least for $d \geq 4$).

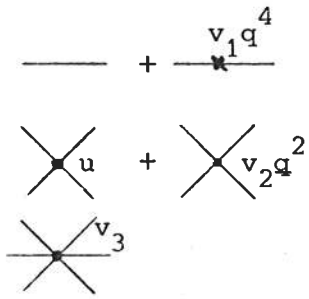
We consider the system defined by $H_\Lambda(S)$ with couplings (taking also the lowest irrelevant terms):

$$u_2 = \frac{1}{2}(m^2 + q^2) + v_1 q^4 =$$

$$u_4 = u + v_2 \underline{q}^2 =$$

$$u_6 = v_3 =$$

$$u_{2n} = o \quad (n>3) \tag{13.1}$$



(where $\underline{q}^2 = (q_1+q_2)^2 + (q_2+q_3)^2 + (q_3+q_1)^2$).

We have to calculate to lowest order (we expect u to be of first order the v_i of higher order)

$$u_2' = \frac{1}{2}(m'^2+q'^2) + v_1' q'^4 + \ldots$$

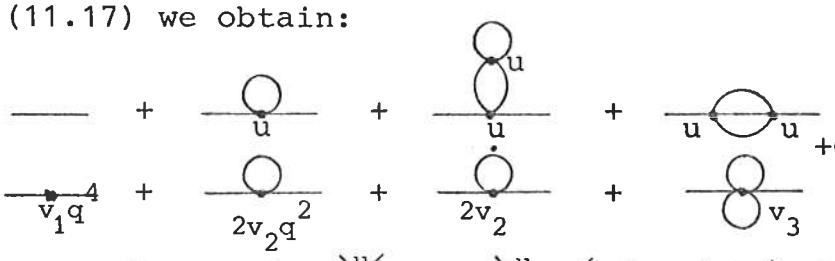
$$u_4' = u' + v_2' \underline{q}'^2 + \ldots$$

$$u_6' = v_3' + \ldots \tag{13.2}$$

$$u_{2n}' = \ldots \quad (n > 3) .$$

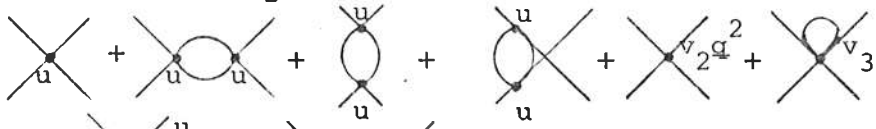
According to (11.17) we obtain:

$$u_2' =$$



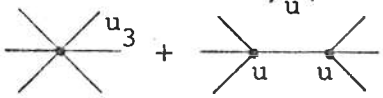
$+O(u^3, v_1^2, v_2^2, v_3^2)$

$$u_4' =$$

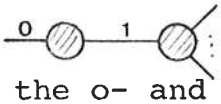


$+O(\ldots)$ (13.3a)

$$u_6' =$$



+ 9 crossed terms + O(...)

(Notice that graphs of the form  = o give no contribution due to the supports of the o- and 1-lines in momentum space.)

Thus using the Feynman rules the RG-transformation reads:

$$u_2' = s^{2x-d}\,\{\tfrac{1}{2}(m^2+q^2) + 6uI_1 - 72u^2I_1I_2 - 48\,u^2\,I_3(q)$$

$$+\; v_1q^4 + 12\,v_2(q^2\,I_1+\dot{I}_1) + 90\;\; v_3\,I_1^2 + \ldots\}$$

(13.3b)

$$u_4' = s^{4x-3d}\,\{u - 12\,u^2\,[I_2(q_1+q_2)+ 2 \text{ crossed}] + \;v_2q^2 + 15\;\; v_3I_1+\ldots\}$$

$$u_6' = s^{6x-5d}\,\{v_3 - \frac{8}{10}\,u^2\,[\frac{\Theta}{(q_1+q_2+q_3)^2+m^2} + 9 \text{ crossed}] + \ldots\ldots\}$$

where we still have to substitute $q = s^{-1}q'$. The integrals I_i are defined in Appendix 13.A. They are functions of Λ, s, m^2 and d.

The approximate RG-transformation for the parameters in (13.1) takes the form

$$\tfrac{1}{2}m'^2 = s^{2x-d}\,\{\tfrac{1}{2}m^2+6uI_1-72u^2I_1I_2-48u^2I_3+90v_2I_1^2+12v_2\dot{I}_1+\ldots\}$$

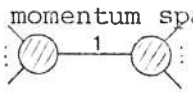
$$\tfrac{1}{2}q'^2 = s^{2x-d-2}\,\{\tfrac{1}{2} - 48u^2I_3' + v_2I_1 + \ldots\;\}\,q'^2$$

$$u' = s^{4x-3d}\,\{u - 36u^2I_2 + 15v_3I_1 + \ldots\}$$

$$v_1' = s^{2x-d-4}\,\{v_1 - 48u^2I_3'' + \ldots\} \qquad (13.4)$$

$$v_2' = s^{4x-3d-2}\{\,v_2 - 36u^2I_2' + \ldots\}$$

$$v_3' = s^{6x-5d}\,\{v_3 + \ldots\}\,.$$

We systematically use the expansion (11.10). This expansion is asymptotic for small momenta. One line reducible graphs (can be cut into two parts by cutting one line) do not contribute to this expansion because their momentum space support is disjoint from the origin q_i=o (expansion point):  = o

This is legitimate in so far as quantities without long range part (no small momenta) are assumed not to affect critical long range behavior. The expansion (11.10) is used for symplicity only. Expansions with respect to a fixed vector in a momentum shell are also manageable in order to get better approximations. for the nonuniversal quantities

The problem now is to find fixed points of the approximate RG-transformation (13.4). Even in this approximation the transformation is of very complicated non linear form and the critical values of m, u, ... are not known. Therefore further approximations are necessary on which consistency tests have to be made.

In the perturbation expansion obviously there is always a Gaussian fixed point

$$m^* = o, \; u^* = o \; , \; v_i^* = o \tag{13.5}$$

present. For d>4 this fixed point is stable for (critical subspace)

$$m = m_c = o.$$

On this subspace (13.5) is instable for d<4. As we will see below d=4 a stable non-Gaussian fixed point appears which is close to Gaussian for d<4, but close to four.

The continuation of the RG-transformation from integer to arbitrary real dimensions d is defined by continuation of the Feynman integrals I_i to generic d (see Appendix 13.A).

The fact that a stable Gaussian fixed point (13.5) is consistent with the transformation (13.4) for d≥4 suggests to look for fixed points in the region

$$\Lambda/_s \gg m^2 > o \; ; \; u, \; v_i \text{ small}$$

and

$$1 \gg \varepsilon = 4-d > o.$$

Thus we may expand the integrals I_i appearing in (13.4) in the parameters

$$ms/_\Lambda \text{ and } \varepsilon \; .$$

In order to restore the connection to the Ising system we would have to consider a different region in parameter space.

In the transition from the lattice system to the cut-off system we roughly have

$$m^2 \to m^2$$

$$u \to u$$

$$\frac{\pi}{a} \to \Lambda$$

and by (10.15) and (10.9):

$$m^2 \to \hat{m}^2 = -\text{const. } u_o \Lambda^2$$

$$u_o = \text{const. } u\Lambda^{d-4} >> 1$$

i.e.

$$\left(\frac{m}{\Lambda}\right)^2 = -\text{const. } u_o \text{ ; large negative}$$

$$\frac{u}{\Lambda^{4-d}} = \text{const. } u_o \text{ ; large positive.}$$

The values lie far away from the fixed point in the curved parameter space and they are beyond the region of application of the linear approximations which we are able to investigate. By linear extension we will only get a qualitatively correct picture for the relation to the original system which concerns nonuniversal quantities like the critical temperature etc..

We now discuss the successive approximations of the RG-transformation (13.4).

1. Gaussian case

Equation (13.4) reads in the Gaussian approximation

$$m'^2 = s^{2x-d}\, m^2$$
$$q'^2 = s^{2x-d-2}\, q'^2 \qquad (13.6)$$

with fixed point

$$x = \frac{d+2}{2} \quad (\eta = o)$$
$$m^* = o. \qquad (13.7)$$

As

$$u' = s^{4-d}\, u \tag{13.8}$$

this fixed point for

$$m = m_c = o \quad \text{(critical point)}$$

is

stable for $d > 4$

instable for $d < 4$.

2. Lowest order non-Gaussian case

The next leading approximation reads

$$m'^2 = s^{2x-d}\,\{m^2 + 12u\, K_d \int_{\Lambda/s}^{\Lambda} \frac{dr\, r^{d-1}}{r^2+m^2}\}$$

$$q'^2 = s^{2x-d-2}\, q'^2 \tag{13.9a}$$

$$u' = s^{4x-3d}\,\{u - 36u^2 K_d \int_{\Lambda/s}^{\Lambda} \frac{dr\, r^{d-1}}{(r^2+m^2)^2}\}$$

which implies to this approximation again a Gaussian value for η:

$$x = \frac{d+2}{2} \quad (\eta = o).$$

Expanding in $\varepsilon = 4-d$ and $\frac{ms}{\Lambda} = O(\varepsilon)$ we have to leading order:

$$m'^2 = s^2\,\{m^2 + 12u\, K_d \frac{\Lambda^2}{2}(1-s^{-2})\} + O(u\varepsilon, u^2)$$

$$u' = (1+\varepsilon \ln s)\,\{u - 36u^2 K_d \ln s\} + O(u^2\varepsilon, u^3) \tag{13.9b}$$

(we expand $s^\varepsilon = e^{\varepsilon \ln s} \simeq 1 + \varepsilon \ln s + O(\varepsilon^2)$).

Fixed point of R_s:

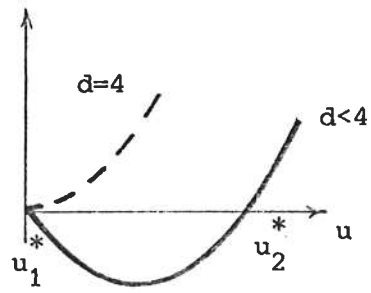
The condition for a fixed point reads

$$m^{*2} = s^2 \{m^{*2} + 6u^* K_d \Lambda^2 (1-s^2)\}$$
$$u^* = u^* - 36u^{*2} K_d \ln s + u^* \varepsilon \ln s \qquad (13.10)$$

with the solutions

1° Gaussian fixed point (instable for $\varepsilon > o$)

$$u^* = o$$
$$m^{*2} = o \; . \qquad (13.11)$$



2° Nontrivial fixed point

$$u^* = \frac{1}{36K_d} \varepsilon + O(\varepsilon^2)$$
$$m^{*2} = - \frac{\varepsilon}{6} \Lambda^2 + O(\varepsilon^2) \; . \qquad (13.12)$$

<u>Linearization of R_s:</u>

We linearize about the nontrivial fixed point. We set

$$m^2 = m^{*2} + \Delta m^2$$
$$u = u^* + \Delta u \qquad (13.13)$$

with $|\Delta m^2| < |m^{*2}|$ and $|\Delta u| < |u^*|$ and expand (13.9b) in Δm^2 and Δu obtaining a linear transformation for $(\Delta m^2, \Delta u)$.
We obtain

$$\Delta m^{2'} = s^2\{\Delta m^2 + 6K_d\Lambda^2(1-s^{-2})\Delta u\}$$
$$\Delta u' = (1+\varepsilon \ln s)\ (1-72u^* K_d \ln s)\Delta u$$

i.e.

$$\begin{pmatrix} \Delta m^{2'} \\ \Delta u' \end{pmatrix} = \begin{pmatrix} s^2 & 6K_d\Lambda^2(s^2-1) \\ o & 1-\varepsilon \ln s \end{pmatrix} \begin{pmatrix} \Delta m^2 \\ \Delta u \end{pmatrix} \; . \qquad (13.14)$$

By the semigroup property the linearized RG-transformation has eigenvalue of the form $\lambda_i = s^{y_i}$: Thus

$$T_s = \begin{pmatrix} \lambda_1 & c \\ o & \lambda_2 \end{pmatrix}$$

with

$$\lambda_1 = s^{y_1} \; ; \quad y_1 = \frac{\ln\lambda_1}{\ln s} = 2 + O(\varepsilon)$$
$$\lambda_2 = s^{y_2} \; ; \quad y_2 = \frac{\ln\lambda_2}{\ln s} = -\varepsilon + O(\varepsilon^2) \tag{13.15}$$

(we used $\ln(1+x) \simeq x + \ldots \; |x| << 1$).

The eigenvectors of T_s are

$$L_1 = \begin{pmatrix} 1 \\ o \end{pmatrix} \quad \text{and } L_2 = \begin{pmatrix} a \\ 1 \end{pmatrix}$$
$$a = -\frac{c}{\lambda_1 - \lambda_2} = 6K_d\Lambda^2 + O(\varepsilon) \tag{13.16}$$

($K_d = K_4 = (8\pi^2)^{-1}$ in the above equations).

<u>Critical subspace</u>

The critical subspace is defined by

$$T_s \begin{pmatrix} \Delta m^2 \\ \Delta u \end{pmatrix}_{crit.} \to \begin{pmatrix} o \\ o \end{pmatrix} \qquad (s\to\infty)$$

i.e. the fixed point (13.12) is approached under the RG-transformation.

Because the eigenvalue $\lambda_1 > 1$

$$T_s \begin{pmatrix} 1 \\ o \end{pmatrix} = s^{y_1} \begin{pmatrix} 1 \\ o \end{pmatrix} \to \begin{pmatrix} \infty \\ o \end{pmatrix} \qquad (s\to\infty)$$

and by $o < \lambda_2 < 1$

$$T_s \begin{pmatrix} a \\ 1 \end{pmatrix} = s^{y_2} \begin{pmatrix} a \\ 1 \end{pmatrix} \to \begin{pmatrix} o \\ o \end{pmatrix} \quad (s \to \infty).$$

Hence writing an arbitrary vector as

$$\begin{pmatrix} \Delta m^2 \\ \Delta u \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ o \end{pmatrix} + A_2 \begin{pmatrix} a \\ 1 \end{pmatrix} \tag{13.17}$$

the critical surface is defined by $A_1 = o$ (see Fig. 21) i.e.

$$\begin{pmatrix} \Delta m_c^2 \\ \Delta u_c \end{pmatrix} = A_2 \begin{pmatrix} a \\ 1 \end{pmatrix} \tag{13.18}$$

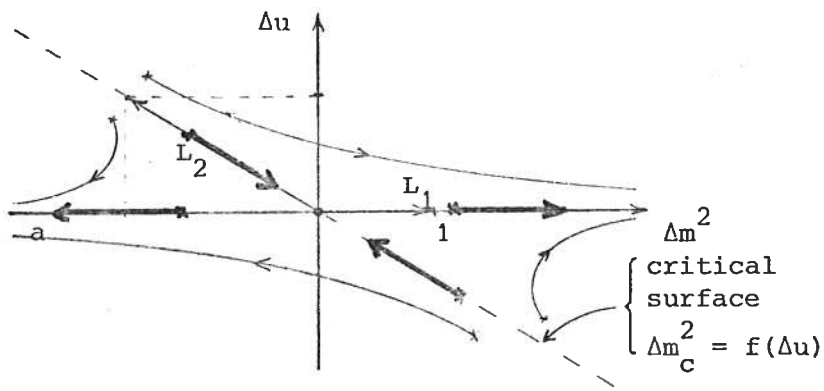


Fig. 21

Just for fun let us check qualitatively by linear extension the relation of our investigation to the Ising model. We use (10.15) with $\Lambda = \pi/a$ and choose $u_o >> 1$:

$$\begin{aligned} \hat{m}^2 &= -2\left(\frac{\Lambda}{\pi}\right)^2 \frac{u_o}{k} \\ u &= \frac{u_o}{4k^2}\left(\frac{\Lambda}{\pi}\right)^{4-d} \simeq \frac{u_o}{4k^2} \end{aligned} \tag{13.19}$$

By our calculation (13.18)

$$a = \frac{\Delta m_c^2}{\Delta u_c} \simeq \frac{\hat{m}_c^2}{u_c} \tag{13.20}$$

for u_o large as m^{*2} and u^* small and

$$a = -\frac{3}{4}\left(\frac{\Lambda}{\pi}\right)^2 .$$

On the other hand (13.19) at the critical point implies

$$\frac{\hat{m}_c^2}{u_c} = -8\, k_c \left(\frac{\Lambda}{\pi}\right)^2 . \tag{13.21}$$

By comparison we would get

$$k_c = \frac{K}{k_B T_c} = \frac{3}{32} \simeq 0.094 \tag{13.22}$$

for the critical Ising coupling.

The qualitative correctness of our approximation is remarkable. u_o can be chosen arbitrarily large (as required) along the critical surface, and $\hat{m}^2$ and u have the correct behaviour and signs.

Critical exponents:

η: To this order of the approximation

$$\eta = 0 + O(\varepsilon^2) \qquad \text{(Gaussian value)}.$$

As we will discuss below $v_2 = O(\varepsilon^3)$ in (13.4) thus we easily get the $O(\varepsilon^2)$ term to η by taking into account the term $u^2 I_3'$ such that

$$q'^2 = s^{-\eta}\{1 - 96u^2\, I_3'\}\, q'^2 .$$

One evaluates

$$I_3' = -\, c_o \ln s + O(\varepsilon) \;;\; c_o = \frac{K_d^{\,2}}{4} . \tag{13.23}$$

Then from

$$1 = (1-\eta \ln s)(1-96u^2\, c_o \ln s)$$

we obtain at the fixed point

$$\underline{\eta = 96\, c_o u^{*2} = \frac{\varepsilon^2}{6\cdot 9} + O(\varepsilon^3)} . \tag{13.24}$$

Notice $\eta = O(\varepsilon^2)$ does not change the other equations in (13.4) to the order considered.

$\underline{\nu}$: By (10.15) with $\Lambda \simeq \frac{\pi}{a}$ we have

$$\Delta m^2 \propto \Lambda^2 u_o (\frac{1}{k} - \frac{1}{k_c}) = \Lambda^2 u_o \frac{k_B}{K}(T-T_c)$$

i.e. Δm^2 is proportional to the reduced temperature $t = (T-T_c)/_{T_c}$. Under the RG-transformation

$$\xi' = s^{-1} \xi_o \quad \text{and} \quad t' = s^{y_1} t_o$$

and hence

$$\xi' = \xi_o \cdot (\frac{t'}{t_o})^{-1/y_1} \propto t'^{-\nu} \Rightarrow \nu = 1/y_1 .$$

To the order of approximation thus

$$\nu = \frac{1}{2} + O(\varepsilon) \quad \text{(Gaussian value)}$$

we can, however, easily calculate the $O(\varepsilon)$ term because T_s is of triangular form and C does not affect the eigenvalue λ_1 we only need consider terms proportional to um^2 in the first equation of (13.4) i.e. the next leading term from $uI_1 = \ldots -u\, K_d m^2 \ln s$. Then in equations (13.9) and (13.14)

$$\begin{aligned} m^2 &\Rightarrow m^2(1 - 12u\, K_d \ln s) \\ \Delta m^2 &\Rightarrow \Delta m^2(1 - 12u^*\, K_d \ln s) \simeq \Delta m^2\, s^{-12u^* K_d} \end{aligned} \tag{13.25}$$

and y_1 is changed to

$$y_1 = 2 - 12u^*\, K_d = 2 - \frac{\varepsilon}{3} + O(\varepsilon^2)$$

and

$$\underline{\nu = \frac{1}{2} + \frac{\varepsilon}{12} + O(\varepsilon^2)} . \tag{13.26}$$

In analogy to the discussion in section 6 we obtain the scaling relations for the free energy density (sect. 4) and the correlation functions (see (7.6) and (7.7)) from the equations (11.12) and (11.11) by inserting the linearized form of R_s and setting the irrelevant variables equal to zero (the generalization of R_s to nonvanishing external field is straight forward) i.e. $A_2 = o$ $(\Delta u = o)$ in (13.17) $\Delta m^2 \propto t$ $(y = y_1)$:

$$f(t,h) = \cancel{G}\,g(t,h) + s^{-d} f(ts^y, hs^x)$$

$$\langle \tilde{S}(q_1)\ldots\tilde{S}(q_n)\rangle \quad (t,h) = s^{nx}\langle \tilde{S}(sq_1)\ldots\tilde{S}(sq_n)\rangle \; (ts^y, hs^x). \tag{13.27}$$

If the leading irrelevant term $\Delta u \neq o$ $(A_2 \neq o)$ we have <u>corrections to scaling</u> governed by the exponent

$$\omega \equiv y_2 = -\varepsilon + O(\varepsilon^2) \; . \tag{13.28}$$

The scaling variables g_1 in which T_s acts diagonally (introduced in (8.45)) are the variables introduced in (13.17)

$$\begin{aligned} g_1 &= A_1 = \Delta m^2 - a\Delta u \\ g_2 &= A_2 = \Delta u \\ T_s\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= \begin{pmatrix} s^{y_1} g_1 \\ s^{y_2} g_2 \end{pmatrix} \end{aligned} \tag{13.29}$$

for $(h = o)$:

$$f(g_1,g_2) = g(g_1,g_2) + s^{-d} f'(s^y g_1 \; , \; s^\omega g_2) \tag{13.30}$$

where f has a power series expansion in g_2 resp. $s^\omega g_2$ i.e.

$$f(g_1,g_2) - g(g_1,g_2) = s^{-d} f(s^y g_1, o) + s^{-d} s^\omega g_2 \, f'(s^y g_1, o) \tag{13.31}$$

and similar for the correlation functions.

3. Higher order terms

We close our discussion on Wilson's RG with a few comments and observations on higher order terms.

Using the results that

$$u^* = O(\varepsilon) \quad \text{and} \quad m^{*2} = O(\varepsilon)$$

we are able to say something on the order of the irrelevant couplings v_i. At the fixed point (13.4) implies

$$v_3^* = s^{-2}\{v_3^* + O(\varepsilon^3)\}$$

and hence

$$v_3^* = O(\varepsilon^3) \ .$$

The integral I_2' is $O(\varepsilon)$ (see Appendix 13.A) and thus

$$v_2^* = s^{-2}\{v_2^* + O(\varepsilon^3)\}$$

so again

$$v_2^* = O(\varepsilon^3) .$$

If $I_3'' = O(1)$ (we have not evaluated this) then

$$v_1^* = s^{-2}\{v_1^* - O(\varepsilon^2)\}$$

and

$$v_1^* = O(\varepsilon^2) \ .$$

As v_1 does not contribute linearly in the other equations (in particular not to the u'-equation) only the u^2 terms contribute to $O(\varepsilon^2)$. With this discussion we have satisfied ourselves that the ε-expansion used above was systematic.

The calculation of the proceeding terms in the ε-expansion is hard to perform by the method discussed here. The main difficulties are the unpleasant cut-off (shell) integrals. Higher order results for critical exponents are given in section 14.

The calculation of the universal (cut-off independent) quantities like critical exponents etc. is much easier if one removes the cut-off from the model. Wilson's RG in the so-called renormalized field theory approach is then replaced by a linear partial differential equation (Gell-Mann-Low renormalization group equation and/or Callan-Symanzik equation). These techniques, however, have their own limitations. It has not been possible up to now to calculate anything beyond the weak coupling approximations (e.g. ε-expansion). The lattice RG calculations which are strong coupling calculations in the field theory sense have a much wider range of applicability.

In Tab. 6 we list the values for critical exponents obtained by continuation to ε = 1 and 2 from the ε-expansion for the S^4-model. The agreement with the Ising values (d=3 series expansions, d=2 exact) is remarkable remembering all the approximations and expansions we have used.

Exp.	d=4 (ε=o) mean field (Gaussian)	d = 3 (ε=1)				d = 2 (ε=2)		
		ε	ε^2	ε^3	Ising (HTE)	ε	ε^2	Ising
α	o	0.167	0.077	0.196	0.125±.015	0.333	-0.025	o
β	$\frac{1}{2}$	0.333	0.340	0.304	0.312±.003	0.167	0.191	0.125
γ	1	1.167	1.244	1.195	1.250±.003	1.333	1.642	1.75
δ	3	6	4.463	6.931	5.15 ± .02	7	6.852	15
ν	$\frac{1}{2}$	0.583	0.627	0.601	0.642± .003	0.667	0.840	1
η	o	o	0.037	0.029	0.041± .01	o	0.235	0.25

Tab. 6: Critical Exponents from the ε-Expansion

The investigation of Wilson's RG for the Landau-Ginzburg-Wilson model again supports the Kadanoff picture of critical behavior discussed in part I. We have found a non trivial fixed point in $4-\varepsilon$ dimensions which confirms a non trivial long range scaling theory. Also within our approximation universality is confirmed. There are two independent critical exponents $y_i > o$ related to two relevant variables, i.e. the temperature and the external magnetic field. The fixed point theory is continuously connected to a Gaussian theory in $d = 4$ dimensions. The critical exponents are functions on d only, i.e. for fixed d they are universal pure numbers.

Appendix 13.A: Feynman integrals

$= \int_k \frac{1}{k^2+m^2} = I_1$

$= \int_k \frac{k^2}{k^2+m^2} = \dot{I}_1$

$= I_1^2$

$= \int_k \frac{1}{k^2+m^2} \frac{\Theta}{(p-k)^2+m^2} = I_2(p) = I_2 + p^2 I_2' + \ldots$

$= I_1 \cdot I_2$

$= \int_{k_1 k_2} \frac{1}{k_1^2+m^2} \frac{1}{k_2^2+m^2} \frac{\Theta}{(p-k_1-k_2)^2+m^2} = I_3(p) = I_3 + p^2 I_3' + p^4 I_3'' + \ldots$

with

$$\int_k \ldots = (2\pi)^{-d} \int_{\Lambda/s<|k|\leq\Lambda} d^d k \ldots ; \quad \frac{\Theta}{x^2+m^2} \equiv \frac{\Theta(\Lambda-|x|)\Theta(|x|-\Lambda/s)}{x^2+m^2}$$

$$I_2 = \int_k \frac{1}{(k^2+m^2)^2}$$

$$p^2 I_2' = \int_k \frac{4(pk)^2 - p^2(k^2+m^2)}{(k^2+m^2)^4}$$

$$I_3 = \int_{k_1,k_2} \Theta \frac{1}{(k_1^2+m^2)(k_2^2+m^2)((k_1+k_2)^2+m^2)}$$

$$p^2 I_3' = \int_{k_1,k_2} \Theta \frac{4(p(k_1+k_2))^2 - p^2((k_1+k_2)^2+m^2)}{(k_1^2+m^2)(k_2^2+m^2)((k_1+k_2)^2+m^2)^3}$$

$$p^4 I_3'' = \int_{k_1,k_2} \Theta \frac{16(p(k_1+k_2))^4 - 12(p(k_1+k_2))^2 p^2((k_1+k_2)^2+m^2) + p^4((k_1+k_2)^2+m^2)^2}{(k_1^2+m^2)(k_2^2+m^2)((k_1+k_2)^2+m^2)^5}$$

The Feynman integrals can be continued as functions of the scalar products from integer to arbitrary real (or complex) dimensions d.

In spherical coordinates

$$\int_{\Lambda/s<|k|\leq\Lambda} d^dk \ldots = \int_{\Lambda/s}^{\Lambda} dr\, r^{d-1} \int d\Omega_d \ldots$$

with $d\Omega_d$ the surface element on the d-1 dimensional sphere:

$$d\Omega_d = (\sin\Theta_1)^{d-2}(\sin\Theta_2)^{d-3}\ldots(\sin\Theta_{d-2})d\Theta_1\ldots d\Theta_{d-2}d\phi$$

$$o \leq \Theta_i \leq \pi \quad ; \quad o \leq \phi \leq 2\pi.$$

Using

$$\int_o^{\pi} d\Theta\, \sin^p\Theta = \sqrt{\pi}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p+2}{2})}$$

we obtain the surface of the d-1 dimensional unit sphere

$$\int d\Omega_d = S_d = 2\,\frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \quad .$$

By K_d we denote the factor

$$K_d = \frac{S_d}{(2\pi)^d} = 2\,\frac{1}{2^d\pi^{d/2}\Gamma(\frac{d}{2})}$$

below.

Evaluation of some integrals for $\Lambda > \Lambda/s >> m > o$:

$$I_1 = K_d \int_{\Lambda/s}^{\Lambda} \frac{dr\, r^{d-1}}{r^2+m^2} \simeq K_d\,\{\frac{\Lambda^{d-2}}{d-2}(1-s^{2-d})-m^2\,\frac{\Lambda^{d-4}}{d-4}(1-s^{4-d})+\Lambda^{d-2}O((\frac{ms}{\Lambda})^4)\}$$

$$\dot{I}_1 = K_d \int_{\Lambda/s}^{\Lambda} \frac{dr\, r^{d+1}}{r^2+m^2}$$

$$\simeq K_d\{\frac{\Lambda^d}{d}(1-s^{-d})-m^2\,\frac{\Lambda^{d-2}}{d-2}(1-s^{2-d})+\Lambda^d\,O((\frac{ms}{\Lambda})^4)\}$$

$$I_2 = K_d \int_{\frac{\Lambda}{s}}^{\Lambda} \frac{dr\ r^{d-1}}{(r^2+m^2)^2}$$

$$\simeq K_d \left\{\frac{\Lambda^{d-4}}{d-4}\ (1-s^{4-d}) - 2m^2\ \frac{\Lambda^{d-6}}{d-6}\ (1-s^{6-d}) + \Lambda^{d-4}\ O((\frac{ms}{\Lambda})^4)\right\}$$

$$I_2' = K_d \int_{\Lambda/s}^{\Lambda} dr\ r^{d-1}\ [\frac{4}{d}\ \frac{r^2}{(r^2+m^2)^4} - \frac{1}{(r^2+m^2)^3}]$$

$$\simeq \frac{4-d}{d}\ K_d\ \{\frac{\Lambda^{d-6}}{d-6}\ (1-s^{6-d}) + \Lambda^{d-6}\ O((\frac{ms}{\Lambda})^2)\}$$

14. Other Models: Survey and Results

14.1. Multicritical systems

Until now we have considered systems of the ferromagnetic short ranged Ising type only, i.e. Hamiltonians

$$H(\sigma) = -\sum_{A\subset G_a} K_\alpha S_A \qquad S_A = \prod_{i\in A} \sigma_i \tag{14.1a}$$

with σ_i classical spin variables with distributions like

$$\rho(\sigma_x^2) = \begin{cases} \sum_{m=1}^{n} \delta(\sigma_x^2 - m)c_m = \delta(P_n(\sigma_x^2)) \\ \sqrt{\frac{u_o}{\pi}}\, e^{-u_o\{P_n(\sigma_x^2)\}^2} \\ \sum_{m=1}^{n} \sqrt{\frac{b}{\pi}}\, e^{-b(\sigma_x^2-m)} \end{cases} \qquad (n=1,2\ldots) \tag{14.1b}$$

For each model out of this class there is an equivalent (i.e. in the same universality class) Landau-Ginzburg-Wilson type model

$$H_\Lambda(S) = \sum_{n=o}^{\infty} \int_{q_1}\ldots\int_{q_n} (2\pi)^d \delta^{(d)}(\Sigma q_i)\, \tilde{S}_{q_1}\ldots\tilde{S}_{q_n}\, u_n(q)$$

with appropriately chosen parameters $u_n(q)$.

From the universality or power counting arguments (section 12) we know that there is a whole hierarchy of possible fixed points that get non-Gaussian in $d=\frac{2n}{n-1} - \varepsilon$ dimensions $(d\geq 2)$ if $P(\sigma)$ or $P'(S)$ is a semi-bounded polynomial of degree 2n

$$P_{2n}(S) = \sum_{i=o}^{2n} a_i\, S^i(x) \;;\; a_{2n} > o\,. \tag{14.2}$$

This polynom determines an effective potential with n the maximal number of possible coexisting phases (Fig. 22). Stability requires the leading term to be even with positive coefficient.

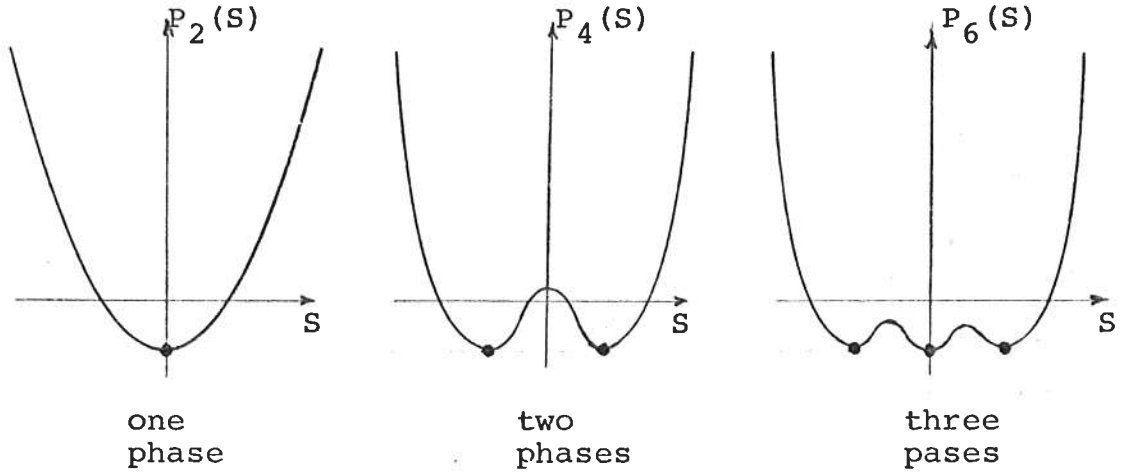


Fig. 22

Critical points at which more than two (bicritical) different phases become identical are called higher order critical points (tricritical, etc.). The corresponding mean field (Gaussian) approximations are obtained by replacing P(S) by $(S+M_i)^2$ at each minimum M_i of P(S).

The rules (they can be established by generalizing the arguments used in part III of these lectures) for the application of the models are as follows:
Starting from a physical (phenomenological) situation (model) we have to specify

(i) <u>the number of relevant parameters</u> (or coexisting phases)

to be adjusted in an experiment (or model) in order to observe the desired transition

(ii) <u>the symmetry of the order parameter</u> (or transformation group relating the different coexisting phases).

In the case considered here the symmetry is a discrete one:

$$S_x \to -S_x. \tag{14.3}$$

These criteria follow more or less directly from the structure of the phase diagram (number of phases, critical points, symmetries near critical points etc. and using universality arguments). In the case of discrete symmetries as a simple phenomenological model, we may always consider a polynomial approximation of the Landau-Ginzburg-Wilson type (continuous classical spin system with cut-off).
For a system with n-th order critical point an appropriate model is defined by a Hamiltonian

$$H_\Lambda(S) = \int d^d x \, \{\tfrac{1}{2}(\partial S)^2 + P_{2n}(S)\} \tag{14.4}$$

Such a system (by arranging suitable the parameters) exhibits always a

line of	n-1-th order
surface of	n-2-nd order
⋮	
n-2 dimensional hyper surface	of 2nd order

transition points.

As an important example (He^3-He^4-mixtures) a system with a tri-critical point can be described by a polynomial

$$P_6(S) = a_6 \, S_x^6 + a_4 \, S_x^4 + a_2 \, S_x^2 + a_o \, .$$

With parameters chosen according to Fig. 23

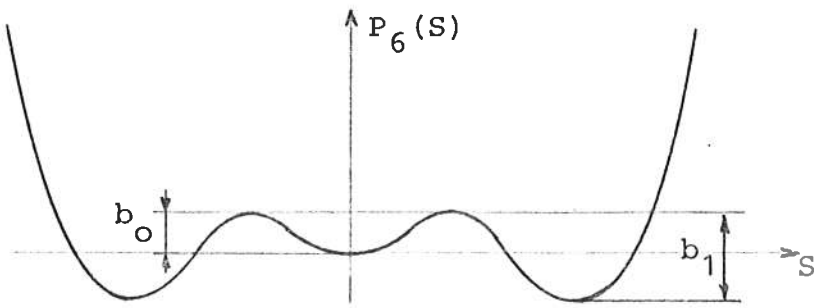


Fig. 23a

(we remind the reader that a critical point always occurs when the difference between two or more coexisting phases disappears) we may approach critical points as

$$b_1 \to o \quad \text{with } \frac{b_o}{b_1} = r > o \text{ fixed.}$$

If $b_1 > o$ we have for

$r > 1$ one stable state	$b_1 \to o$ is noncritical	
$r = 1$ three stable states	$b_1 \to o$ is tricritical	
$r < 1$ two stable states	$b_1 \to o$ line segment of bicritical points parametrized by $o<r<1$.	

b_1 is a "reduced temperature like" parameter.

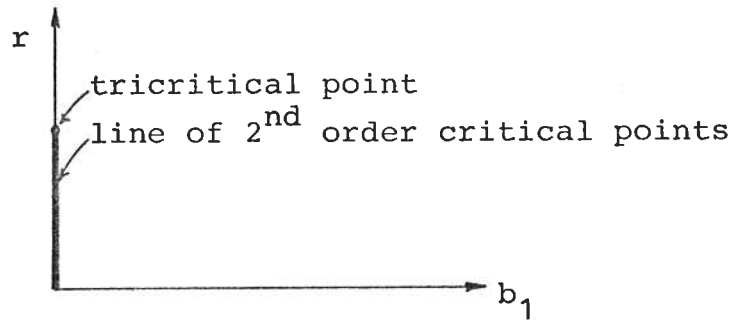


Fig. 23b

The tricritical point is the end point of the line of bicritical points. Obviously the critical behavior is depending on the order of the transition. In d=3 dimensions (this has been verified by means of the ε-expansion in d=3-ε dimensions) we expect a Gaussian tricritical point (compare the power counting argument given in section 12); approaching the bicritical line, however, we expect the d=3 Ising behavior.

The bicritical Hamiltonians are parametrized by the (with respect to $H^*_{bicrit.}$) irrelevant parameter $o<r<1$ (critical exponents independent on r). With respect to $H^*_{tricrit.}$ r is a relevant parameter with critical value $r_{crit.} = 1$. b_1 is relevant with respect to both types of fixed points. (Notice that the terms relevant, marginal and irrelevant always refer to a particular critical point.)

Due to the presence of more than one type of fixed points the physically interesting cross over phenomenon occurs. This will be discussed to some extent below. Here we only mention that since r is relevant with respect to $H^*_{tricrit.}$ this fixed point is instable with respect to infinitesimal perturbation in r. If $r = r_{crit.} + \delta r$ with $\delta r < o$, however, a fixed point is still approached, namely $H^*_{bicrit.}$. If $\delta r < o$ is numerically "very small" then as $b_1 \overset{>}{\to} |\delta r| \simeq o$ H seems to approach $H^*_{tricrit.}$ and tricritical exponents are observed. In the region $b_1 \simeq |\delta r|$ a cross over to $H^*_{bicrit.}$ as $|\delta r| > b_1 \to o$ takes place and the bicritical exponents finally show up.

In $d = \frac{2n}{n-1} - \varepsilon$ dimensions the n-th order critical point becomes non-Gaussian in exactly the same manner the 2nd order transition gets non-Gaussian in $d = 4 - \varepsilon$ dimensions. The exponent η has been calculated by Wegner for arbitrary n to order ε^2.

$$\eta = \frac{(n!)^6}{(2n)!^3} (2(n-1)\varepsilon)^2 + O(\varepsilon^3) \quad . \tag{14.5}$$

14.2. n-component spin systems

In the case of continuous, in particular rotational symmetry of the order parameter, a simple generalization of the models considered so far enables us to describe the situation: The classical lattice spin systems with n-component spin variables $\vec{\sigma}_x$, Hamiltonian

$$H(\vec{\sigma}) = - K \sum_{|x-y|=a} \vec{\sigma}_x \cdot \vec{\sigma}_y \ , \tag{14.6a}$$

and site spin distribution

$$\rho(\vec{\sigma}^2) = \delta(|\vec{\sigma}|-1) \quad . \tag{14.6b}$$

Particular cases are the models

$n = 1$	Ising
$n = 2$	XY (planar rotator)
$n = 3$	Heisenberg.

The limit $n \to \infty$ corresponds to the exactly solvable spherical model.

As in the Ising case (n=1) one can relate (using universality arguments) to any of these models a Landau-Ginzburg-Wilson model (n-vector model) by replacing the real spin variable S_x in the Ising case by an isotopic spin vector $\vec{S}_x = (S^1,\ldots,S^n)$ and

$$S^2 \to \vec{S}^2$$
$$S^4 \to (\vec{S}^2)^2$$

etc.

The n-vector model Hamiltonian thus takes the form

$$H_\Lambda(\vec{S}) = \sum_n \int_{q_1}\ldots\int_{q_{2n}} (2\pi)^d\, \delta^{(d)}(\Sigma q_i)\, \tilde{\vec{S}}_{q_1}\cdot\tilde{\vec{S}}_{q_2}\ldots\tilde{\vec{S}}_{q_{2n-1}}\cdot\tilde{\vec{S}}_{q_{2n}}\, u_{2n}(q)$$

Obviously for n > 1 anisotropic interactions are possible. In the isotropic case $H_\Lambda(\vec{S})$ only depends on rotation invariant quantities, i.e. it is a function of $|\vec{S}|$ only. Anistropic perturbations change the symmetry of the model (smaller symmetry group that leaves H_Λ invariant) and accordingly the leading critical behavior (fixed points, exponents etc.).

Due to the presence of fixed points of different symmetry type in the n-vector models for n > 1, cross over phenomena between the different fixed points appear in the case of small anisotropic perturbations quite in the same way as discussed above for systems with higher order critical points (see below),

We may generalize the methods developed in section III for the case n=1 in an obvious way for the general n-vector model. Essentially only the combinatorics of graphs change.

The power counting arguments of section 12 apply without modification to these models. Accordingly the leading system is described by an n-vector S^4-Hamiltonian

$$H_\Lambda(S) = \int d^dx\, \{\tfrac{1}{2}(\partial\vec{S})^2(x) + \frac{m^2}{2}\vec{S}^2(x) + u(\vec{S}^2)^2(x)\}. \quad (14.7)$$

In the isotropic case a simple way to take into account the isotropy of the interaction is to replace e.g. the S^4-vertices

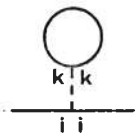
$\sum_{i,k=1}^{n}$ by where $= \sum_{i=1}^{n}$

such that "i-charge" conservation applies to each .
The isotropic propagators are diagonal in i-space, i.e.

$$\langle S_x^i \, S_y^k \rangle = \delta_{ik} \, G(x-y)$$

and so only vertices with identical i-spin components are joined by lines <u>i i</u> . (The dotted line has only a mnemotechnical meaning namely that no i-charge flows along it.)

<u>Example</u>: 1st order terms



Excercise: 1) Draw all possible 2nd order contributions with 2 and 4 external lines and write down the associated Feynman integrals.

2) Evaluate the RG-transformation (13.9) for the isotropic n-vector model and calculate to lowest nontrivial order the critical exponents.

It should also be noted that for particular values of n the order parameter and $\vec{S}$ can be given the geometrical meaning of a vector field, an antisymmetric tensor field, a symmetric traceless tensor field, and so on over real space:

$n = d$	vector
$n = \frac{d(d-1)}{2}$	antisymmetric tensor
$n = \frac{d(d+1)}{2} - 1$	symmetric tracelss tensor etc.

The limit $n \to \infty$ corresponds to the exactly solvable spherical model and an expansion around this model can be set up: the $^1/_n$-expansion. As the model depends only on the iso-space scalars $S_x^2 = \vec{S}_x^2$ etc. one can analytically continue the model to continuous real iso-space dimensions n. This is analogous to the analytic continuation in the space dimension d possible for rotation invariant systems (which depend only on scalars x^2, q^2 etc.). The continuation in n is possible down to $n = -2$. The point $n = -2$ is the exactly solvable Gaussian model. The case $d = 1$ can be exactly solved for $-2 \leq n \leq 1$. For arbitrary n the critical behavior of the S^4 models is (expected to be) Gaussian in $d \geq 4$ and the ε-expansion in $\varepsilon = 4-d > o$ is possible.

We thus have a large class of models characterized by (d,n) with range

$$1 \leq d \leq \infty \quad \text{and} \quad -2 \leq n \leq \infty \quad .$$

On the boundaries of the region with $d \leq 4$ the critical behavior is known to a large extent Tab. 7 and Fig. 24.

Gaussian	Spherical
$n = -2;\ 2 \le d \le 4$	$n = \infty \quad 2 < d \le 4$
$\alpha = \frac{4-d}{2}$	$\alpha = \frac{4-d}{d-2}$
$\beta = \frac{d-2}{4}$	$\beta = \frac{1}{2}$
$\gamma = 1$	$\gamma = \frac{2}{d-2}$
$\eta = 0$	$\eta = 0$
Mean field	**1-dimensional**
$1 \le n \le \infty \quad d \ge 4$	$-2 \le n \le 1 \quad d = 1$
$\alpha = 0$	-
$\beta = \frac{1}{2}$	$\beta = 0$
$\gamma = 1$	-
$\eta = 0$	$\eta = 1$

Tab. 7

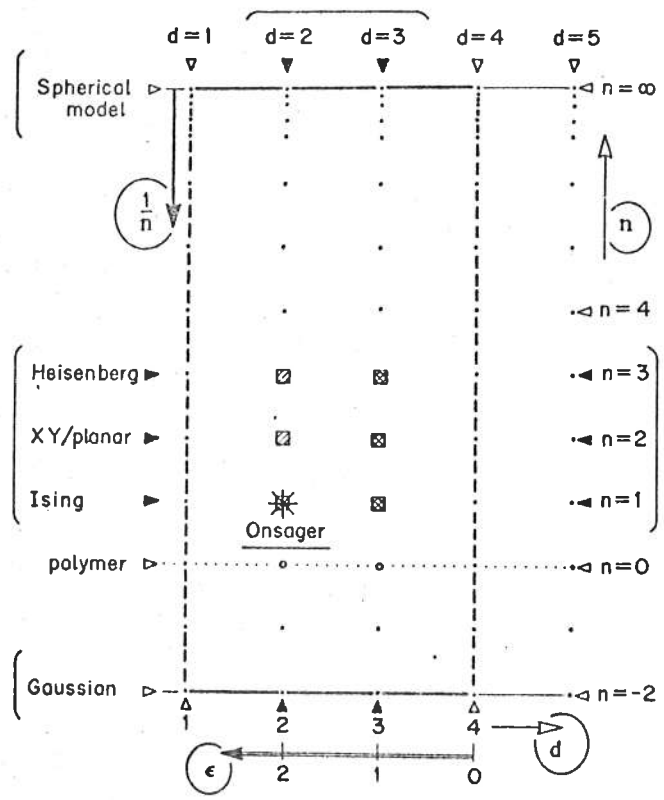


FIG. Diagram of the (d, n) plane showing the expansion variables $\epsilon = 4 - d$ and $1/n$, the boundaries at $n = \infty$ and -2, and $d = 1$ and 4, and various physically relevant cases.

Fig. 24

In the interior of this region only the Ising case (d,n) = (2,1) is exactly solved. All the other physically interesting cases (see Tab. 8)

$$d = 1,2,3 \quad , \quad n = o,1,2,3$$

can be reached by the ε- and $^1/_n$-expansion (of course also direct investigations of the physical cases are possible and have been performed to some extent (see part II)).

Physically realized systems	
d = 3:	
n = 1,2,3	magnetic materials
n = 2	super fluid helium, He^3 - He^4 mixtures
n = 1	normal fluids, fluid mixtures, alloys
n = o	polymer chains in solution
d = 2	films, monolayers, submonolayers, layered magnetic systems
d = 1	linear chain of magnetic ions in crystals

Tab. 8

Below we give the values of some exponents calculated by the 1/n- and the ε-expansion. As expected from the universality they only depend

(i) on the dimension d(ε)
(ii) on the symmetry index n.

TABLE I. Third order ϵ expansion[a] for the exponent γ.

$$\gamma = 1 + \frac{(n+2)}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2 + \frac{(n+2)}{8(n+8)^3}\left[(n+2)^2 + 24\frac{(n+2)(n+3)-(10n+44)\zeta(3)}{(n+8)} + 4\frac{55n^2+268n+424}{(n+8)^2}\right]\epsilon^3 + O(\epsilon^4).$$

[a] From E. Brézin, J. C. LeGuillou, J. Zinn-Justin, and B. G. Nickel (1973).

TABLE II. Fourth order expansion[a] for the exponent η.

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + \frac{n+2}{8(n+8)^3}\left[\frac{24(3n+14)}{(n+8)^2} - 1\right]\epsilon^3 + \frac{n+2}{2(n+8)^2}\left[\frac{-5n^2+234n+1076}{16(n+8)^2} - 8\frac{3n^2+53n+160+3(5n+22)\zeta(3)}{(n+8)^3} + 45\frac{(3n+14)^2}{(n+8)^4}\right]\epsilon^4 + O(\epsilon^5).$$

[a] From E. Brézin, J. C. LeGuillou, J. Zinn-Justin, and Nickel (1973).

TABLE III. First order $1/n$ expansions for exponents for $2 < d < 4$.[a]

$$\gamma = \frac{2}{d-2}\left[1 - \frac{3A_d}{n} + O\left(\frac{1}{n^2}\right)\right]$$

$$\alpha = -\frac{4-d}{d-2}\left[1 - \frac{4(d-1)}{(4-d)}\frac{A_d}{n} + O\left(\frac{1}{n^2}\right)\right]$$

$$\eta = \frac{2(4-d)}{d}\frac{A_d}{n} + O\left(\frac{1}{n^2}\right)$$

where

$$A_d = \frac{2\Gamma(d-2)\sin(\frac{1}{2}d-1)\pi}{[\Gamma(\frac{1}{2}d-1)]^2(\frac{1}{2}d-1)}, \qquad A_3 = 4/\pi^2,$$

and

$$A_{4-\epsilon} \approx \tfrac{1}{2}\epsilon \text{ as } \epsilon \to 0, \qquad A_{2+\theta} \approx \tfrac{1}{2}\theta \text{ as } \theta \to 0$$

[a] From M. E. Fisher, S. -K. Ma, and B. G. Nickel (1972); M. Suzuki (1972), S. -K. Ma, (1973); R. Abe, (1972, 1973); R. Abe and S. Hikami, (1973).

Tab. 9

A comparison of the numerical values obtained from these expansions with data from numerical analysis are given in Tab. 10. The reader may convince himself of the quality of asymptotic 1/n- and ε-expansion results which do not allow us to estimate errors.

Fisher has plotted contours of constant values of the exponents α, β, γ and η (from ε-expansion) in the (d,n)-plane Fig. 25. These plots give an interesting qualitative picture of the (d,n)-dependence of critical exponents; in particular one gets an impression of the rate of change of the exponents with d and n, and peculiar points and lines are observed. The one dimensional Ising point (1,1) turns out to be a confluence point of the contours.

Table 10.I

Exponent	Defining Relations		n=1 ε^2	n=1 s=1/2 Ising nearest-neighbor (a)	n=2 ε^2	n=2 plane rotor	n=2 XY	n=3 ε^2	n=3 Heisenberg
γ	$\chi \sim t^{-\gamma}$	$T>T_c$ $h=0$	1.642	1.75	1.800	3.0±0.5 (b)	s=1 (c) 2.15±? s=∞ 1.75±?	1.932	s=1 3.2±0.1 (d) 3.8±? (c) s=∞ 2.5±0.1 (d)
ν	$\xi \sim t^{-\nu}$	$T>T_c$ $h=0$	0.840	1	0.920	$2.0\pm^{1.0}_{0.5}$ (b)		0.986	
α	$C \sim t^{-\alpha}$	$T>T_c$ $h=0$	-0.025	$0(\ln)$	-0.280			-0.491	
Δ	$\frac{\partial^{2n} F}{\partial h^{2n}} \sim t^{2-\alpha-n\Delta}$	$T>T_c$ $h=0$	1.833	1.875	2.040			2.211	
β	$M \sim t^{\beta}$	$T<T_c$ $h=0$	0.191	0.125	0.240			0.280	
δ	$M \sim h^{1/\delta}$	$t=0$	6.852	15.04±0.07 (e)	6.840			6.835	
η (f)	$\Gamma(\underset{\sim}{r}) \sim r^{-d+2-\eta}$	$h=0$ $t=0$	0.235	0.25	0.232			0.230	

Selected critical exponents in two lattice dimensions (d=2). n labels the dimensionality of the order parameter. $t \equiv |T-T_c|/T_c$. ε-expansion results are from standard sources.

Tabl. 10.II: Selected critical exponents in three lattice dimensions (d = 3). Notation is the same as for Table I. Quantities shown in parentheses are derived by the use of scaling relations.

Exponent	n = 1			n = 2				n = 3				
	ε^2	ε^3	s = 1/2 Ising Nearest-neighbor	ε^2	(a) Grover $\eta=0$	s=1/2 XY	s=∞ XY and plane rotor	ε^2	ε^3	(b) Grover et al. $\eta = 0$	s = 1/2 Heisenberg	s = ∞ Heisenberg
γ	1.244	1.195	1.250 ±0.002 (c)	1.300	1.29	1.35 ±0.03 (d)	1.318 ±0.010 (e)	1.347	1.325	1.356	1.43 ± 0.01 (f) 1.36 ± 0.04 (g)	1.405±0.020 (h) 1.375+0.020 −0.010 (i)
ν	0.627		$0.638^{+0.002}_{-0.001}$ (c) 0.643±0.002 (j)	0.650	(0.645)		0.670 ±0.006 (e)	0.678		(0.678)	0.70 ± 0.03 (i)	0.717±0.007 (h) 0.703±0.010 (i)
α	0.077	(0.196)	0.125±0.015 (k)	−0.020	(0.065)		−0.02 ±0.03 (e)	−0.100		(−0.034)	(−0.20 ± 0.01)	−0.14 ±0.06 (h
Δ	1.583		1.563±0.003 (l)	1.660	(1.613)		1.67 ±0.01 (e)	1.724		(1.695)	1.82 ± 0.02 (f)	1.77 ±0.02 (h
β	0.340	(0.304)	$0.312^{+0.002}_{-0.005}$ (m)	0.360	(0.323)			0.377		(0.339)	0.38 ± 0.03 (n)	(0.373±0.014)
δ	4.463		5.00 ±0.05 (o)	4.460	(5.0)			4.459		(5.0)	(4.75 ± 0.25)	(4.9 ±0.4)
η (p)	0.037	0.029	$(0.041^{+0.006}_{-0.003})$	0.039	0		(0.04 ±0.01)	0.039	0.032	0		(0.040±0.008)

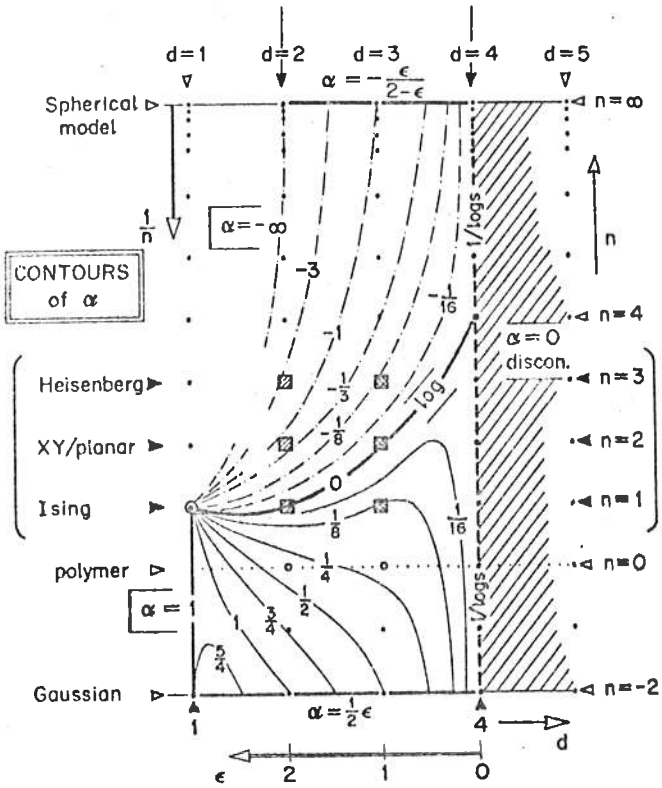


FIG. 2 Diagram showing contours of constant exponent α in the (d, n) plane. The dash-dot contours indicate negative α; the solid contours are for $\alpha \geq 0$.

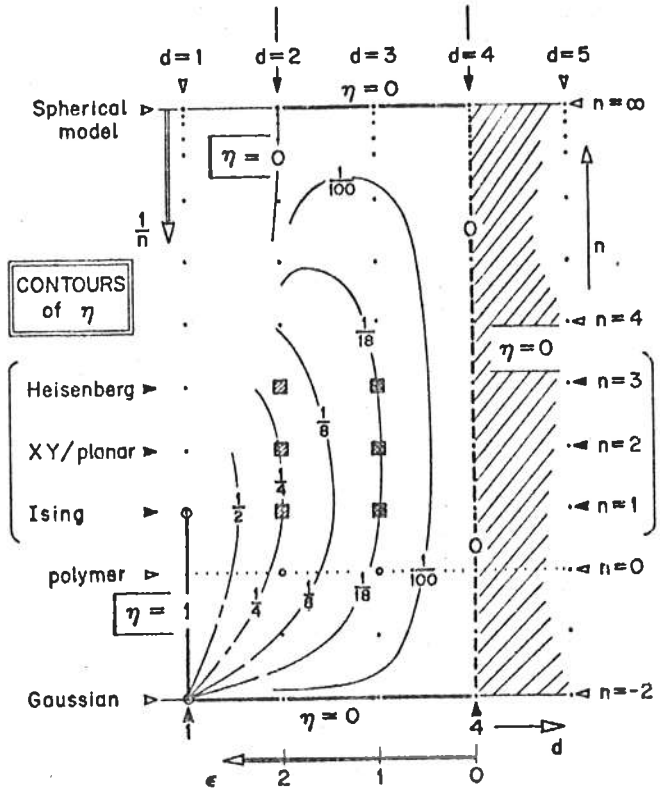


FIG. 4 Contours of constant exponent η in the (d, n) plane. Note the smoothly rising trend and the near vertical contours for $d = 3$, $n = 1$, 2, and 3.

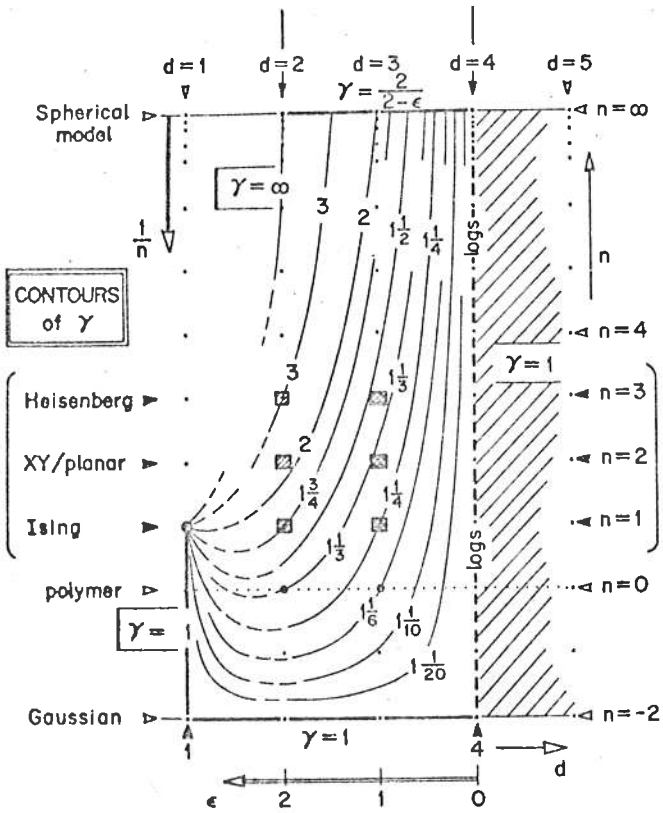


FIG. 3 Contours of constant susceptibility exponent γ in the (d, n) plane. The location of the $\gamma = \infty$ and other contours is uncertain in the region $d \simeq 2$, $n \geq 2$.

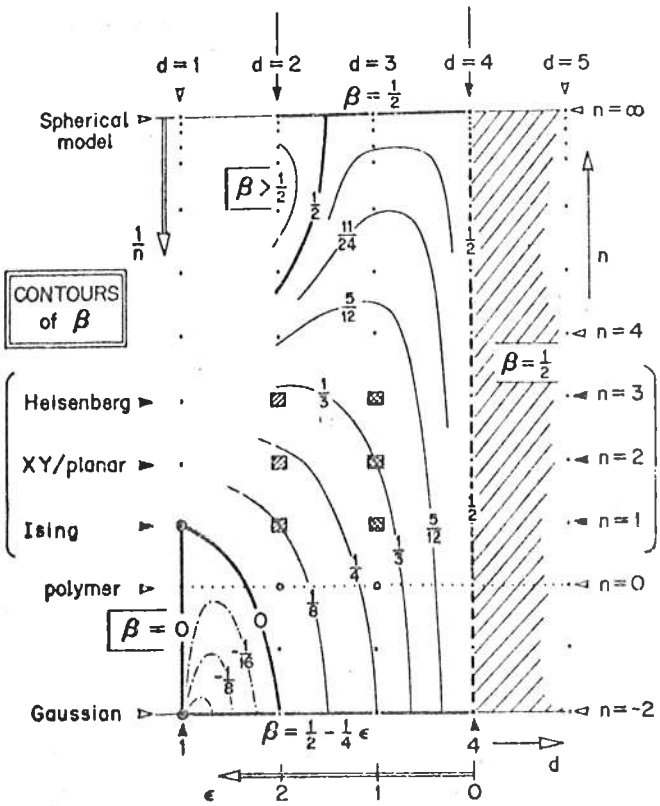


FIG. 5 Diagram of the (d, n) plane showing contours of constant β. There is a region where $\beta > \frac{1}{2}$, and a nonphysical region of negative β.

14.3. Anisotropic Systems

Real physical systems in general exhibit anisotropies in real space (anisotropic lattice structures) as well as in the space of the order parameter (i-space anisotropies). It is therefore very important to know the influence of such anisotropies on the critical behavior of a system. Here we restrict ourselves to considering (i-space-)anisotropies in the ferromagnetic n.n. interactions (assuming that n.n.n, four spin and more complicated interactions do not change the critical behavior).

Real space anisotropies do not affect the leading critical behavior. By universality the lattice structure (e.g. lattice anisotropies) is expected to be irrelevant. Nevertheless for very strong anisotropies quasi d'(<d) dimensional behavior may be observed in "pre"-critical regions with a cross over to d-dimensional behavior at criticality (see section 15). Thus we shall only consider an anisotropic system described by a Hamiltonian of the form

$$H(\vec{\sigma}) = - \sum_{\substack{|x-y|=a \\ i,k}} K_{x-y,ik}\, \sigma_x^i\, \sigma_y^k \tag{14.8}$$

with $\vec{\sigma}$ an n-component spin variable. The site spin distribution may be either symmetric $\rho = \rho(|\vec{\sigma}|)$ or anisotropic $\rho = \prod_{i=1}^{n} \rho_i(\sigma^i)$.

A. Quadratic Anisotropy

The most important anisotropic systems are those with anisotropic K and isotropic $\rho(|\sigma|)$ (quadratic anisotropy).

The n-isotropic coupling

$$K_{x-y,ik} = K_{x-y}\, \delta_{ik}$$

is replaced by an anisotropic coupling

$$K_{x-y,ik} = (a_i K_{x-y} + b_i\, \delta_{x-y})\, \delta_{ik}$$

with $a_i > o$ and K_{x-y} the n.n. Ising coupling. We will assume K to be isotropic in an m-dimensional subspace and in its orthogonal complement, i.e.

$$a_1,\ldots, a_m = a \; ; \; a_{m+1},\ldots,a_n = a'$$

$$b_1,\ldots, b_m = b \; ; \; b_{m+1},\ldots,b_n = b' \; .$$

Furthermore we assume that $\rho(|\vec{\sigma}|)$ is isotropic in the variables σ^i for which K has diagonal form.

Again the translation into the Landau-Ginzburg form goes along the lines discussed in part III. The new spin variables $S^i(x)$ are (re)normalized such that the $\frac{1}{2}|\tilde{\vec{S}}_q|^2 q^2$-term in $H_\Lambda(S)$ is symmetric with coefficient 1. The leading part of $H_\Lambda(S)$ then takes the form

$$H_\Lambda(S) = \int d^dx \, \{\tfrac{1}{2}(\partial\vec{S})^2(x) + \tfrac{1}{2}\sum_i m_i^2 (S^i)^2(x) + \sum_{i,k} u_{ik}(S^i)^2(S^k)^2(x) \tag{14.9}$$

with
$$S_x^i = \sqrt{2ka^i \, a^{2-d}} \; \sigma_x^i$$

$$m_i^2 = -2a^{-2}\left(\frac{u_o - b/2}{ka_i} + d\right) \tag{14.10}$$

$$u_{ik} = \frac{u_o}{4k^2 a_i a_k} \, a^{d-4} \qquad \text{(compare (10.15))}.$$

Stability requires the matrix u_{ik} to be positive. By a suitable choice of parameters this system exhibits a hierarchy of fixed points each characteristic of the isotropic m-vector like model. Apart from the n-isotropic critical point there is a

line of		n-1	isotropic
surface of		n-2	isotropic
:		:	
n-m dimensional hypersurface	of	m	isotropic
:		:	
n-1 dimensional hypersurface	of	1	Ising

critical points.

Only the "end points" characterized by $u_{ik} = o$ for $i,k = \{m+1,\ldots,n\}$ of the m-isotropic critical domains (m<n) correspond to a usual m-isotropic critica point. The components $S_x^{m+1},\ldots,S_x^n$ are then decoupled and independent Gaussian.

In the parameter space of the Hamiltonian (14.8) this corresponds to critical points with $a_i = b_i = o$ for $i = m+1,\ldots,n$ and the remaining components $\sigma_x^{m+1},\ldots,\sigma_x^n$ form an independent site spin model.

In general the lower index critical points are "<u>m-isotropic like</u>" in the sense that only the fluctuations $\sigma^1,..\sigma^m$, in the m-dimensional subspace become critical whereas the $\sigma^{m+1},\ldots,\sigma^m$ have non critical fluctuations (i.e. do not scale for large distances).

An important example of an anisotropic system of the type considered here is the anisotropic Heisenberg ferromagnet

$$H(\vec{\sigma}) = H_{iso}(\vec{\sigma}) + g\, H_{aniso}(\vec{\sigma})$$

$$H_{iso} = - k \sum_{|x-y|=a} \vec{\sigma}_x \cdot \vec{\sigma}_y$$

$$H_{aniso} = - k \sum_{|x-y|=a} [\sigma_x^3\, \sigma_y^3 - \tfrac{1}{2}(\sigma_x^1\, \sigma_y^1 + \sigma_x^2\, \sigma_y^2)].$$

The form of H_{aniso} is dictated by the continuity of the system at $g = o$:

$$\langle H(\vec{\sigma})\rangle_{g=o} = \langle H_{iso}(\vec{\sigma})\rangle_{g=o} .$$

The couplings are

$$k(1+g) \text{ for } \sigma^3$$

$$k(1-\tfrac{g}{2}) \text{ for } \sigma^1 \text{ and } \sigma^2$$

accordingly we have the isotropic points:

$g = o$ Heisenberg

$g = -1$ XY

$g = 2$ Ising.

The Heisenberg critical point $g = o$ is instable with respect to anisotropic perturbations. For $2 > g > o$ an Ising like critical point exists i.e. σ^3 fluctuations get critical whereas the σ^1, σ^2 fluctuations do not similarly for $-1 < g < o$ an XY-like critical point exists i.e. σ^1, σ^2 fluctuations get critical whereas the σ^3 fluctuations do not.

In the corresponding Landau-Ginzburg form the effective potential is:

$$P_4(\vec{S}) = \sum_{i,k=1}^{3} u_{ik} (S_x^i)^2 \, (S_x^k)^2 + \sum_{i=1}^{3} m_i^2 \, (S_x^i)^2 \, .$$

$P_4(\vec{S})$ may be diagnolized in the $(S_x^i)^2$ variables. The even relevant parameters may then be chosen according to Fig. 26 which shows the profiles of $\bar{P}_4$ in the directions $i = 1,2,3$.

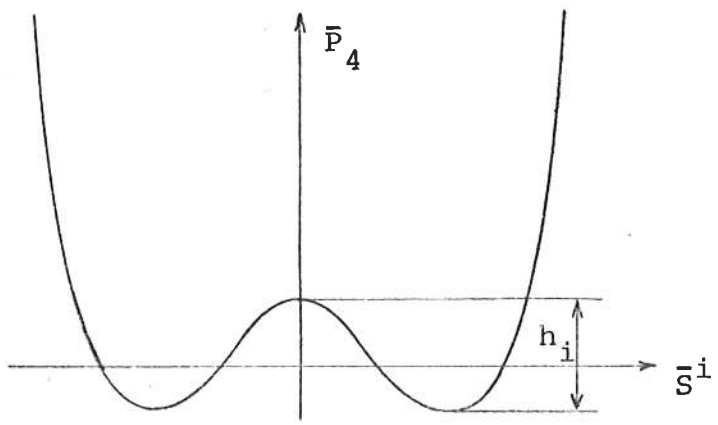


Fig. 26a

Without loss of generality we may assume $h_1 \geq h_2 \geq h_3 > o$ for temperatures below T_c. Critical points may be approached as

$$h_1 \to o \quad \text{with} \quad \frac{h_2}{h_1} = r_1 \, , \; \frac{h_3}{h_2} = r_2 > o \quad \text{fixed.}$$

If $h_1 > o$ we have for

$r_1 < 1$, $r_2 \leq 1$ two stable phases; $h_1 \to o$ m=1 critical

$r_1 = 1$, $r_2 < 1$ 1-sphere of stable phases; $h_1 \to o$ m=2 critical

$r_1 = 1$, $r_2 = 1$ 2-sphere of stable phases; $h_1 \to o$ m=n=3 critical

h_1 is a "reduced temperature like" parameter.

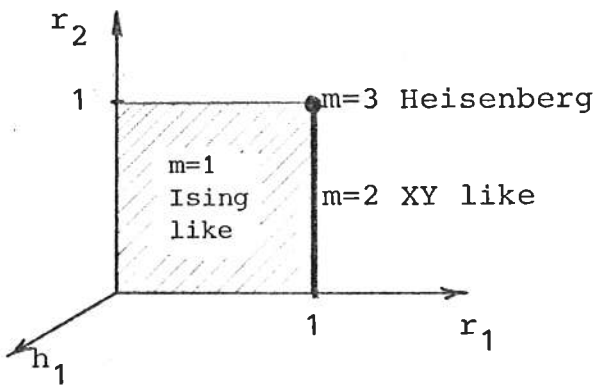


Fig. 26b

With respect to the Heisenberg fixed point H_3^* r_1 and r_2 are relevant variables with $r_{13\text{-crit}} = r_{23\text{-crit}} = 1$.
Small perturbations in r_2 : $r_2 = 1+\delta r_2$, $\delta r_2 < o$ lead to a cross over to the XY fixed point H_2^* whereas small perturbations in $r_1 : r_1 = 1+\delta r_1$, $\delta r_1 < 0$ lead to a cross over from $H_3^*(r_1=1)$ or $H_2^*(r_2<1)$ to the Ising fixed point H_1^*. $o<r_2<1$ is irrelevant relative to H_2^* and $o<r_1, r_2<1$ are irrelevant relative to H_1^*.

The relation between the geometrical variables h_1, r_1, r_2 and the physical parameters may be worked out in a straightforward manner. In general this relation does not have a simple form.

B. Cubic Anisotropy

Another interesting anisotropic system (observed in magnetic and structural phase transitions) is the one with so-called (hyper-) cubic anisotropy. It may be defined in terms of an isotropic

Hamiltonian

$$H(\vec{\sigma}) = - \sum_{|x-y|=a} K_{x-y} \vec{\sigma}_x \cdot \vec{\sigma}_y \tag{14.11a}$$

and a particular anisotropic spin distribution

$$\rho(\vec{\sigma}) \propto e^{-u_o(\vec{\sigma}^2-1)^2} e^{-v_o \sum_{i=1}^{n} (\sigma_x^i)^4} . \tag{14.11b}$$

The corresponding Landau-Ginzburg Hamiltonian thus reads

$$H_\Lambda(S) = \int d^dx\{\tfrac{1}{2}(\partial\vec{S})^2(x) + \frac{m^2}{2}\vec{S}^2(x) + u(\vec{S}^2)^2(x) + v\sum_{i=1}^{n}(S^i)^4(x)\}. \tag{14.11d}$$

Below T_c this system orders in a <u>discrete</u> set of states along the axes and diagonals of a (hyper-)cube i.e. for $m^2 < o$ the effective potential

$$P(\vec{S}) = v\sum_{i=1}^{n}(S_1^i)^4 + u(\vec{S}_x^2)^2 + \frac{m^2}{2}\vec{S}_x^2$$

has minima at

$$S_a = (o,o,\ldots o,\pm 1,o,\ldots o)\left(\frac{c}{u+v}\right)^{1/2}$$

$$S_d = \left(\pm\frac{1}{\sqrt{n}}, \pm\frac{1}{\sqrt{n}},\ldots,\pm\frac{1}{\sqrt{n}}\right)\left(\frac{c}{u+\frac{v}{n}}\right)^{1/2} \tag{14.12}$$

$c = -\frac{m^2}{4} > o$ and

$$P(\vec{S}_a) = -\frac{c^2}{u+v} ; \; P(\vec{S}_d) = -\frac{c^2}{u+\frac{v}{n}} . \tag{14.13}$$

Stability requires $u+v$, $u+\frac{v}{n} > o$ $(n > 1!)$

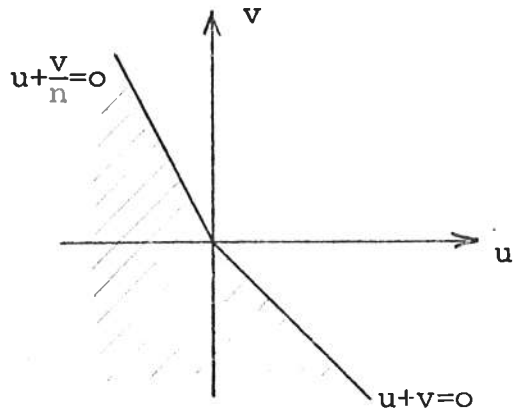


Fig. 27

If $v > o$ the stable states are represented by the 2^n end points of the diagonals; if $v < o$ the stable states are represented by the 2n end points of the axes. If v changes sign a first order transition axial $\Leftrightarrow$ diagonal takes place below T_c. This situation suggests that there should be a new type of critical behavior. Indeed a new type of fixed point, the cubic fixed point, is found in the ε-expansion of the RG-transformation for this system.

As the system is

n-isotropic for $u > o$, $v = o$

and (n decoupled copies)

Ising for $u = o$, $v > o$

there are (apart from the instable Gaussian) three fixed points in competition. The fixed points and their associated critical exponents to lowest order in $\varepsilon = 4-d$ are given in Tab. 11.

	u^*	v^*	y_1	y_n	y_v
Gaussian	o	o	2	ε	ε
n-isotropic	$\frac{\varepsilon}{16(n+8)}$	o	$2-\frac{n+2}{n+8}\varepsilon$	$-\varepsilon$	$\varepsilon\frac{n-4}{n+8}$
Ising	o	$\frac{\varepsilon}{16\cdot 9}$	$2-\frac{\varepsilon}{3}$	$\frac{\varepsilon}{3}$	$-\varepsilon$
cubic	$\frac{\varepsilon}{48n}$	$\frac{\varepsilon}{16\cdot 9}(\frac{n-4}{n})$	$2-2\frac{n-1}{3n}\varepsilon$	$-\varepsilon$	$\frac{4-n}{3n}\varepsilon$

Tab. 11

It is important to note that the stability properties ($y_{u,v} < o$ stable, $y_{u,v} > o$ instable) of the isotropic and cubic fixed point and the sign ov v^* depend on n.

The value of n for which the stability of the cubic fixed point changes is defined by

$$y_v(n_c(d),d) = o \tag{14.14a}$$

and one obtains

$$n_c(d) = 4-2\varepsilon + \frac{5}{2}\varepsilon^2(\zeta(\varepsilon) - \frac{1}{6}) + O(\varepsilon^3) \tag{14.14b}$$

with $\zeta(x)$ the Riemann ζ-function.

The topology of the two different regimes $n < n_c(d)$ and $n > n_c(d)$ are depicted in Fig. 28.

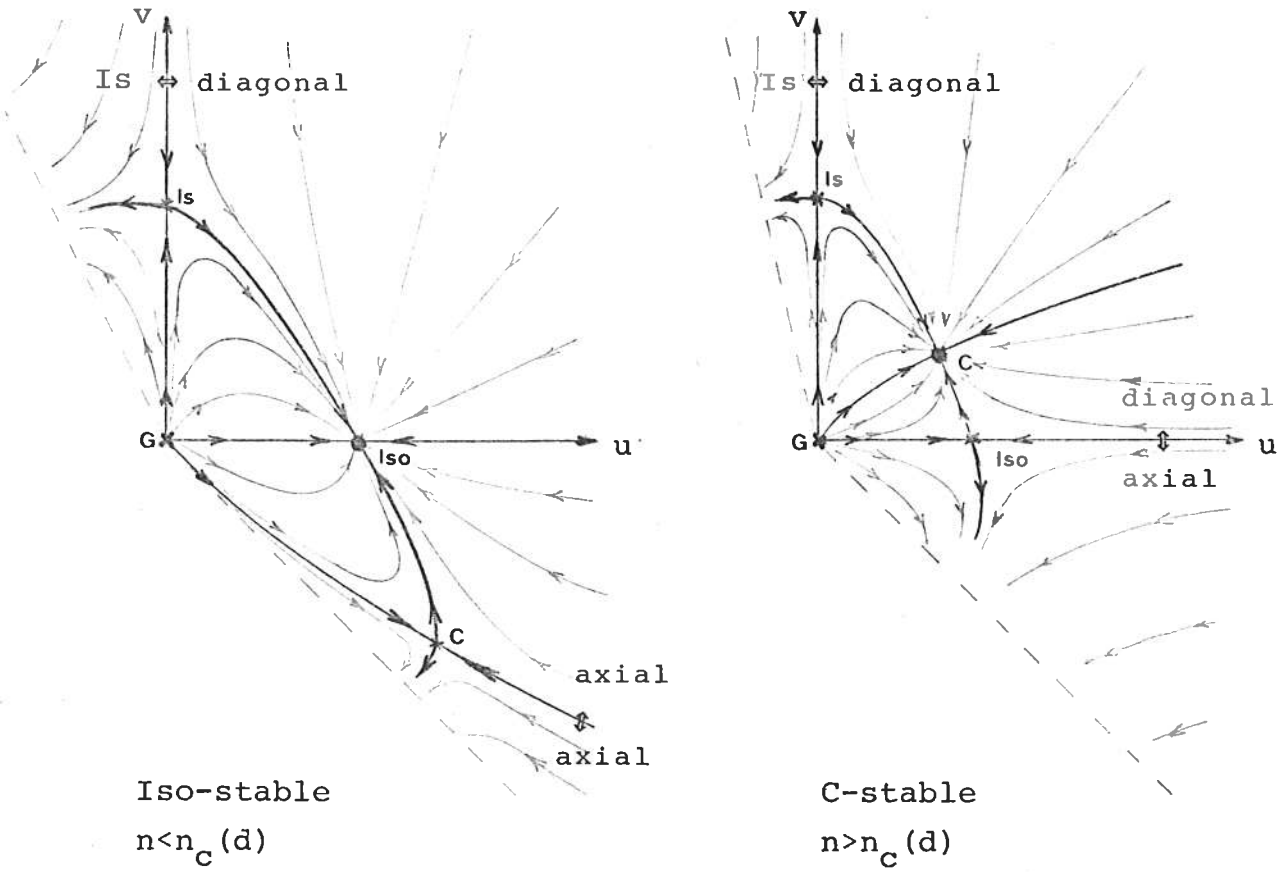


Fig. 28

The boarder lines G-Is and G-C resp. G-Is and G-Iso are lines of 1st order transitions ⇔. In the "attractive regions" of the fixed points the system clearly only orders below T_c.

In the case $n < n_c(d)$ the iso-fixed point is stable whereas for $n > n_c(d)$ the cubic-fixed point is stable.

In the border case $n = n_c(d)$ the iso- and the cubic-fixed points coincide. In each case small perturbations of the instable fixed points lead to cross-over phenomena.
The critical exponents for the cubic-fixed point are:

$$\eta_c = \frac{(n+2)(n-1)}{54\ n^2}\ \varepsilon^2 + O(\varepsilon^3)$$

$$\nu_c = \frac{1}{2} + \frac{n-1}{6n} + \frac{(n-1)(17n^2+290n-424)}{648\ n^2}\ \varepsilon^2 + O(\varepsilon^3)\ . \tag{14.15}$$

As $\eta_c < \eta_{iso}$ for $n < n_c(d)$ and $\eta_c > \eta_{iso}$ for $n > n_c(d)$ the stable fixed point is always the one with the largest value of η!.

For $n \to \infty$ we notice

$$\eta_c = \eta_{Is}$$

$$\nu_c = \frac{\nu_{Is}}{1-\alpha_{Is}}\ . \tag{14.16}$$

To conclude the cubic anisotropic system exhibits a excitingly rich structure. However, as $n_c(d=3) \simeq 3.13$ it seems that for all physically interesting cases the iso-fixed point is stable. Also the cubic and isotropic exponents for $n = 3$ are numerically close to each other and cross over phenomena would be very hard to observe experimentally.

14.4. Long Range Interactions

So far we only considered systems with short range forces. The RG-methods developed in part III apply equally well, however, to systems exhibiting long range interactions. We briefly summarize some results.

A. Isotropic long range exchange forces

We consider the influence on critical behavior of an interaction term

$$H(\vec{\sigma}) = -\sum_{x,y} K_{x-y,\sigma}\, \vec{\sigma}_x \cdot \vec{\sigma}_y$$

with (14.17)

$$K_{x-y,\sigma} \simeq \frac{c}{|x-y|^{d+\sigma}} \; ; \; |x-y| \gg a$$

and

$$\rho(\vec{\sigma}) = \delta(\vec{\sigma}^2-1) \text{ resp. } \sqrt{\frac{u_o}{\pi}}\, e^{-u_o(\vec{\sigma}^2-1)^2} \quad .$$

The leading contribution of the long distance tail is given by

$$\tilde{K}_{q,\sigma} = \mu_o a^{-\sigma} - \mu_\sigma |q|^\sigma + O(q^2 a^{2-\sigma}) \tag{14.18}$$

$$\tilde{K}_{q,\sigma} = a^d \sum_x \frac{e^{-iqx}}{|x|^{d+\sigma}} \propto \int_{-\infty}^{\infty} d^d x\, \frac{e^{-iqx}}{(x^2+a^2)^{\frac{d+\sigma}{2}}} \tag{14.19a}$$

for $N \to \infty$ and $|x| \gg a$. The replacement $x^2 \to x^2+a^2$ is necessary in order not to pick up artificial singular short range interactions in the continuum form. For $o < \sigma < 2$

$$\int_{-\infty}^{\infty} d^d x\, \frac{e^{-iqx}}{(x^2+a^2)^{\frac{d+\sigma}{2}}} = S_d\, \Gamma(\tfrac{d}{2})\, \left(\frac{2}{|q|}\right)^{\frac{d-2}{2}} \int_o^\infty dr\, \frac{r^{\frac{d}{2}}\, J_{\frac{d-2}{2}}(r|q|)}{(r^2+a^2)^{\frac{d+\sigma}{2}}}$$

$$= S_d \Gamma(\tfrac{d}{2}) \frac{a^{-\frac{\sigma}{2}} |q|^{\frac{\sigma}{2}}}{2^{\frac{\sigma}{2}} \Gamma(\frac{d+\sigma}{2})} K_{\frac{\sigma}{2}}(a|q|) \tag{14.19b}$$

with $S_d = 2 \frac{\pi^{d/2}}{\Gamma(d/2)}$,

$$K_{\frac{\sigma}{2}}(a|q|) = \frac{\pi}{2} \frac{1}{\sin\sigma\frac{\pi}{2}} \{I_{-\frac{\sigma}{2}}(a|q|) - I_{\frac{\sigma}{2}}(a|q|)\}$$

and

$$I_\nu(z) = (\tfrac{z}{2})^\nu \sum_{k=o}^{\infty} \frac{(\frac{z^2}{4})^k}{k!\Gamma(\nu+k+1)} \; ; \; \frac{\pi}{\sin\sigma\frac{\pi}{2}} = \Gamma(\tfrac{\sigma}{2})\ \Gamma(1-\tfrac{\sigma}{2}) .$$

$I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions. Thus

$$\tilde{K}_{q,\sigma} = \frac{\pi^{d/2}}{\Gamma(\frac{d+\sigma}{2})} \{\Gamma(\tfrac{\sigma}{2})\ a^{-\sigma} + \Gamma(-\tfrac{\sigma}{2})\ (\tfrac{|q|}{2})^{\sigma} + O(q^2)\} . \tag{14.19c}$$

The existence of the thermodynamical limit requires $\sigma > o$ and as the leading n.n. term is $O(q^2)$ we are interested in the regime with

$$o < \sigma < 2.$$

The leading terms in the related Landau-Ginzburg Hamiltonian then read

$$H_\Lambda(\vec{S}) = H_o + H_{int}$$

$$H_o(\vec{S}) = \tfrac{1}{2}\int_q |\tilde{\vec{S}}_q|^2\ (m^2+\mu_\sigma|q|^\sigma) \tag{14.20}$$

$$H_{int}(\vec{S}) = u\int_x (\vec{S}_x^2)^2 .$$

Again the leading bilinear term defines H_o, the classical (free) long range (l.r.) Hamiltonian which can be solved exactly (see section 10). We have the generating functional (n=1):

$$Z_o\{J\} = \exp \frac{1}{2}(J,GJ) \tag{14.21a}$$

with

$$(J,GJ) = \frac{1}{(2\pi)^d} \int_{|q|\leq\Lambda} d^dq \; |\tilde{J}_q|^2 \frac{1}{|q|^\sigma+m^2} \quad . \tag{14.21b}$$

The two point correlation is

$$\langle S_x \; S_y \rangle = G_{x-y} = \frac{1}{(2\pi)^d} \int_{|q|\leq\Lambda} d^dq \; \frac{e^{iq(x-y)}}{|q|^\sigma+m^2} \quad . \tag{14.21c}$$

The connected higher correlations all vanish. Obviously the cut off limit $\Lambda \to \infty$ exists in the sense of distributions. In field theoretical terms S_x is a <u>non-local</u> scalar field (also for $\Lambda = \infty$).

For the interacting model $u > o$ the perturbation expansion with respect to the new H_o has precisely the same old form with a replacement of the propagator

$$\frac{1}{m^2+q^2} \quad \text{by} \quad \frac{1}{m^2+|q|^\sigma\mu_\sigma} \; .$$

Apart from this change the Feynman rules given in chapter 10 remain the same. The RG-transformation is defined in exactly the same way as before and it can be evaluated in a perturbation expansion with respect to the new H_o (see section 11).

Obviously the coefficients $\hat{u}_{2n}(q)$ defined in (11.10a) are no longer analytic in q. They have, however, a double power series expansion in $|q|^\sigma$ and q i.e.

$$\hat{u}_{2n}(q) = \sum_{n,l,m,\alpha} P_{\alpha lm}\,(q,|q|^\sigma)\; g_{nlm\alpha} \tag{14.22}$$

with P_α a polynomial of degree l in q and of degree m in $|q|^\sigma$.

Accordingly the power counting given in section 12 holds with obvious modification. We obtain

$$g'(s) \propto s^{\omega}$$

with (14.23)

$$\omega = 2nx - (2n-1)d - 1 - m\sigma = d - 2nd_S - 1 - m\sigma \ .$$

We thus have apart from the terms discussed in section 12 (see p. 90) the new terms:

	n = 1	n = 2	
m=1, l=o	$S\,\partial^{\sigma}\,S$ $2-\eta-\sigma$	$S^3\,\partial^{\sigma}\,S$ $4-d-2\eta-\sigma$	
m=1, l=1:	$S\,\partial^{\sigma+2}\,S$ $-\eta-\sigma$	$S^3\,\partial^{\sigma+2}\,S$ $2-d-2\eta-\sigma$	
m=2, l=o	$S\,\partial^{2\sigma}\,S$ $2-\eta-2\sigma$	$S^3\,\partial^{2\sigma}\,S$ $4-d-2\eta-2\sigma$	

The S-field renormalization exponent x defined in (11.3) and (11.7) has to be determined in such a way that the RG-transformation leads to a non singular limit for $\langle S_x\, S_o\rangle$. Thus we require

$$\omega_{S\partial^{\sigma}S} = 2x-d-\sigma = 2-\eta-\sigma \leq o \ . \tag{14.24}$$

In this case the other extra terms are irrelevant. First we notice that the free (classical) l.r. system exhibits a trivial fixed point ($m^*=o$) with

$$\eta_{lr} = 2-\sigma \ . \tag{14.25}$$

Let us denote by η_{sr} and η_{lr} the exponents η corresponding to the short range and long range fixed points.

In the interacting case the following cases must then be distin guished:

(i) If $\eta_{sr} > \eta_{lr}$ classical i.e.

$$\sigma > 2-\eta_{sr} \tag{14.26}$$

then the parameter μ_σ is irrelevant with respect to the s.r. fixed point, i.e. the s.r. fixed point is stable with respect to the l.r. perturbations with $\sigma > 2 - \eta_{sr}$.

(ii) in the border case $\eta_{sr} = \eta_{lr}$

$$\sigma = 2 - \eta_{sr} \tag{14.27}$$

μ_σ is a persistent marginal variable relative to the s.r. point.

(iii) In the range

$$o < \sigma < 2 - \eta_{sr}$$

μ_σ is relevant relative to the s.r. fixed point, i.e. the s.r. fixed point is instable with respect to l.r. perturbations with $o < \sigma < 2-\eta_{sr}$. In this case there is a stable l.r. fixed point with

$$\eta_{lr} = 2-\sigma \ , \ (o < \sigma < 2-\eta_{sr}). \tag{14.28}$$

at least for the classical case.

The possible influence of the S^4-interaction term follows from a consideration of its associated exponent

$$y_u = \omega_{S^4} = d-4d_S = 2\sigma-d \tag{14.29}$$

with d_S the classical l.r. dimension of S

$$d_S = d-x = \frac{d-\sigma}{2} \ ; \ x = x_{cl} = \frac{d+\sigma}{2} \ .$$

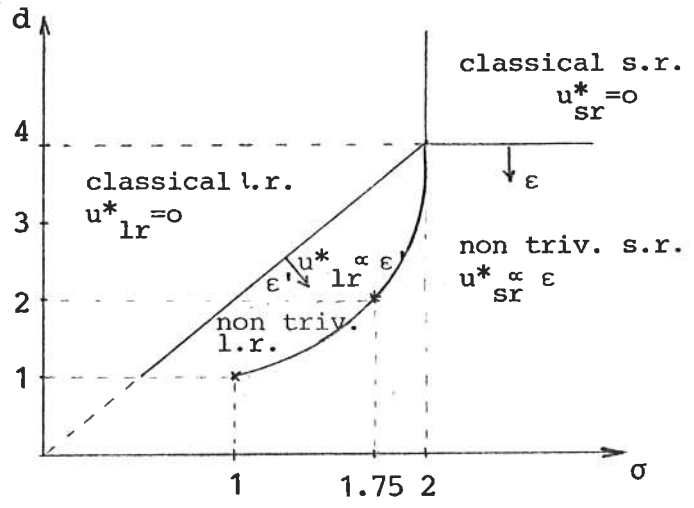
It follows

$$y_u = \begin{cases} < o & d > 2\sigma \\ = o & d = 2\sigma \\ > o & d < 2\sigma \end{cases} \tag{14.30}$$

i.e. the classical l.r. fixed point $u^*=o$ is stable for $d > 2\sigma$, actually including $d = 2\sigma$. For $d < 2\sigma$, however, a non trivial fixed point appears as S^4 gets classically relevant. This fixed point can be studied by means of an ε-expansion in

$$\varepsilon = 2\sigma - d.$$

The different regimes in the (d,σ) plane are shown in Fig. 29.



$$\varepsilon = 4-d \ , \ \varepsilon' = 4-\frac{2d}{\sigma}$$

Fig. 29

The classical l.r. exponents are

$$\nu = 1/\sigma \quad , \eta = 2-\sigma \quad , \gamma = 1 \quad , \alpha = 2-d/\sigma \qquad (d \geq 2\sigma). \qquad (14.31a)$$

The non classical l.r. exponents

$$\eta_{lr} = \eta_{lr} \text{ classical} = 2-\sigma$$

$$\gamma_{lr} = 1 - \frac{n+2}{n+8} \frac{\varepsilon}{\sigma} - \frac{(n+2)(7n+20)}{(n+8)^3} A(\sigma) \left(\frac{\varepsilon}{\sigma}\right)^2 + O(\varepsilon^3) \qquad (14.31b)$$

with

$$A(\sigma) = \sigma\{\psi(1) - 2\psi(\tfrac{\sigma}{2}) + \psi(\sigma)\}$$

and $\psi(\sigma)$ the digamma function.

The other exponents follow from the scaling relations. It is interesting to notice that η_{lr} always takes its classical value (confirmed by the ε- and $1/n$-expansion).

For $n \to \infty$ in particular

$$\eta_{lr} = 2 - \sigma$$

$$\gamma_{lr} = \frac{d-\sigma}{\sigma} \quad . \qquad (14.31c).$$

The physical conclusion of these considerations is that small μ_σ perturbations of the s.r. system lead to a cross over to a l.r. fixed point if $\sigma < 2 - \eta_{sr}$.

B. Dipolar forces

Dipolar forces are long range forces that are not invariant under separate rotations in space and i-space. In a fundamental way they appear in systems where the order parameter is a vector field over real space i.e. n=d. Dipole-dipole interactions have a demagnetization effect in ferromagnetic systems and one expects that they therefore affect the critical behavior of these systems. On the

other hand there is no demagnetization in the antiferromagnetic case and thus critical behavior of such systems is expected not to be changed. The dipole-dipole interaction Hamiltonian is (*)

$$H(\vec{A}) = - \sum_{\substack{x,y \\ |x-y|=a \\ i,k}} K_{x-y,ik} \, A_x^i \, A_y^k$$

with (14.32)

$$K_{x-y,ik} = \beta\mu_B^2 \, g_i g_k \frac{1}{|x-y|^d} \left(d \frac{(x-y)_i (x-y)_k}{|x-y|^2} - \delta_{ik}\right) .$$

We have replaced the d-component spin variable $\vec{\sigma}$ (resp. $\vec{S}$) by

$$\vec{A} = (A^1, \ldots\ldots, A^d) .$$

We suppose the vector field $\vec{A}$ to have an isotropic distribution as in (14.17). Furthermore we only consider the isotropic form $g_i = g_k = g_o$. H is then invariant under space rotations by which $\vec{A}$ transforms as a vector field.

The dominant small q-term (exact for a=o) is

$$\tilde{K}^{dip}_{q,ik} = -\bar{g}_o \frac{q_i q_k}{q^2} + O(a^2 q^2 \ln q^2) \qquad ((14.33)$$

(*) This form of the interaction is well known from classical electrodynamics. The dipole moment of a neutral system of charges q_n is

$$\vec{p} = \Sigma q_n \vec{r}_n .$$

The electromagnetic field of a point like dipole reads

$$\vec{E} = \frac{3(\vec{p}\cdot\vec{r})\vec{r} - r^2\vec{p}}{4\pi\varepsilon_o r^5}$$

and the interaction energy of the dipole field with a dipole is $\vec{p}\cdot\vec{E}$. In a dipolar medium the field strength is the sum of the external field $\vec{E}_o$ and the dipole induced "deelectrization" field $\vec{E}_p$ which for ferroelectrics (antiferroelectrics) is antiparallel (parallel) to $\vec{E}_o$. In the magnetic case $(\vec{p}, \vec{E})$ is replaced by $(\vec{\mu}, \vec{B})$, $\vec{B}$ the magnetic field and $\vec{\mu}$ the magnetic moment. The magnetic moments of electrons or atoms are given by

$$\vec{\mu} = -g\mu_B \vec{J} .$$

g is the g-factor, μ_B the Bohr magneton and $\vec{J}$ the angular momentum. With the replacement of $\vec{J}$ by the classical spin variables $\vec{\sigma}$ the above form of the interaction energy follows in the generalization to d dimensions.

$$\tilde{K}^{dip}_{q,ik} \propto \int_0^\infty d^dx\{d\frac{x_i x_k}{(x^2+a^2)^{\frac{d+2}{2}}} - \frac{\delta_{ik}}{(x^2+a^2)^{\frac{d}{2}}}\}\, e^{-iqx} \tag{14-34a}$$

$$= -\{d\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_k}\tilde{K}_{q,\sigma=2} + \delta_{ik}\tilde{K}_{q,\sigma=o}\}$$

with $\tilde{K}_{q,\sigma}$ given by (14.19).

Let $z = a|q|$ then

$$\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_k} f(z) = a^2\,\delta_{ik}\,\frac{1}{z}\frac{d}{dz} f(z) + a^4 q_i q_k (\frac{1}{z}\frac{d}{dz})^2 f(z)$$

in particular

$$\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_k} z\,K_1(z) = -a^2\delta_{ik}\,K_o(z) + a^4 q_i q_k\, z^{-1}\,K_1(z)\ .$$

It follows

$$\tilde{K}^{dip}_{q,ik} \propto -\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\,a^2\,q_i q_k\,\frac{K_1(a|q|)}{a|q|} = -\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\frac{q_i q_k}{q^2} + O(a^2 q_i q_k \ln a|q|) \tag{14.34b}$$

In the limit $a \to o$ the leading term is exact.

In the Landau-Ginzburg form including the leading s.r. terms, we have:

$$H_\Lambda(\vec{A}) = H_o + H_{int}$$

$$H_o(\vec{A}) = \frac{1}{2}\sum_{i,k}\int_q \tilde{A}^i_q\,\tilde{A}^k_{-q}\,\left((m^2+cq^2)\delta_{ik} + \bar{g}\,\frac{q_i q_k}{q^2}\right) \tag{14.35}$$

$$H_{int}(\vec{A}) = u\int_x (\vec{A}^2_x)^2\ .$$

Also in this case the leading bilinear term defining H_o has changed. In field theory language the pure dipole term i.e. c=o defines a

system of self interacting vector bosons. The free (classical) case again is of "Gaussian type" i.e. formulas (10.26, 27) resp. (14.21) hold with obvious modifications; in particular the two point correlation is

$$\langle A_x^i A_y^k \rangle = G_{x-y,ik} = \frac{1}{(2\pi)^d} \int_{|q|\leq\Lambda} d^d q \; e^{iq(x-y)} \; \tilde{G}_{q,ik} \tag{14.36a}$$

with $\tilde{G}_{q,ik}$ the inverse of

$$\tilde{G}^{-1}_{q,ik} = (m^2 + cq^2)\delta_{ik} + \bar{g} \; \frac{q_i q_k}{q^2}$$

which is

$$\tilde{G}_{q,ik} = \frac{\delta_{ik} - \frac{\bar{g}}{\bar{g}+m^2+cq^2} \frac{q_i q_k}{q^2}}{m^2 + cq^2} \; . \tag{14.36b}$$

The presence of the s.r. term c>o is crucial because the critical small q-behavior of $\tilde{G}_q$ is discontinuous in c:

$$c > o : \quad \tilde{G}_q \propto \frac{1}{cq^2} \left(\delta_{ik} - \frac{q_i q_k}{q^2}\right) \propto q^{-2} \; . \tag{14.37a}$$

If c = o

$$\tilde{G}_{q,ik} = \frac{1}{m^2} \delta_{ik} - \frac{\bar{g}}{\bar{g}+m^2} \frac{1}{m^2} \frac{q_i q_k}{q^2} \propto \text{const.} \tag{14.37b}$$

We assume the fields to be normalized according to the s.r. term with c = 1 in the following i.e.

$$\tilde{G}_{q,ik} = \frac{\delta_{ik}}{m^2+q^2} - \bar{g} \; \frac{q_i q_k}{q^2(m^2+q^2)(\bar{g}+m^2+q^2)} \; . \tag{14.38}$$

Again with replacement of the $\bar{g}$ = o propagator by the one with $\bar{g} \neq$ o we may set up the corresponding perturbation expansion in u for the correlation functions and the RG-transformation. The power counting (section 12) remains valid also in this case, the only change being that the coefficient functions $u_{2n}(q)$ are no longer

rotation invariant. The scalars $\vec{x} \cdot \vec{A}_x$, $\vec{q} \cdot \tilde{\vec{A}}_q$ etc. again can be continued to non-integer dimensions.

The new operators appearing in addition to $\bar{g} = o$ terms are:

	$n = 1$	$n = 2$	
$l = o$	$\sum_{i,k} A^i \partial^{-2} \partial_i \partial_k A^k$ $\omega = 2-\eta$	$\sum_{i,k,l,m} A^i A^k A^l \partial^{-4} \partial_i \partial_k \partial_l \partial_m A^m$ $4-d-2\eta$	
$l = 2$	$\sum_{i,k} A^i \partial_i \partial_k A^k$ $-\eta$	$\sum_{i,k,l,m} A^i A^k A^l \partial^{-2} \partial_i \partial_k \partial_l \partial_m A^m$ $2-d-2\eta$	
$l = 4$	$\sum_{i,k} A^i \partial^2 \partial_i \partial_k A^k$ $-2-\eta$	$\sum_{i,k,l,m} A^i A^k A^l \partial_i \partial_k \partial_l \partial_m A^m$ $-d-2\eta$	

Thus we have a new relevant field relative to the s.r. fixed point. For the pure dipole interaction we would have a trivial (classical) dipole fixed point provided

$$\eta^{dip} = 2 \ .$$

In the presence of s.r. interactions $\bar{g}$ is a relevant field; $\bar{g}$, however, is not independent of the other relevant parameter m^2. m^2 and $\bar{g}$ are both proportional to $\beta = \frac{1}{k_B T}$ and hence $\bar{g} - \bar{g}_c \propto m^2 - m_c^2$. We therefore introduce as independet parameters $\Delta m^2 = m^2 - m_c^2$ and $g = \bar{g} - \bar{g}_c / \Delta m^2$. g is a dimensionless parameter, i.e. it transforms canonically with $\omega = o$ (canonically marginal). Therefore there is a competition between the operators

$$(\partial \vec{A})^2 \ , \ \sum_{i,k} A^i \partial^{-2} \partial_i \partial_k A^k \quad \text{and} \quad (\vec{A}^2)^2 .$$

From (14.37) already it follows that for the "Gaussian" fixed point $m^* = o$, $u^* = o$, g arbitrary, the two point correlation does not coincide with the Gaussian s.r. theory. A typical non rotation invariant term survives. In $d < 4$ the critical long range behavior again gets non-"Gaussian" and does not coincide with the s.r. fixed point theory.

From the ε-expansion it actually follows that the s.r. fixed point is in any case instable with respect to dipolar perturbations and a non trivial stable dipolar fixed point exists for $d < 4$. The critical exponents relative to this fixed point are:

$$\eta^{dip} = \frac{20}{867}\,\varepsilon^2 + O(\varepsilon^3)$$

$$\gamma^{dip} = 1 + \frac{n+2}{2[n+8-\frac{4}{n+2}]}\,\varepsilon + O(\varepsilon^2)\ .$$

We notice that $\eta^{dip} > \eta_{iso}$, i.e. as in all other cases the stable fixed point is the one with largest value of η.

It should be noted that the values of the dipolar exponents in the physically interesting case $d = 3$ are again very close to the s.r. exponents (e.g. $\gamma^{dip} \simeq 1.372$, $\gamma_{sr} \simeq 1.365$ and $\alpha^{dip} \simeq -0.135$, $\alpha_{sr} \simeq -0.125$) and hence it would be very difficult to detect the difference experimentally.

For the results on anisotropic dipolar systems (in particular the interesting uniaxial dipolar case) we refer the reader to the literature.

The critical behaviour of many other systems has been investigated by means of the RG-method (compressible ferromagnets, semi-infinite systems (surface scaling), time dependent systems, quantum spin systems, percolation systems, Kondo system etc.). The discussion of these systems goes beyond the aim these lectures and we refer the interested reader to the relevant literature.

15. Cross-Over Phenomena

From our discussions in the foregoing section we already know that the critical behavior of a system may be instable with respect to several types of perturbations as anisotropies, long range forces, dipolar forces etc.. If the perturbations are "small" the system may show the critical properties of the unperturbed system in a preasymptotic region. When the critical point of the perturbed system is approached, however, the perturbation becomes important and a cross-over to the asymptotic critical behavior takes place.

Let us describe this situation for the simplest case in a more quantitative way. The unperturbed system is described (in zero external field) by a Hamiltonian $\dot{H}$ with one relevant field, the reduced temperature $\dot{t}$, and fixed point $\dot{H}^*$. $H = \dot{H} + g\,\mathcal{O}_g$ is the Hamiltonian of the perturbed system. g is relevant relative to $\dot{H}^*$. H has another fixed point H^* relative to which g is an irrelevant variable. Thus the RG-transformation transforms H in linear approximation as

$$R_s H = \dot{H}^* + s^{\dot{y}}\dot{t} + s^{\dot{y}_g} g + \ldots\ldots \tag{15.1}$$

relative to $\dot{H}^*$ and

$$R_s H = H^* + s^{y} t(g) + s^{y_g} g + \ldots. \tag{15.2}$$

relative to H^*. The reduced temperature of the perturbed system of course depends on g. With

$$\dot{t} = \frac{T-T_c(o)}{T_c(o)} \; ; \quad t(g) = \frac{T-T_c(g)}{T_c(g)}$$

we may write

$$\dot{t} = \dot{t}(g) + \frac{T_c(g)}{T_c(o)}\, t(g)$$

where

$$\dot{t}(g) = \frac{T_c(g)-T_c(o)}{T_c(o)}$$

is the deplacement of the critical temperature relative to the unperturbed system. A cross-over situation is present if $\dot{y}$, $\dot{y}_g > o$ and $y > o$ however $y_g < o$. The free energy density then transforms in the fields $\dot{t}$ and g as

$$\dot{f}_{sing}(\dot{t},g) = s^{-d}\,\dot{f}'(s^{\dot{y}}\dot{t},\ s^{\dot{y}_g}g)$$
$$= |\dot{t}|^{d/\dot{y}}\,\dot{f}'(\pm 1,\ \frac{g}{|\dot{t}|^{\dot{\phi}}}) \qquad (15.3)$$

where

$$\dot{\phi} = \dot{y}_g/\dot{y} > o \qquad (15.4)$$

is called the <u>cross-over exponent</u>. In the fields $t(g)$ and g on the other hand we have

$$f_{sing}(t,g) = |t|^{d/y}\,f'(\pm 1,\ \frac{g}{|t|^{\phi}}) \qquad (15.5)$$

where $\phi = y_g/y < o$ such that

$$g' = g|t|^{-\phi} \to o \quad \text{as } |t| \to o\ . \qquad (15.6)$$

$f'(\pm 1,g')$ is regular at $g' = o$ and we can set $g = o$ for fixed $t = t(g) \neq \dot{t} = t(o)$ for which case g is an irrelevant field. Obviously the leading scaling form (15.5) must have a singularity at $t = \dot{t}$.

If $g \neq o$ and $g|\dot{t}|^{-\dot{\phi}} << 1$ the scaling form (15.3) describes the behavior of the system and unperturbed exponents $\dot{y}$ etc. are observed. As $\dot{t} \to o$, however, in the region $g|\dot{t}^*|^{-\dot{\phi}} \simeq 1$ the $\dot{t}$-dependence is no longer described by the scaling factor in front of $\dot{f}'$ as $\dot{f}'$ itself substantially depends on $\dot{t}$. This is the cross-over region. At $\dot{t} = \dot{t}(g)$ the function $\dot{f}'(\pm 1,\ g|\dot{t}|^{-\dot{\phi}})$ which is regular at $g = o$ must have a singularity

$$\dot{f}'(\pm 1,\ g|\dot{t}|^{-\dot{\phi}}) = c\{g|\dot{t}|^{-\dot{\phi}} - g|\dot{t}(g)|^{-\dot{\phi}}\}^{d/y}$$

to match form (15.5). Depending on whether $\dot{\phi} > 1$ or $o < \dot{\phi} < 1$ the cross-over region is depicted in Fig. 30.

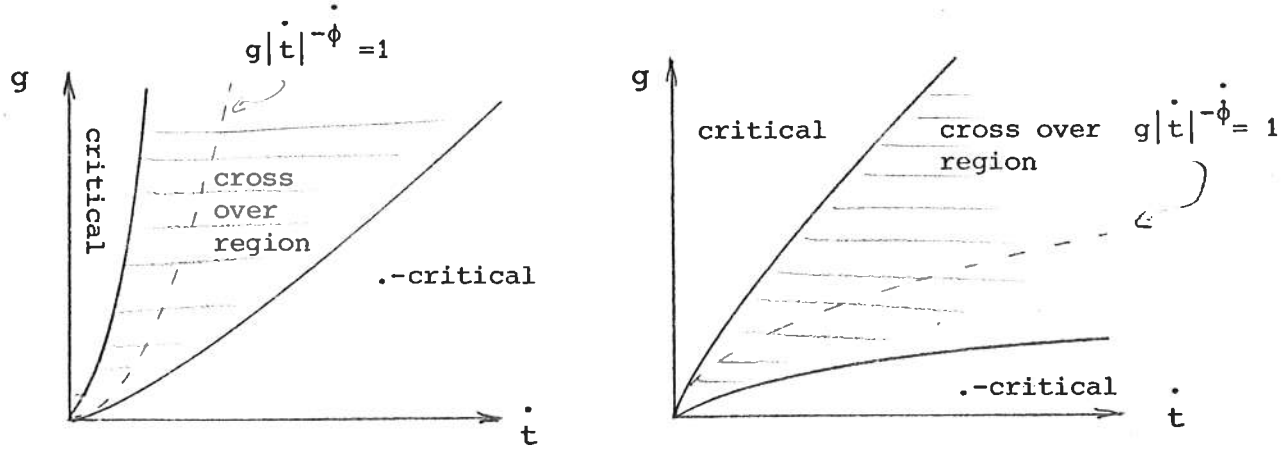


Fig. 30

The fields conjugate to the relevant variables have critical fluctuations at the corresponding critical point. So for the unperturbed critical point $(\dot{t},g) = (o,o)$

$$\langle \mathcal{O}_{\dot{t}x}\ \mathcal{O}_{to}\rangle \quad \text{and} \quad \langle \mathcal{O}_{gx}\ \mathcal{O}_{go}\rangle$$

scale at large distances and the related "susceptibilities" diverge

$$"\chi" = \int d^d x\ \langle \mathcal{O}_x\ \mathcal{O}_o\rangle_{crit} = \infty.$$

At the perturbed critical point only

$$\langle \mathcal{O}_{t(g)x}\ \mathcal{O}_{t(g)o}\rangle$$

scales at large distances at $t(g) = o$ whereas

$$\langle \mathcal{O}_{gx}\ \mathcal{O}_{go}\rangle$$

has non-critical fluctuations and the corresponding "susceptibility" stays finite Fig. 31.

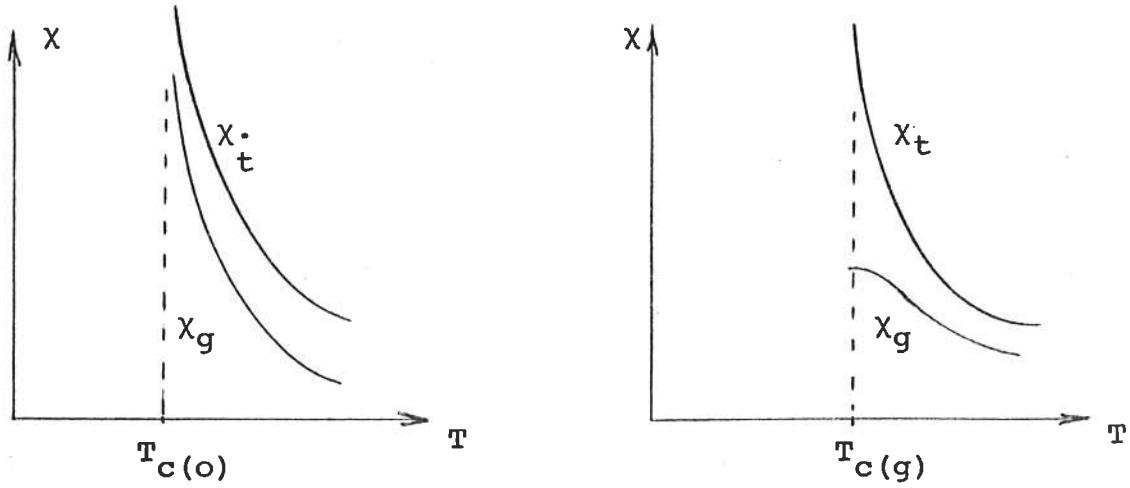


Fig. 31

In the flow diagram of the Hamiltonians, cross-over situations are characterized by a near-by passage of trajectories at an instable fixed point Fig. 32.

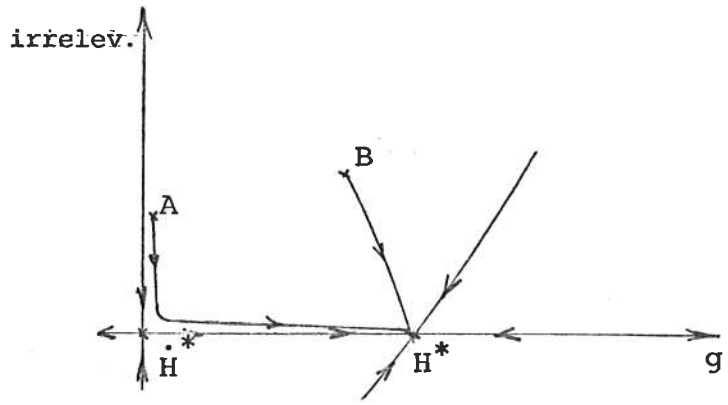


Fig. 32 A: cross-over

B: non cross-over

The cross-over exponents are always given by the exponent of the perturbation term relative to the unperturbed fixed point normalized by the temperature exponent $\dot{y}$.

For the quadratic anisotropy $\dot{y}_g > \dot{y}$ such that $\dot{\phi} > 1$

$$\dot{\phi}_{anis} = 1 + \frac{n}{2(n+8)}\,\varepsilon + \frac{n^2+24n+68}{4(n+8)^3}\,\varepsilon^2 + O(\varepsilon^3).$$

In general $\dot{y}_g < \dot{y}$ and hence $\dot{\phi} < 1$ so for the tricritical cross-over

$$\dot{\phi}_{tricrit} = \frac{1}{2} + \frac{1}{10}\,\varepsilon + O(\varepsilon^2) \quad ; \quad \varepsilon = 3-d$$

and the cubic cross-over

$$\dot{\phi}_c = \frac{n-4}{2(n+8)}\,\varepsilon + \frac{n^3+16n^2+4n+240}{2(n+8)^3}\,\varepsilon^2 + O(\varepsilon^3).$$

Further cross-over situations are shown in the following diagram Fig. 33.

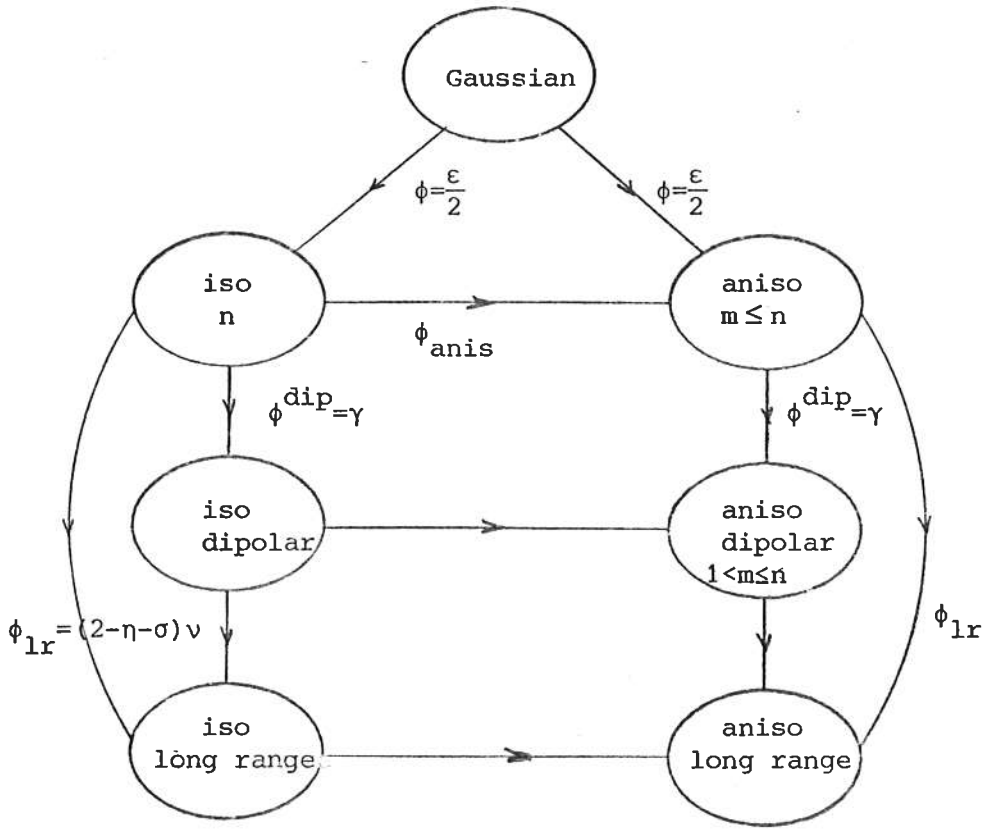


Fig. 33

Cross-over phenomena play an important role in the experimental determination of critical exponents. If there are certain small perturbations in a system the true asymptotic critical behavior may show up only very close to T_c whereas scaling may show up with other exponents in a larger temperature range. A beautiful example is shown in Fig. 34. The cross-over shows up as a rather sharp

kink in the plot where the exponents are given by the ascent of the straight lines.

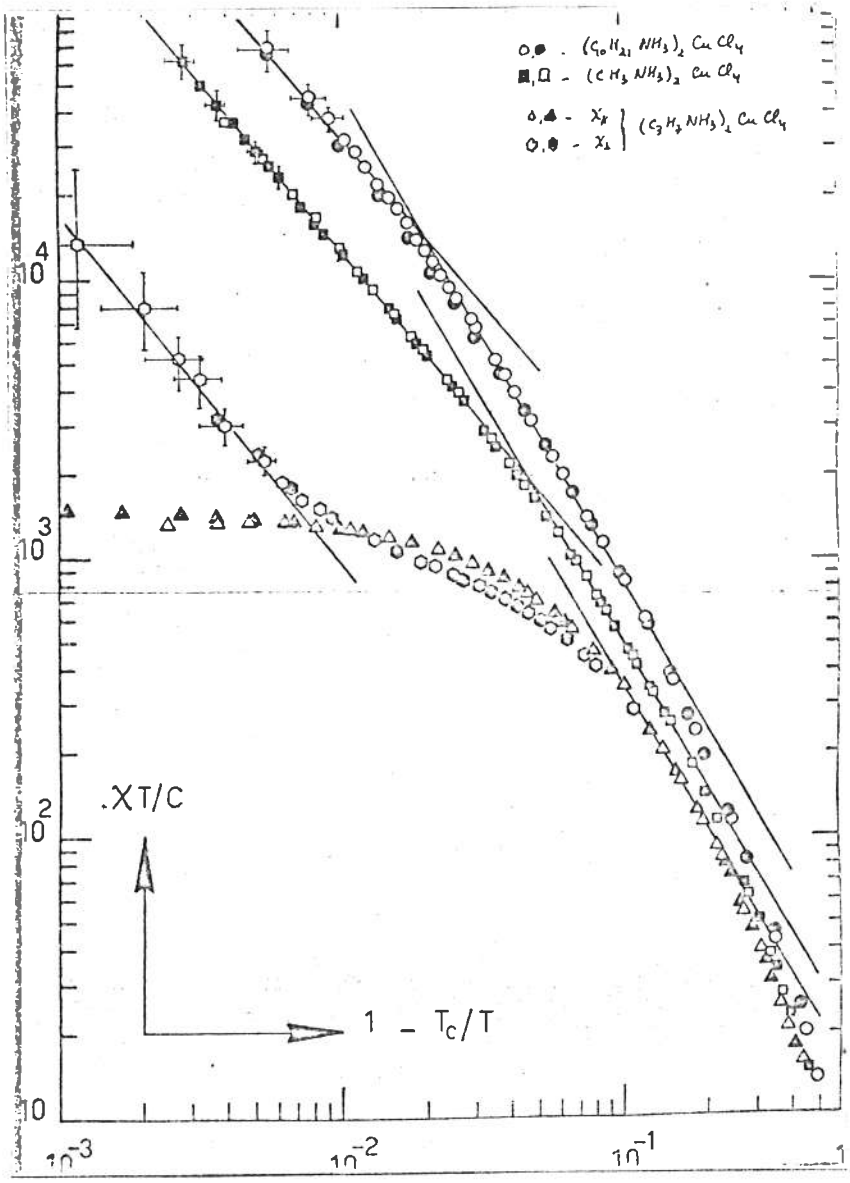


Fig. 34

As indicated by the two sets of parallel lines drawn through the data the lattice-dimensionality crossover is fom 2-dimens. Ising behaviour (γ = 1.75) to 3-dimens. Ising behaviour (γ = 1.25).

16. Universality and Corrections

16.1. Universal Ratios

The RG-methods as developed in part III. where only the leading long ranged fluctuations are treated "exactly" (in the asymptotic sense) only allow us to calculate universal quantities as e.g. the critical exponents which do not depend on short distance (lattice) details of a given system. In scaling forms like

$$\chi = C_{\pm}|t|^{-\gamma} \ , \ C = A_{\pm}|t|^{-\alpha} \quad ; \ t \to \pm o \tag{16.1}$$

the amplitudes $C_{\pm}$, $A_{\pm}$ etc. are not universal, e.g. they depend on the details of the system. Formally this is related to the fact that the normalizations of the fluctuation fields S_x, E_x etc. or its conjugate parameters are ambiguous and these enter directly into $C_{\pm}$, $A_{\pm}$ etc. via

$$\chi = \int d^d x \ \langle S_x \ S_o \rangle^{conn} , \ C = \int d^d x \ \langle E_x \ E_o \rangle^{conn} \text{ etc..} \tag{16.2}$$

It is important to know whether there are other universal quantities apart from the exponents. Indeed there are many other universal numbers namely the <u>universal ratios</u> characteristic of a given critical point. These ratios can be easily obtained by forming ratios of quantities in such a way that the ambiguous normalizations of the S_x, E_x etc. fields drop out. In particular

$$\frac{C_+}{C_-} = 2^{\gamma-1} \frac{\gamma}{\beta} + O(\varepsilon^3) \tag{16.3}$$

and

$$\frac{A_+}{A_-} = 2^{\alpha}(1+\varepsilon)\frac{n}{4} + O(\varepsilon^2) \tag{16.4}$$

are universal. Similarly from the correlation length defined as the normalized second moment

$$\xi^2 = \frac{\int d^d x \; x^2 \langle S_x \; S_o \rangle^{conn}}{\int d^d x \; \langle S_x \; S_o \rangle^{conn}} \tag{16.5}$$

$$\xi = f_{\pm} |t|^{-\nu}$$

we obtain the universal ratio

$$\frac{f_+}{f_-} = 2^{\nu}\{1 + \frac{5}{24}\varepsilon + \frac{1}{432}\,(\frac{295}{24} + 2\,I)\varepsilon^2 + O(\varepsilon^3)\} \tag{16.6}$$

with $\quad I = \int_0^1 dx \, \frac{\ln x(1-x)}{1-x(1-x)} \simeq -2.3439.$

Obviously universal ratios may also be defined by ratios of amplitudes which appear when the critical point is approached from other directions, e.g. along the critical isotherme. For d=3 the numerical values of the above ratios are given in Tab. 12.

		ε-expansion	mean field	series expansion	experiments
A_+/A_- :	n=1	0.55	0.25	0.62-0.7	0.35
	n=2	0.99	0.5	-	1.05
	n=3	1.36	0.75	1.11	1.36
C_+/C_-		4.8	2	5.03±0.05	-
f_+/f_-		1.91	1.41	1.96±0.03	2.05±0.22

Tab. 12

In this case also the ε-expansion results improve the mean field results and to order ε^2 the corrections seem to have the right order of magnitude. This is a further confirmation of the idea of universality and the applicability of RG-methods.

If one introduces <u>global scaling fields</u> à la Wegner such that

$$H = H^* + \sum_\alpha g_\alpha \mathcal{O}_\alpha^* \tag{16.7a}$$

holds globally with

$$R_s H = H^* + \sum_\alpha s^{y_\alpha} g_\alpha \mathcal{O}_\alpha^* \tag{16.7b}$$

then up to normalizations of the g_α or $\mathcal{O}_\alpha^*$ all quantities are universal. (In degenerate situations logarithms or (broken-) powers of logarithms appear in place of s^y). This fact is useful if quantities are expanded in terms of scaling fields.

Examples:

Equation of state: If we set the irrelevant variables equal to zero we have

$$h = cM^\delta f(t/_M 1/\beta)$$

where with normalization $f(o) = 1$ $f(x)$ is a universal function (e.g. depending on d and n only for the isotropic n vector model). The expansion coefficients of $f(x)$ in x for small x or x^{-1} for large x respectively are universal.

Two point correlation: In zero magnetic field above T_c setting the irrelevant field equal to zero we have

$$G^{(2)}(q;t) = D\, q^{-(2-\eta)} g(q/_t \nu).$$

The function g normalized by $g(o) = 1$ again is universal with universal expansion coefficient, if we expand for small q t-fixed or for small t q-fixed respectively. For details we refer to the review article by Brezin-Le Guillou-Zinn-Justin.

16.2. Corrections to Scaling

It is important to evaluate the corrections to scaling in particular because of its relation to the size of the critical region.

a) Thermodynamical quantities

For the thermodynamical quantities the leading correction terms are given by the irrelevant variable g with the largest exponent

$$\omega = -y_i > o \ .$$

The free energy density then has a power series expansion in g and one easily obtains the corrections. A brief discussion was already given in section 13 (see equations (13.30) and (13.31)).

For the n-vector model

$$\omega = d_{S^4} - d = \varepsilon - \frac{3(3n+14)}{(n+8)^3}\varepsilon^2 + O(\varepsilon^3)$$

or

$$\omega = \varepsilon - \frac{\varepsilon^2}{n}\ \frac{(2-\varepsilon)(3-\varepsilon)^2}{2(4-\varepsilon)} A(\varepsilon) + O(\frac{1}{n^2}) \qquad (16.8)$$

and we have for $\varepsilon > o$

$$\begin{aligned} f_{sing}(t,h,g) &= s^{-d} f'(s^y t,\ s^x h,\ s^\omega g) \\ &= |t|^{d\nu} f'(\pm 1,\ h|t|^{-x/y},\ g|t|^{-\omega/y}) \\ &= |h|^{d/x} f'(t|h|^{-y/x},\ \pm 1,\ g|h|^{-\omega/x}) . \end{aligned}$$

By expansion in g the leading corrections easily follow:

$$\chi(t) = C_\pm |t|^{-\gamma}\{1 + gC_{1\pm}|t|^{-\omega\nu} + \ldots\}$$

$$C(t) = A_\pm |t|^{-\nu}\{1 + gA_{1\pm}|t|^{-\omega\nu} + \ldots\}$$

$$\xi(t) = f_\pm |t|^{-\nu}\{1 + gf_{1\pm}|t|^{\omega\nu} + \ldots\}$$

$$h = cM^\delta \quad \{1 + gC_1 M^{-\omega\nu/\beta} + \ldots\} \ ; \ \frac{t}{M^{1/\beta}} << 1$$

$$M = B(-t)^\beta \{1 + gB_1(-t)^{\omega\nu} + \ldots\} \ ; \ h = o$$

Up to the normalization of g the coefficients of the correction terms are universal, i.e. C_{1-}, $A_{1\pm}$, $f_{1\pm}$, C_1, B_1 etc. are determined from say C_{1+} and appropriate universal ratios.

As long as we cannot calculate at least one of the non-universal amplitudes for the physically relevant microscopic model we are not able, however, to determine the size of the critical region. A step in this direction could be the application of the lattice RG-methods (see part II) which in principle allow us to calculate non-universal quantities for the microscopic model. Calculations from the Landau-Ginzburg model by means of the ε- and $^1/_n$-expansion at least give an important structural insight. If the size of the critical region is determined experimentally for one of the above quantities the theory predicts the size of the critical region of the others provided there are no other irrelevant variables which come into play.

We finally have to mention that in the degenerate cases where $\omega = o$ as in $d = 4$ (tricritical $d = 3$) <u>logarithmic corrections</u> are present (see e.g. Wegner 1972).

b) <u>Correlation functions</u>:

We only consider the two point functions $G^{(2)}$ where $G^{(2)}(q;t,h,g)$ stands for $\langle\tilde{S}_q\ \tilde{S}_o\rangle$ or $\langle\tilde{E}_q\ \tilde{E}_o\rangle$.

The critical correlations can be expanded as the free energy in powers of g yielding

$$G^{(2)}(q;o,o,g) = D\ q^{2-\eta}\ \{1+gD_1\ q^{\omega}+\ldots\}.$$

It is much more complicated to evaluate finite temperature or external field corrections because $G^{(2)}(q;t,h)$ is highly non-analytic at $(t,h) = (o,o)$. The problem is to find the correction terms to asymptotic scaling due to finite correlation length i.e. in the region

$$a \ll |x| \ll \xi < \infty$$

ξ is the measure of the distance to the critical point $(t,h) = (o,o)$.

What we considered above was for $\xi = \infty$ an expansion for large distances, i.e. in $\frac{x}{a} \gg 1$. Similarly if we could set $a = o$ we could obtain the critical region expansion $o < |x| \ll \xi$ as an expansion in $\frac{|x|}{\xi} \ll 1$ which is a Wilson short distance expansion. In the lattice and Landau-Ginzburg models discussed here we cannot set $a = o$ resp. $\Lambda = \infty$ without performing singular (at $a = o$) renormalizations on the fields and parameters. For fixed Λ correlation RG-transformations like (11.11) break down for $|q| \gtrsim \Lambda/s$. As $|q| \lesssim \Lambda/s$ the non-linear terms of the RG-transformation dominate and a product of two fields behaves as

$$\mathcal{O}_{\alpha x} \; \mathcal{O}_{\beta y} = \sum_{\alpha} f_{\alpha\beta\gamma}(x-y) \; \mathcal{O}_{\gamma \, \frac{x+y}{2}}$$

where the singular coefficients behave as

$$f(x-y) \propto |x-y|^{d_\gamma - d_\alpha - d_\beta}.$$

Using this expansion one obtains in the region $\xi^{-1} \ll |q| \ll \Lambda$ expansions of the form

$$G^{(2)}(q;t,o,o) = D\, q^{2-\eta}\{1+D_1(\frac{t}{q^{1/\nu}})^{1-\alpha} - D_2(\frac{t}{q^{1/\nu}}) + \ldots\}.$$

In the n-vector model below T_c anisotropic correction terms appear (see Brezin et al.).

With these remarks we refer the interested reader to the references quoted where many more details on the problems touched here can be found.

References

There are a number of excellent lecture notes and review articles available and with some exceptions we restrict to quote these. References to the original literature can be found therein, see in particular M.E. Fisher, Rev. Mod. Phys. 46 597 (1974).

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